Identification of a non-linear damping coefficient characteristics in the free decay test of a single pendulum with friction
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Abstract: A pendulum in form of an equal arms angle body being a part of a two degrees-of-freedom mechanical system with friction is identified with respect to the observed influence of some resistance of its rotational motion in ball bearings. It is damped in a much more complex manner, what could be considered as a non-linear damping. There is supposed between others, that the effective non-linear damping characteristics depends on a few effects such as fluid friction caused by vibrations of the pendulum with two springs in the air, as well as unknown kinds of a frictional resistance existing in ball bearings. The model under investigation finds its real realization on a laboratory rig designed for experimental investigations of viscous and structural frictional effects. A transient response oscillations of the pendulum are described by the explicitly state-dependent free decay. A free decay test of the pendulum with the state-dependent non-linear parameters of damping and stiffness has been performed in this paper. It provided interesting observations that led to elaboration of a method of the overall damping coefficient identification. Effects of application of the proposed semi-empirical method of identification of the overall damping and stiffness coefficients have been illustrated and discussed.

1. Introduction

In the theory of waves the linear models are commonly applied to predict the performance of various engineering objects. In offshore engineering and naval architecture it is common practice to determine damping coefficients, both linear and non-linear, from free decay tests [10]. For example, in ocean engineering it is common practice to obtain damping coefficients of floating structures from free decay tests [1]. Authors of the paper present some work on the determination of non-linear damping coefficients for flap-type oscillating wave surge converters from free decay tests. Simulations of free decay tests in computational fluid dynamics are presented as well as their validation against experimental results is performed. Analysis of the obtained data reveals that linear quadratic-damping, as commonly used in time domain models, is not able to accurately model the occurring damping over the whole regime of rotation amplitudes. The authors concluded that a hyperbolic function is most suitable to express the instantaneous damping ratio over the rotation amplitude.

In the design process, it is essential to use accurate numerical simulation tools to predict the complex aero-hydro-servo-elastic response of a floating wind turbine [11]. Cited paper focuses on the use of the open-water test data of the SWAY prototype wind turbine to calibrate a floating offshore wind turbine numerical model for future validation efforts. After turbine deployment and installation of the NREL instrumentation, five free decay tests were conducted on the SWAY prototype by displacing the system and allowing it to return to equilibrium. The inability to model frictional damping in the universal joints of the system contributes to discrepancies between measured and simulated results. The inability to model frictional damping in the universal joints in the tension rod became significant in affecting the overall motion of the system.

Presented examples and many problems in other measurements of damping [6–9] confirm the need of continuation of investigations on techniques related to identification of parameters of oscillating bodies, especially, when some difficult properties like viscous damping are necessary to determine [12–14].

Our paper is focused on identification of a non-linear damping coefficient characteristics during free vibrations of a single pendulum with friction [3].

The angle body – a single pendulum, a part of the 2-DoF mechanical system with friction shown in Fig. 1a is damped in a much more complex manner, what could be considered as a non-linear damping of some characteristics that is by assumption dependent on angular displacement and velocity of the pendulum rotating about the pivot point s (see Fig. 1).

One assumes in the research an existence of an effective non-linear damping characteristics, which could be dependent on system states affected by a few possible phenomena:

