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Implementation of the New Control Methods in Simplification of a Multidimensional Control and Optimization of a Control System Parameters

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The main purpose of this text is to present application of the Largest Lyapunov Exponent (LLE) as a criterion for optimization of the new type of simple controller parameters. Investigated controller is the part of numerically simulated control system. The calculation of LLE was done with a new method [2].

Introduction contains reference to previous publications on inverted pendulum control and Lyapunov stability. Application of the new simple formula for LLE estimation in control systems is discussed. In the next part simulated dynamical system is described and new type of simple controller allowing to control multidimensional system is introduced. In the last part results of the simulation are shown along with conclusions to whole dynamics analysis. Comparison of the proposed regulator with the linear-quadratic regulator (LQR) was verified and its better effectiveness with respect to LQR was proved.

Keywords: Largest Lyapunov exponent, robot control, stability, nonlinear dynamics.

1. Introduction

Typical criteria of control performance assessment (CPA) are widely described in variety of publications.In this text application of LLE as CPA criterion will be investigated. Definitions form [1] will be used and calculation of LLE will be done by means of simple numerical method [2].

There are few main multidimensional control methods widely used in control systems. One of them , linear quadratic control provides linear–quadratic regulator (LQR)[20]. The best results for such a regulator can be achieved for linear or linearized systems with small regulation errors. This method gives decent results but is uneasy to compute, especially without professional programs. The whole procedure of constructing such a controller for inverted pendulum and acrobot (system researched in this paper) is described by Russ Tedrake [17]. There were also developed intelligent controllers for this system using Direct Fuzzy Control [20] and spiking neural network [21]. Another interesting type of controllers is Energy Shaping [17] that is commonly used in swing-up actions. The main assumption in this approach is to move the actuator in this direction so that Lagrangian would have lower value than if the actuator would move in the opposite direction. However this method is very effective in swing-up actions but it leads to very high overshoots in the control systems.

In this paper simplification of a multidimensional control will by described.

Control system parameters optimization process was carried out using Lyapunov exponents. There have been developed few types of invariants characterizing dynamical systems. Depending on what kind of information is useful in investigations of the system one can use for instance Kolmogorov entropy [3] or correlation dimensions [4] to evaluate complexity or chaotic level of the system, but the most commonly used are Lyapunov exponents, because allow to predict behavior of the real systems,especially regulated one. This is because its value tells how the state vector should behave in long term action for example determines exponential convergence or divergence of trajectories that start form close initial conditions. There have been developed many algorithms for calculation of Lyapunov exponents by Benettin et al. [5] and Shimada and Nagashima [6], later improved by Benettin et al. [7] and Wolf [8]. All those methods are correct for continuous systems. Numerical algorithms have been developed by Wolf et al. [9], Sano and Sawada [10], and later improved by Eckmann et al. [11], Rosenstein et al. [12] and Parlitz [13].

In the long term behavior only the largest Lyapunov exponent plays significant role in determination of the predictability of the dynamical system. It is especially frequently referred to as a most important evidence of chaos[14], because by definition at least one positive Lyapunov exponent is a confirmation of existence of chaos the system. Methods of calculation of LLE have been propsed by Rosentein et al. [12] and Kantz [15] and later on it has been improved by Kim and Choe [16].

In this paper method that allow calculation of instantaneous values of estimated Lyapunov exponents [18–19] is used in control system parameters optimization process:

$$\lambda^* = \frac{\mathbf{z} \cdot \frac{d\mathbf{z}}{dt}}{|\mathbf{z}|^2} \tag{1}$$

where $\mathbf{z}(t)$ is a perturbation vector that can be defined as difference between reference vector and state vector. An averaged value of $(\tilde{\lambda^*})$ of λ^* in time is an approximation of LLE.

2. Control systems

2.1. *Dynamics and control of the acrobot*

There are many types of the controllers the most basic one is PID controller that is described with an equation:

$$u(t) = k_p \left[e(t) + \frac{1}{T_l} \int_0^t e(\tau) d\tau + T_D \frac{de(t)}{dt} \right] + u(0) \tag{2}$$

where k_p, T_l, T_D are constant coefficients, $e(t)$ is an error of regulation and $u(t)$ is an output signal. A drawback of this regulator is that in multidimensional control systems one would have to construct n equations for the controller, where n is a dimension of the controlled phase space. That would give up to $3n$ constant coefficients to optimize. On the other hand linear–quadratic regulator allowing for multidimensional control needs the system to be linear or linearized with small regulation errors. In the presented article we propose a new type of simple controller for multidimensional control of the nonlinear system.