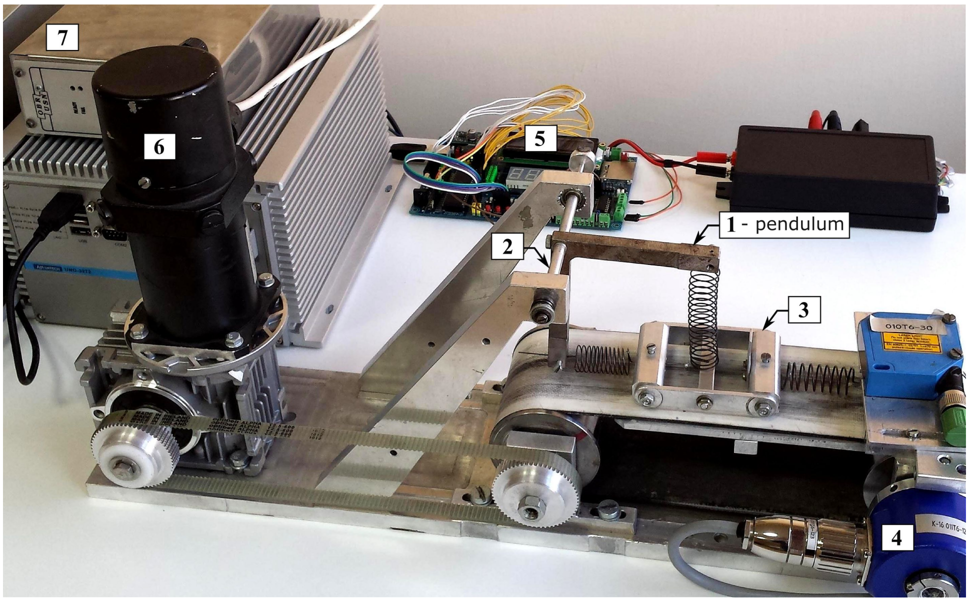
– fluid friction caused by vibrations of the pendulum $\underline{1}$ with two springs in the air (see Fig. 1a);

– unknown kinds of resistance existing in ball bearings, in which both ends of an aluminium shaft $\underline{2}$ are mounted to allow rotation of the pendulum about the joint s.

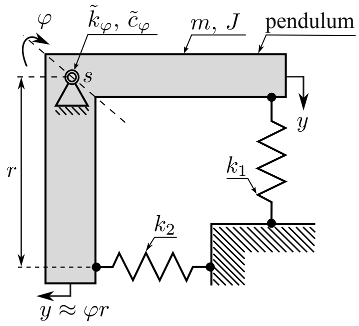
It is likely a hypothesis, but there will be proved in a free decay test an existence of a non-linear explicitly state-dependent characteristics of damping and stiffness of the pendulum.

To get knowledge about the true characteristics of both parameters, the block $\underline{3}$ (see in Fig. 1a) has been stopped and a transient response (see Fig. 2) of pendulum $\underline{1}$ in the free decay test was analysed using some parameter identification method presented below.

Originally, self-excited vibrations of the block $\underline{3}$ sliding on the moving belt cause some irregularly forced response of the pendulum [15]. The pendulum $\underline{1}$ is coupled with the block by means of two springs, and therefore, it changes the normal and tangent contact forces in the frictional connection created by the stick-slip contact of the block sliding on the moving belt. A precise mathematical description of dynamical behaviour of the the single pendulum 1 is very important in the context of identification of the static as well as kinematic characteristics of a frictional contact in the block-on-belt model [16].



a) experimental stand



b) a physical model of single pendulum

Figure 1. Picture of the experimental stand (a) for measurement of friction characteristics with the particular physical model (b): 1 – the pendulum under investigation, 2 – an aluminium shaft rotating in ball bearings fixed in the frame, 3 – the block sliding on the belt (temporarily stopped in the experiment), 4 – incremental encoder, 5 – a microcontroller for data acquisition, 6 – a direct current motor with gear, 7 – a direct current motor driver.

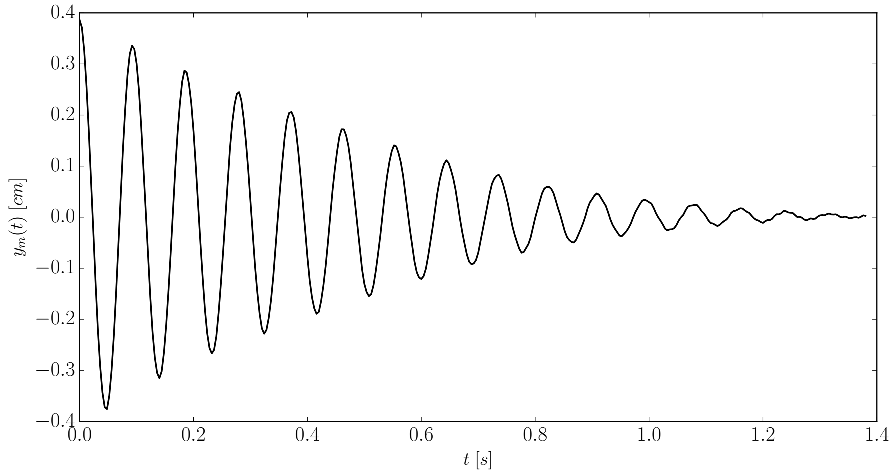


Figure 2. Time history of a transient response of the pendulum in the free decay test. The time series $y_m(t)$ has been acquired from measurements on the laboratory rig (see Fig. 1).

2. A semi-empirical method of estimation of the non-linear characteristics of overall damping and stiffness

It would be interesting to check if the damping and stiffness coefficients of the investigated free decay oscillations are constant.

At small enough angles of rotation, the starting equation in our research follows

$$m\ddot{y} + c\dot{y} + ky = 0, \quad \text{for } y \approx \varphi r \text{ and } \varphi < \pi/36, \tag{1}$$

where: y [m] is the linear displacement of the pendulum, φ [rad] – angle of rotation, a virtual mass $m = J/r^2$ [kg], mass moment of inertia $J = 2.4423 \cdot 10^{-4}$ [kg·m^2], arm length $r = 0.078$ [m], unknown overall damping $c \to \tilde{c}_\varphi(t)$ [N·s/m], unknown overall stiffness $k \to \tilde{k}_\varphi(t) + k_1 + k_2 = \tilde{k}_\varphi(t) + 145.82$ [N/m], where: k_1, k_2 – constant stiffness coefficients estimated from static characteristics of both elastic elements (linear springs), $\tilde{k}_\varphi(t)$ – unknown implicitly state-dependent function of stiffness of the rotational connection (see the joint s in Fig. 1b), which is created by the ball bearings and the pendulum mounted in the bearings, $\tilde{c}_\varphi(t)$ – unknown implicitly state-dependent function of overall coefficient of damping in both symmetrically situated rotational joints.

Equation (1) can be represented in a classic form of a non-forced traditionally damped harmonic oscillator

$$\ddot{y} + \frac{\omega}{Q}\dot{y} + \omega^2 y = 0, \tag{2}$$

where the damping component at $\dot{y}$ is written in canonical form by means of a quality factor Q – a dimensionless parameter of strength of viscous friction in motion of the pendulum [1].

Our oscillator has small mass and is fairly small damped, so factor Q is defined by $2\pi E/|\Delta E|$, where E is the energy of oscillation, ΔE is the energy loss per cycle of the oscillation because of dissipation [17]. The dissipation is expressed either in terms of the dimensionless quality factor or by a damping ratio δ, which has the dimension of frequency.

Then, by a definition [17]

$$2\delta\dot{y} = \frac{\omega}{Q}\dot{y}. \tag{3}$$

Applying in Eq. (3) the definition of period $T = 2\pi/\omega$ between time instances of every two adjacent turning point amplitudes A_i and A_{i+1}, one finds, that constant logarithmic decrement of damping, called the damping ratio δ is given by

$$\delta = \frac{1}{T}\ln\frac{A_i}{A_{i+1}} = \frac{\pi}{Q}.$$

If we check the successive period durations of the free decay oscillations shown in Fig. 2, then all points are irregularly distributed as shown in Fig. 5. Therefore, some usage of any constant damping ratio of the real free response of our single pendulum is not well justified. In consequence, constant damping ratio δ is to be substituted by $\tilde{\delta}(t)$, which will exhibit a non-linear characteristics dependent on time going in the free decay test.