An application for system simulations has been written in C++ programming language in order to test LLE as CPA criterion. The goal of the program is to simulate behavior of inverted double–link pendulum called acrobot controlled with the new type of simple controller. Name of the system came from combination of two words "acrobat" and "robot" due to its similarities with an acrobat trying to maintain his stability on a rope. Scheme of the control system is presented on Fig. 1.

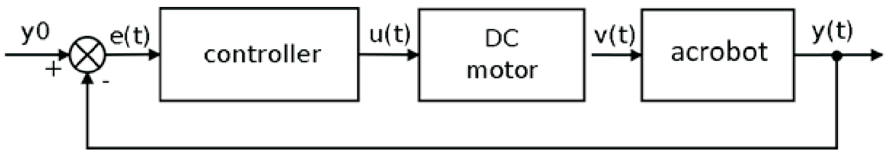

Figure 1 Scheme of the control system

Fig. 2 shows schematic representation of the acrobat with motor allowing stabilization of the system.

System parameters:

l_i – lengths of the links,

l_{ci} – length form the base of the link to its center of mass, length to the center of mass of the motor is equal to l_1,

m_i – masses,

q_i – angular position of the links,

τ – torque of the motor.

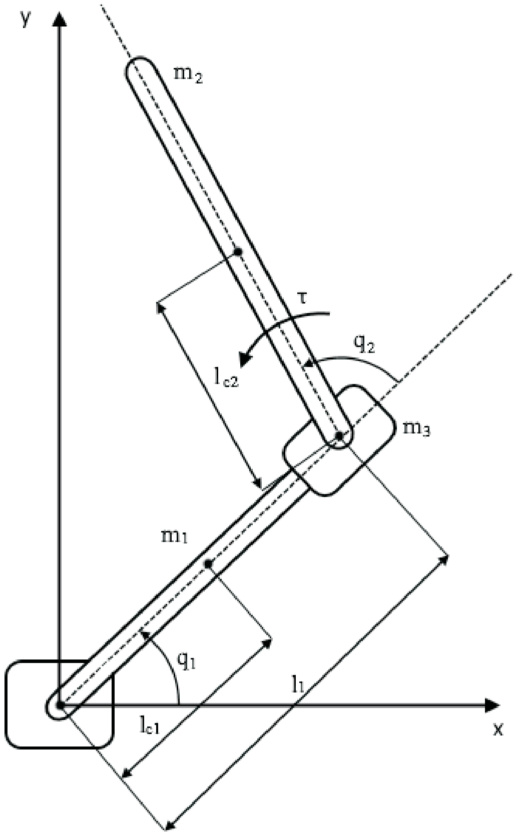

Figure 2 Scheme of the acrobot – control object

As presented in the Fig. 2 the base of the first link m_1 is placed in a bearing and at the end of this link is attached a motor m_3 that moves second link m_2 relative to the first.

The equations of motion of the system are as follows:

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + h_1 + \phi_1 = 0 \tag{3}$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + h_2 + \phi_2 = \tau \tag{4}$$

where:

$$
\begin{aligned}
d_{11} &= m_1 l_{c1}^2 + m_3 l_1^2 + m_2(l_1^2 + l_{c1}^2 + 2l_1 l_{c2}\cos(q_2)) + I_1 + I_2 + I_3 \\
d_{22} &= m_2 l_{c2}^2 + I_2 \\
d_{12} &= d_{21} = m_2(l_{c2}^2 + 2l_1 l_{c2}\cos(q_2)) + I_2 \\
h_1 &= -m_2 l_1 l_{c2}\sin(q_2)(\dot{q_2}^2 - 2\dot{q_2}\dot{q_1}) \\
h_2 &= -m_2 l_1 l_{c2}\sin(q_2)\dot{q_1}^2 \\
\phi_1 &= (m_1 l_{c1} + m_2 l_1 + m_3 l_1)g\cos(q_1) + m_2 l_{c2} g\cos(q_1 + q_2) \\
\phi_2 &= m_2 l_{c2} g\cos(q_1 + q_2)
\end{aligned}
$$

I_1, I_2 is the moment of inertia of the links, I_3 is the moment of inertia of the motor with respect to the base of the first link, $\ddot{q_1}$ is the second derivative of q_1.