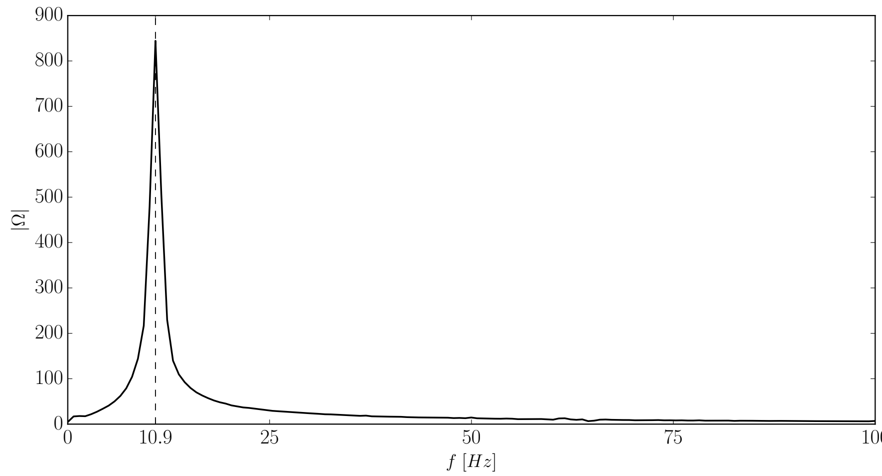


Figure 3. A discrete FFT of a time series $y_m(t)$ of the pendulum's linear displacement acquired from the measurement of angle φ on the experimental stand. One dominant frequency is found as marked by the dashed line.

Analysing in Fig. 3 the frequency spectrum of vibrations from the experimental time series $y_m(t)$ of our single pendulum, one observes, that the body vibrates with the dominant frequency $f_F \approx 10.9$ [Hz] corresponding to $\omega_F \approx 68.45$ [rad/s].

Let us now select two adjacent turning point amplitudes separated by one period T of motion, i.e.: $A_1 = 0.421$, $A_2 = 0.371$ [cm].

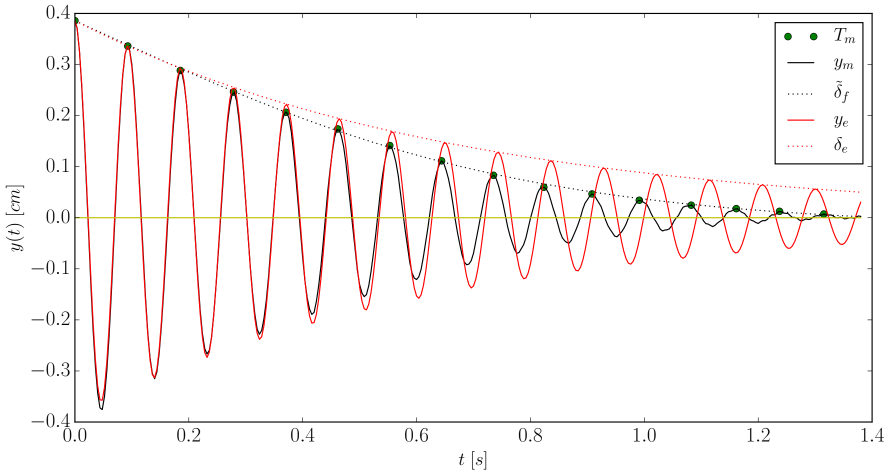


Figure 4. A comparison of two time histories confirming discrepancy between the experimental trajectory $y_m(t)$ and correspondingly the analytical solution $y_e(t)$.

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Then, let us calculate the angular frequency ω with respect to the period T related to the time elapsed between two successive measurements of peak amplitudes A_1 and A_2. We get constant angular frequency, period of oscillations and a damping ratio as below:

$$\omega = \frac{2\pi}{T} = 67.630 \text{ [rad/s]}, \quad T = t_{A_2} - t_{A_1} = 0.093 \text{ [s]}, \quad \delta = \frac{1}{T} \ln \frac{A_1}{A_2} = 1.481 \text{ [1/s]}.$$

Comparable values of both angular frequencies, i.e. $\omega \approx \omega_F$ are confirmed, so after that simple calculus the exact transient response $y_e(t)$ [cm] of the oscillator with exponential decay

$$\delta_e(t) = A_1 \exp(-\delta t), \tag{4}$$

and constant angular frequency ω takes the analytical form

$$y_e(t) = \delta_e(t) \cos(\omega t) = A_1 \exp(-\delta t) \cos(\omega t) = 0.421 \exp(-1.481t) \cos(67.63t). \tag{5}$$

The solution $y_e(t)$ given in Eq. (5) is drawn in Fig. 4 using a red line.

We see in Fig. 4, that the obtained estimate solution $y_e(t)$ bounded by the line of the exponential decay $\delta_e(t)$ passing through turning point amplitudes of the solution, does not coincide with the experimental trajectory $y_m(t)$ (black line) representing the measurement.

Drawing a conclusion, the estimated parameters of the investigated transient response given by Eq. (5) are not valid at each successive constant period T. Therefore, the highest inaccuracy in the coverage of both compared time trajectories is visible at the end of the pendulum's free decay oscillations, i.e. at final time $t_k = 1.4$ [s], when $y_e(t)$ should much closely tend to zero.

The procedure of searching for the new approximating function $\tilde{\delta}_f(t)$ – a state-dependent polynomial decay, which replaces the standard exponential decay with a constant damping ratio, as well as for an approximation of $\tilde{\omega}(t)$, resulting from $\tilde{T}(t)$ is proposed below. Tilde over the symbols make them distinguishable from constants ω, T and function $\delta_e(t)$.

2.1. Estimation of a polynomial decay $\tilde{\delta}_f$ of oscillations

The polynomial decay is an important function in context of the initiated dynamical analysis and the frictional phenomena observed on a stick-slip contact surface of the block-on-belt model investigated on the experimental stand. Moreover, it has significant influence on evaluation of damping properties of the pendulum vibrating at significant velocity variations as observed during experiments [2, 4, 5, 18].

The quality factor introduced in Eq. (2) is expressed by [17]

$$Q(t) = \frac{\pi}{\tilde{T}(t)\left(a_1\tilde{\delta}_f(t) + a_2 + \dfrac{a_3}{\tilde{\delta}_f(t)}\right)}. \tag{6}$$

We obtain the first sought approximation in a form of the polynomial decay (see black doted line in Fig. 4 and 8)

$$\tilde{\delta}_f(t) = \frac{a_2(p(t)-1) + r(p(t)+1)}{2a_1(1-p(t))}, \tag{7}$$

where: $\tilde{\delta}_f(t)$ is the polynomial decay of oscillations, $p(t) = \frac{2a_1A_1+a_2-r}{2a_1A_1+a_2+r}\exp(-rt)$, $r = \sqrt{a_2^2 - 4a_1a_3}$, and $a_1 = -3.45$, $a_2 = 2.52$, $a_3 = 0.05$ are numerically estimated.

2.2. Estimation of variable angular frequency $\tilde{\omega}$

Now, we search for an estimation of the second system parameter that is directly connected with the unknown variable stiffness coefficient, i.e. $\tilde{\omega}$ – the implicitly state-dependent angular frequency of motion.

As it is shown in Fig. 5, the polynomial function $\tilde{T}(t)$ of period instances can be obtained from a continuous approximation of measurement points $T_m(i) = t_{A(i)} - t_{A(i+1)}$

for $i = 0 \ldots 16$ with the use of the following polynomial of third degree

$$\tilde{T}(t) = \gamma_3 t^3 + \gamma_2 t^2 + \gamma_1 t + \gamma_0 \quad \text{for } \bar{\gamma}_{(3\ldots0)} = [-2,\ 28,\ -147,\ 9385] \times 10^{-5}. \tag{8}$$