2.2. *Dynamics of the motor*

The motor provides torque that moves the second link. Such a motor should be as small as possible so that it would not disturb the whole system and that the system as whole could be controllable. This constraint significantly limits the torque that the motor has at its disposal. For simulations a simple DC motor was taken. Torque of the motor was modeled as linearly dependent on the angular velocity of the rotor. The torque has been calculated form the equation:

$$\tau = \tau_{max}\left(1 - \frac{n_{max}}{n}\right) \tag{5}$$

where n_{max} is a maximal angular velocity of the motor, n is angular velocity at time t and τ_{max} is a maximal torque of the motor.

2.3. *Controller*

The goal of the controller is to keep both links vertically upward. It indicates that the reference signal is four dimensional:

$$Y_0 = \begin{bmatrix} q_1 \\ \dot{q_1} \\ q_2 \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Regarding the fact that the system is described by two second order differential equations (3, 4) the reference signal can be reduced to just two dimensions:

$$Y_0 = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \end{bmatrix}$$

For the well optimized control system achieving Y_0 values would cause that values of the velocities $\dot{q_1}$ and $\dot{q_2}$ tend to converge to zero. Single PID equation is able to control only one variable. In case of the acrobot it is not viable to use simple PID controller, because one would get a set of two equations describing the controller that have to be somehow combined to give single output that controls the motor.

Even if this approach would work, one would have to optimize values of up to six constants in the controller.

In this article a different approach is presented. One can notice that the stability of acrobat can be described by the torque of gravity forces with respect to the base of the first link. This torque is given by equation:

$$\tau_0 = l_{c1}\cos(q_1)g(m_1 + 2m_3) + m_2 g[l_1 \cos(q_1) + l_{c2}\cos(q_1 + q_{2stab})]$$

where q_{2stab} is a required position of the second link in a stable position. For the stable position of the links $\tau_0 = 0$. Based on this condition for actual value q_1 the position of the second link can be calculated as follows:

$$q_{2stab} = \arccos\left(\frac{l_{c1}\cos(q_1)g(m_1 + 2m_3) + m_2 g l_1 \cos(q_1)}{m_2 g l_{c2}}\right) - q_1 \tag{6}$$

The value of q_2 could now be taken as a reference signal for the controller but what can be noticed is that equation (6) does not take into account angular velocity $\dot{q}_1$. Assuming that the kinetic energy associated with $\dot{q}_1$ is equal to:

$$K = \frac{I\dot{q_1}^2}{2}$$

where I is a moment of inertia of the whole acrobot with respect to an axis perpendicular to the plane of motion and going through the base of the first link. The kinetic energy associated with $\dot{q}_2$ was deliberately omitted in order to increase effectiveness of the regulator. This fact was verified experimentally.

Any move of the second link changes location of the center of mass of the whole system. It gives the condition to create the torque τ_0 that decelerates movement of the first link. As it is assumed that if this torque is kept constant than the first link decelerates with constant acceleration and work done by the torque can be obtained from:

$$W = \int \tau_0 dx = \left(\dot{q_1}\Delta t - \frac{\ddot{q_1}\Delta t^2}{2}\right)\tau_0$$

where Δt is the time it would take the first link decelerate to 0. Taking into account:

$$\ddot{q_1} = \frac{\dot{q_1}}{\Delta t} \tag{7}$$

following equation can be obtained:

$$W = \frac{\dot{q_1}\Delta t}{2}\tau_0 \tag{8}$$

Now comparing equations (7) and (8) τ_0 can be calculated:

$$\tau_0 = \frac{I\dot{q_1}}{\Delta t} = \frac{I\dot{q_1}}{T} \tag{9}$$

where T is a constant parameter of regulator. By implementing calculated torque into equation (6) new value of q_{2stab} is calculated:

$$q_{2stab} = \arccos\left(\frac{l_{c1}\cos(q_1)g(m_1 + 2m_3) + m_2 g l_1 \cos(q_1) - \frac{I\dot{q_1}}{T}}{m_2 g l_{c2}}\right) - q_1$$

If an error of regulation is defined as:

$$e(t) = q_{2stab}(t) - q_2(t)$$

than a simple proportional regulator can be constructed:

$$u(t) = k_p e(t)$$

In the very first stage of simulations it was observed that this kind of k_p regulator requires very high values of to stabilize the system. To obtain better characteristics of $e(t)$, especially for small values, the regulator was changed into nonlinear one (arctan):

$$u(t) = k_p \arctan[ce(t)] \tag{10}$$

where c is just a scaling constant. With those simple transformations a final equation for the regulator is reached. This regulator has two constants coefficients k_p and T.