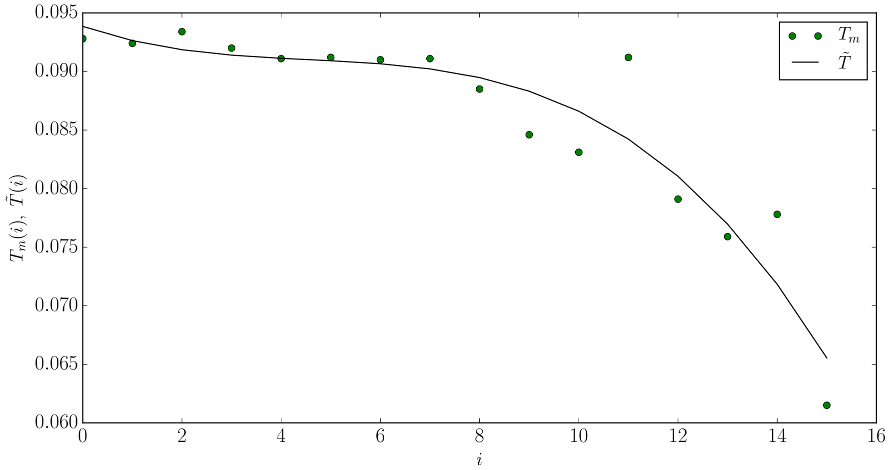


Figure 5. Non-smooth distribution of periods $T_m(i)$ (green circles) calculated between time $t_{A(i)}$ and $t_{A(i+1)}$ of appearance of the successive peak amplitudes $A(i)$ and $A(i+1)$ of oscillations in the experimentally acquired series $y_m(t)$ versus i-th cycle number. Third degree polynomial approximation $\tilde{T}(t)$ of the distribution is matched by solid line.

We have observed that fitting of the trajectory $y_m(t)$ at its final stage for $t \in [1.2, 1.4]$ [s] regarded to small-amplitude vibrations of the pendulum could be more precise. Therefore, the free response's approximation was checked for a replacement of the polynomial approximation $\tilde{\omega}(t)$ by a logarithmic one $\tilde{\omega}_l(t)$ (see Fig. 6), that reads

$$\tilde{\omega}_l(t) = b_1 \tilde{\omega}(t) + b_2 \log \tilde{\omega}(t) + b_3 \quad \text{for } \bar{b}_{(1\ldots3)} = [2.57,\ 0.13,\ 93] \times 10^{-3}. \tag{9}$$

Figure 6 illustrates a time-dependent function $\tilde{\omega}(t)$ that results from the polynomial approximation $\tilde{T}(t)$ given by Eq. (8) with its logarithmic fit $\tilde{\omega}_l(t)$ for the purpose of improvement of the stage of small-amplitude vibrations of the pendulum. After checking the resulting effectiveness of approximations $\tilde{\omega}(t)$ and $\tilde{\omega}_l(t)$, the first one has been selected.

If $\tilde{\delta}_f(t)$ given by Eq. (7) states for the desired approximation of turning point amplitudes of free decay oscillations of the analysed pendulum, then using the obtained polynomial approximation (8) an implicitly state-dependent angular frequency reads

$$\tilde{\omega}(t) = \frac{2\pi}{\tilde{T}(t)} \quad \text{for } t \in [0, t_k]. \tag{10}$$

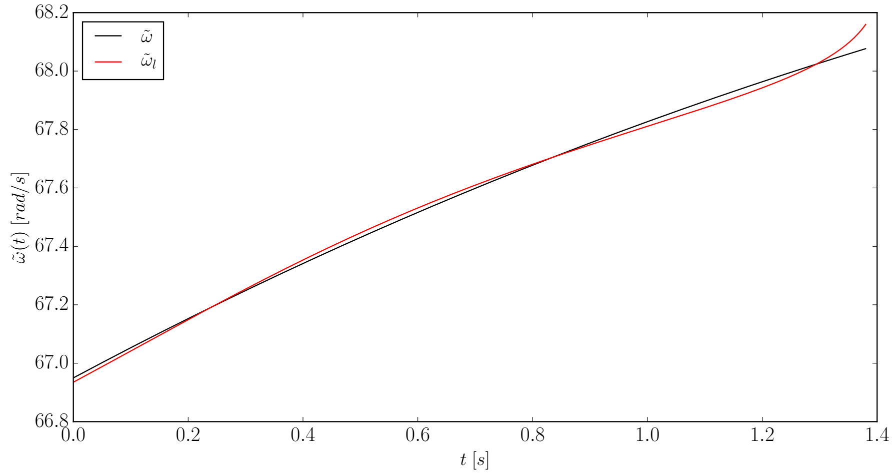


Figure 6. A polynomial approximation of the angular frequency $\tilde{\omega}(t)$ (black line) and its logarithmic fit $\tilde{\omega}_l(t)$ (red line).

2.3. Estimation of implicitly state-dependent parameters of the pendulum

This section takes into account the obtained estimates (6) and (10) to provide definitions of parameters for the investigated dynamical system.