3. Numerical simulations of the control system

The control system is simulated by an application written in C++. The first step of the program is numerical integration of equations (3, 4) using Runge–Kutta method of the fourth order (RK4), than using the formula (9) the torque on the motor is computed, but it cannot be greater than motors capabilities so it is checked with equation (5). In each step of integration values of λ^* is calculated with formula (1) and also its average in time $\tilde{\lambda}^*$. After the average value of λ^* stabilizes it is assumed to be LLE.LLE is applied to verify performance of the control system and the value of the LLE is estimated on basis of state vector using formula (1).

In simulations different coefficients of the regulator have been chosen and for each combination LLE was calculated. Later on, most optimal values of parameters were chosen that corresponded to the smallest value of LLE.

Parameters of the system were chosen based on the measurements of acrobot which is in phase of construction. Its physical parameters are as follows:

$$\begin{aligned}
&m_1 = m_2 = 0,18\,\mathrm{kg},\\
&m_3 = 0,05\,\mathrm{kg},\\
&l_1 = l_{c1} = l_2 = l_{c2} = 0,42\,\mathrm{m}
\end{aligned}$$

Dynamic parameters of the motor are:

$$\begin{aligned}
\tau_{max} &= 1,25\,\mathrm{Nm}\\
n_{max} &= 0,5\,\frac{\mathrm{rad}}{\mathrm{s}}
\end{aligned}$$

and have been determined experimentally.

Initial conditions are:

$$q_1(0) = 5^o = \frac{\pi}{30}\,\mathrm{rad}, \quad q_2(0) = -16^o = \frac{4\pi}{45}\,\mathrm{rad}, \quad \dot{q_1}(0) = 0, \quad \dot{q_2}(0) = 0$$