Comparison of terms at state variables y and $\dot{y}$ in Eq. (1) divided by m and in Eq. (2) yields:

$$\frac{\tilde{c}_\varphi(t)}{m} = \frac{\tilde{\omega}(t)}{Q(t)} \quad \text{and} \quad \frac{\tilde{k}(t)}{m} = \tilde{\omega}^2(t),$$

and after rearrangement:

$$\tilde{c}_\varphi(t) = \frac{m\tilde{\omega}(t)}{Q(t)} \quad \text{and} \quad \tilde{k}(t) = m\tilde{\omega}^2(t). \tag{11}$$

The non-linear functions of system parameters, i.e. the variable damping coefficient $\tilde{c}_\varphi(t)$ and variable stiffness coefficient $\tilde{k}(t)$ have been drawn in Fig. 7. These parameters are denoted as functions of time, but it is only valid for the time going in the free decay test. Therefore, the parameters will be implicitly state-dependent when one will need to apply them in the simulation of dynamics of full mechanical system shown in Fig. 1 composed of the block-on-belt subsystem and the identified pendulum forced by motion of the block.

It is worth reminding that by obtaining the stiffness coefficient $\tilde{k}(t)$ in Eq. (11) we have identified the overall stiffness of the pendulum (see Sec. 2) that includes constant components k_1 and k_2 of springs. Unknown at the beginning the state-dependent stiffness $\tilde{k}_\varphi(t)$ of the rotational connection of the pendulum at point s (see Fig. 1b) is found $\tilde{k}_\varphi(t) = \tilde{k}(t) - k_1 - k_2$.

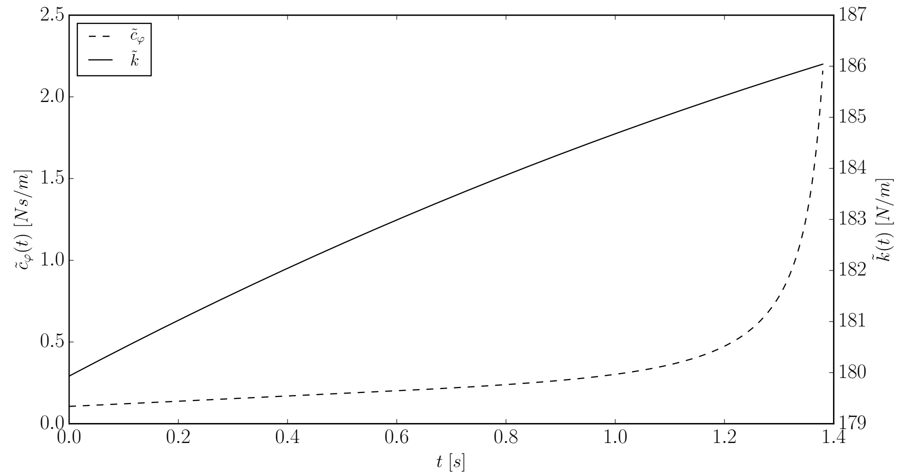


Figure 7. Non-linear functions of system parameters: a) the variable damping coefficient $\tilde{c}_\varphi(t)$ (dashed line), b) the variable stiffness coefficient $\tilde{k}(t)$ (solid line).

3. Numerical verification of the non-linear approximations of parameters of the pendulum

First case. Verification of accurateness of the analytical formula (see blue line in Fig. 8)

$$y_f(t) = \tilde{\delta}_f(t)\cos(w_1\tilde{\omega}(t)t), \tag{12}$$

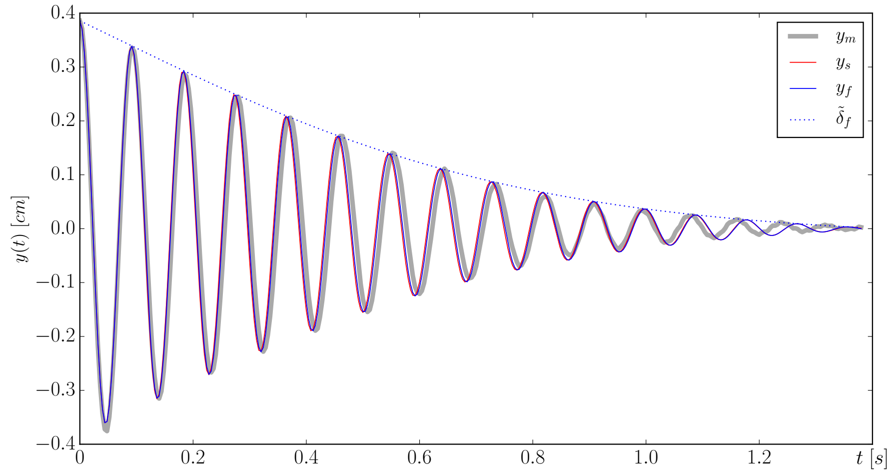
where: $\tilde{\delta}_f(t)$ is defined by expression (7), $\tilde{\omega}(t)$ is defined by (10), and $w_1 = 1.02$ is a non-dimensional fitting parameter of angular frequency.