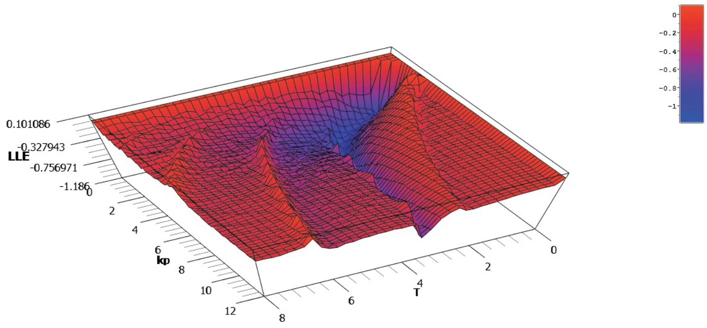

Figure 3 Dependence of LLE for combinations of regulator coefficients k_p and T

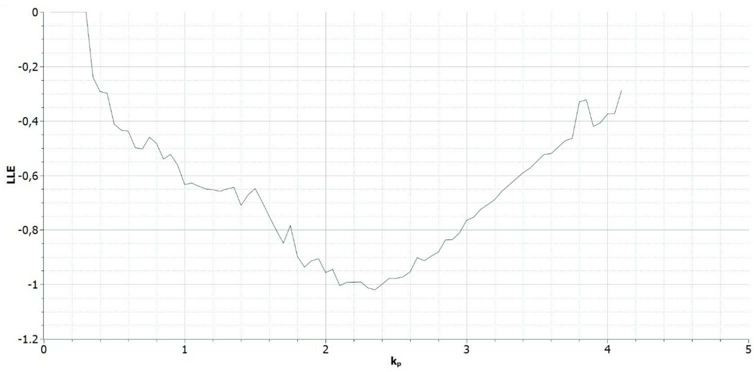

Figure 4 Dependence of LLE from k_p for $T = 2,4$

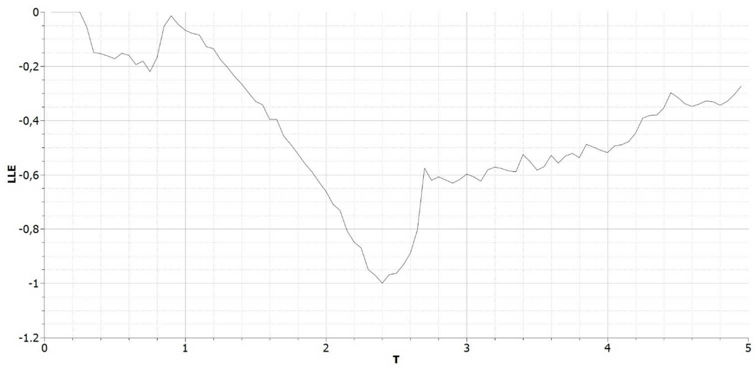

Figure 5 Dependence of LLE from T for $k_p = 2,4$

One step of RK4 integration is equal to $2 \cdot 10^{-4}$s and the process of integration is terminated if the absolute value of difference of $\tilde{\lambda}^*$ for 1000 estimated values does not exceed 10^{-3} or if the first link crosses horizontal position which makes this system no longer controllable with proposed method.

Fig. 3 presents results of calculation of LLE for following values of k_p, T:

$$k_p \in \{0;\ 0,2;\ 0,4;\ \dots 12\}$$
$$T \in \{0;\ 0,2;\ 0,4;\ \dots 12\}$$

In minimum is visible and the lowest value of LLE is obtained for k_p = 2,6±0,2 and T = 2,4±0,2, but to determine closer value for k_p and T another simulations have been done for T = 2,4 and $k_p \in \{0;\ 0,05;\ 0,1;\ \dots 4,1\}$.

On Fig. 4 it can be observed that that the lowest value of LLE is achieved for $k_p = 2,35 \pm 0,05$. Similar simulation has been done but for $k_p = 2,4$ and $T \in (0;\ 0.05 :\ 0.1 \dots;\ 5)$ (Fig. 5).

Also a bifurcation diagram can be sketched (Fig. 6). Points for this diagram were taken after omission of at least 200 oscillations of the second link.

It is clearly seen that that there is neither chaotic nor quasiperiodic behavior which implies that negative Lyapunov exponent should be expected.

Form Figs 4 and 5 one can draw conclusion that optimal constant coefficients for this regulator are $k_p = 2,4 \pm 0,05$ and $T = 2,4 \pm 0,05$. For this set of parameters q_1 as a function of time can be obtained (Fig. 7).

Into this data an exponential decay can be fitted and form its parameters one can read that a decay time is equal to 1,057 s where calculated LLE is equal to -1,013 and because between a decay time and Lyapunov exponent for simple harmonic oscillations there is a relation:

$$\tilde{\lambda^*} = -\frac{1}{T}$$

Then, it can be assumed, that this method of calculation of LLE is well–defined.

To see if parameters of the regulator have been chosen correctly one can draw a plot of $q_1(t)$ for different values of T and k_p but fairly close to $T = 2,4$ and $k_p = 2,4$ (Fig. 8). Values of T and k_p have been chosen as follows:

$$k_p \in (2,2;\ 2,4;\ 2,6)$$
$$T \in (2,2;\ 2,4;\ 2,6)$$

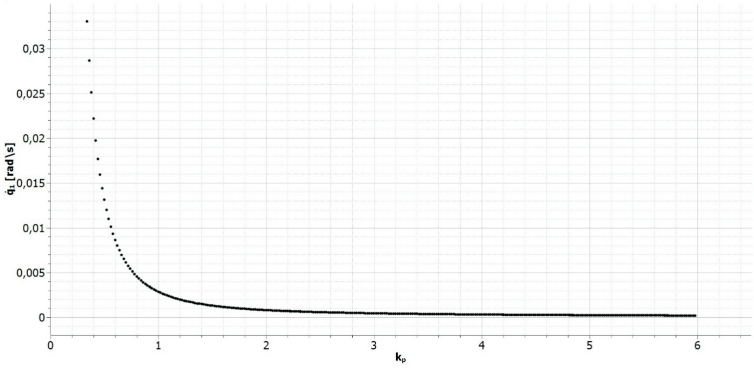

Figure 6 Bifurcation diagram as a function of k_p for $T = 2,4$

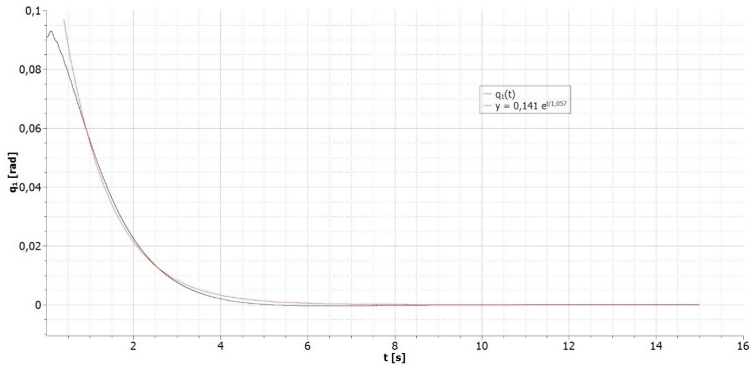

Figure 7 Time series plot of q_1 for $k_p = 2,4$ and $T = 2,4$ with approximation with exponential function

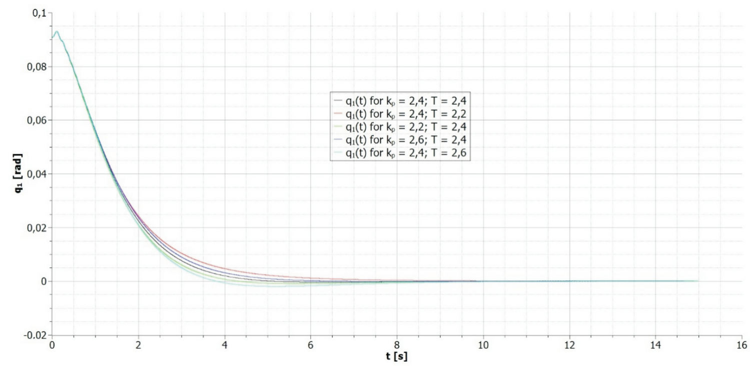

Figure 8 Time series plot of q_1 for different values of regulator parameters k_p and T

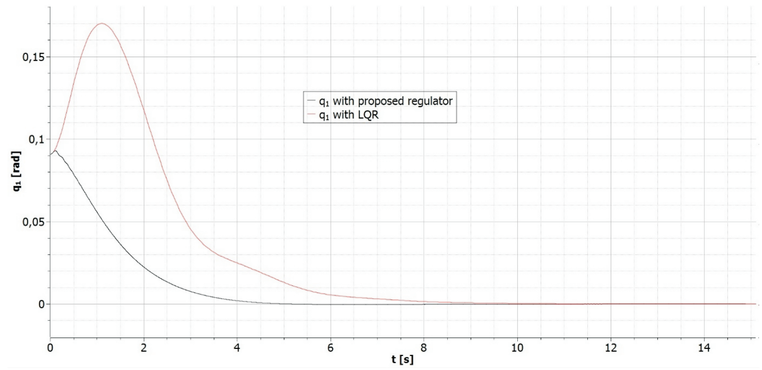

Figure 9 Time series plot of q_1 for LQR and proposed regulator with optimized constant coefficients

What is clearly visible values of q_1 tend to 0 for $T = 2,4$ and $k_p = 2,4$. An interesting fact is that with those optimal parameters of regulation critical damping of error of regulation has been achieved, because for $T = 2,4$, $k_p = 2,2$ and $T = 2,6$, $k_p = 2,4$ and overshoot is visible whereas for $T = 2,2$, $k_p = 2,4$ and $T = 2,4$, $k_p = 2,6$ values of $q_1$tend to 0 slower. This is also an evidence that the change in parameters or in the system itself does not cause the system to be unstable or uncontrollable.

At last LQR controller [17] has been created to compare quality of proposed regulator. A time series plot of q_1 has been done for both controllers with the same initial conditions (Fig. 9).

One can clearly see that with proposed regulator system much faster reaches demanded position of the first link. Also accurately obtained LLE for LQR is equal to -0,68 while for proposed regulator LLE is equal to -1,14.

4. Conclusions

Method of calculation of LLE proposed in [2] can be classified as very effective, however by numerical calculations of λ^* a problem with discontinuity occurs while $|\mathbf{z}| = 0$ especially in systems with few oscillations averaging the values of λ^* tend to be problematic. Nevertheless it was possible to optimize the parameters of the regulator with satisfying precision.

As for the regulator it turned out to be effective and what is more very simple to construct. It was possible to use this kind of regulation because the demanded position of the acrobot was a stable position which automatically makes it an attactor in a phase space. It was possible to reach this position since the system was actually overregulated, because in equation (7) kinetic energy of the second link with respect to the first one and also potential energy of the whole system was not taken into account.

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References

[1] **Dębowski, A.**: Automatyka podstawy teorii, *WNT*, **2008**.

[2] **Dąbrowski, A.**: Estimation of the the Largest Lyapunov exponent–like (LLEL) stability measure parameter from the perturbation vector and its derivative dot product. (Part 2) experiment simulation, *Nonlinear Dynamics*, 78:1601–1608, **2014**.

[3] **Bennettin, G., Froeschle, C. and Scheidecker, J. P.**: Kolmogorov entropy of a dynamical system with increasing number of degrees of freedom, *Phys. Rev. A*, 19:2454–60, **1979**.

[4] **Grassberger, P. and Procaccia, I.**: Characterization of strange attractors, *Phys. Rev. Lett.*, 50:346–9, **1983**.

[5] **Benettin, G., Galgani, L. and Strelcyn, J. M.**: Kolmogorov entropy and numerical experiment, *Phys. Rev. A*, 14:2338–45, **1976**.

[6] **Shimada, I. and Nagashima, T.**: A numerical approach to ergodic problem of dissipative dynamical systems, *Prog. Theor. Phys.*, 61(6):1605–16, **1979**.

[7] **Benettin, G., Galgani, L., Giorgilli, A. and Strelcyn, J. M.**: Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, part I: theory, *Meccanica*, 15:9–20, **1980**.

[8] **Wolf, A.**: Quantifying chaos with Lyapunov exponents. In: Holden V, editor. Chaos, *Manchester University Press*, p. 273–90, **1986**.

[9] **Wolf, A., Swift, J. B., Swinney, H. L. and Vastano, J. A.**: Determining Lyapunov exponents from a time series, *Physica D*, 16:285–317, **1985**.

[10] **Sano, M. and Sawada, Y.**: Measurement of the Lyapunov spectrum from a chaotic time series, *Phys. Rev. Lett.*, 55:1082–5, **1985**.

[11] **Eckmann, J. P., Kamphorst, S. O., Ruelle, D. and Ciliberto, S.**: Lyapunov exponents from a time series, *Phys. Rev. Lett.*, 34(9):4971–9, **1986**.

[12] **Rosenstein, M. T., Collins, J. J. and De Luca, C. J.**: A practical method for calculating largest Lyapunov exponents from small data sets, *Physica D*, 65(1,2):117–34, **1993**.

[13] **Parlitz, U.**: Identification of true and spurious Lyapunov exponents from time series, *J. Bifurcation and Chaos*, 2(1):155–65, **1992**.

[14] **Young, L.**: Entropy, Lyapunov exponents, and Hausdorff dimension in differentiable dynamical systems, *IEEE Trans. Circuits Syst.*, CAS-30:599–607, **1983**.

[15] **Kantz, H.**: A robust method to estimate the maximal Lyapunov exponent of a time series, **Phys. Lett. A**, 185:77–87, **1994**.

[16] **Kim, B. J. and Choe, G. H.**: High precision numerical estimation of the largest Lyapunov exponent, *Commun. Nonlinear Sci. Numer. Simulat.*, 15:1378–1384, **2010**.

[17] **Tedrake, R.**: Underactuated Robotics: Learning, Planning, and Control for Efficient and Agile Machines, *Cours Notes for MIT*, 6.832, **2009**.

[18] **Balcerzak, M., Dąbrowski, A., Kapitaniak, T. and Jach, A.**: Optimization of the Control System Parameters with Use of the New Simple Method of the Largest Lyapunov Exponent Estimation, *Mechanics and Mechanical Engineering*, 17, (4), 325–339, **2013**.

[19] **Szaniewski, M., Dąbrowski, A., Zawiasa, P. and Balcerzak, M.**: Application of the Lyapunov exponent based Current Vibration Control (CVC) parameter in control of an industrial robot, *Mechanics and Mechanical Engineering*, Vol. 19, No. 1, 51–61, **2015**.

[20] **Brown, S. C. and Passino, K. M.**: Inteligent Control for Acrobot, *Journal of Intellignet Robotic Systems*, 18: 209–248, **1997**.

[21] **Wiklendt, L., Calup, S. and Midelton, R.**: A small spiking neural network with LQR control applied to the acrobot, *Neural. Comput. & Applic.*, 18:369–375, **2009**.