Second case. The identified non-linear approximations (7) and (10), respectively for $\tilde{\delta}_f(t)$ and $\tilde{\omega}(t)$, are put into a numerical model of the analysed single pendulum to check accurateness of the numerical solution $y_s(t)$ in comparison to measurement $y_m(t)$.

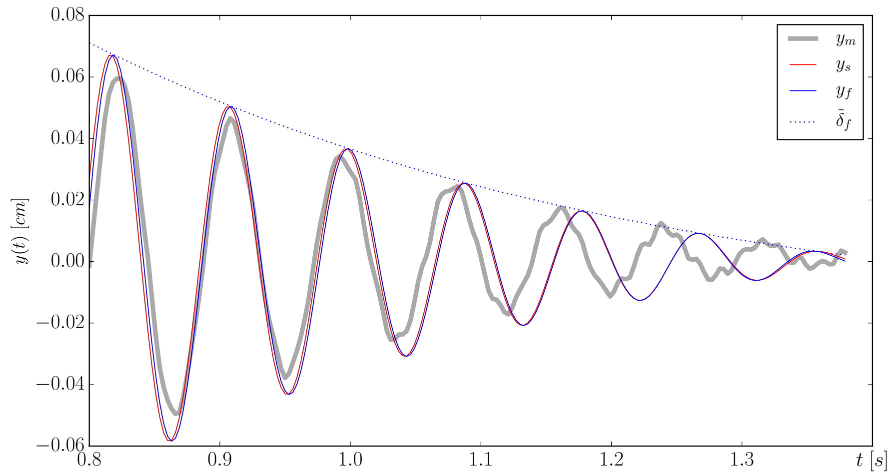
A state-space representation of the single pendulum dynamics described by one second order differential equation (2) is as follows:

$$\begin{aligned} \dot{y}_1(t) &= y_2(t), \\ \dot{y}_2(t) &= -\frac{\tilde{\omega}(t)}{Q(t)}y_2(t) - w_2^2\tilde{\omega}^2(t)y_1(t). \end{aligned} \tag{13}$$

Solving numerically system (13) we obtain a numerical solution $y_s(t) = y_1(t)$ – a linear displacement of the pendulum matched in Fig. 8 by red line. Also here, a non-dimensional fitting parameter of angular frequency $w_2 = 1.029$ is applied in the numerical model to obtain the computed time history $y_s(t)$ better fit to the experimental counterpart $y_m(t)$.



a) $t \in [0, t_k]$



b) $t \in [0.8, t_k]$

Figure 8. Time histories of a transient response of the pendulum in the free decay test: $y_m(t)$ – measurement (thick grey line), $y_s(t)$ – numerical solution (red line), $y_f(t)$ – analytical solution (blue line) that takes into account the identified parameters of the pendulum.

Third case. Average displacement of the pendulum in i-th period of oscillations of the time trajectories $y(t)$ visible in Fig. 8 can be taken into account in the qualitative assessment of the obtained approximations.

The average displacements $\bar{y}(i)$ in i-th cycle period have been computed numerically for the three time series $y_m(t)$, $y_s(t)$ and $y_f(t)$ by means of the formula (bar over the symbol denotes the average value)

$$\bar{y}(i) = \frac{1}{\tilde{T}(i)} \int_0^{\tilde{T}(i)} y(t)dt \quad \text{for } i = 1\ldots 16 \text{ and } y = \{y_m,\, y_s,\, y_f\}. \tag{14}$$

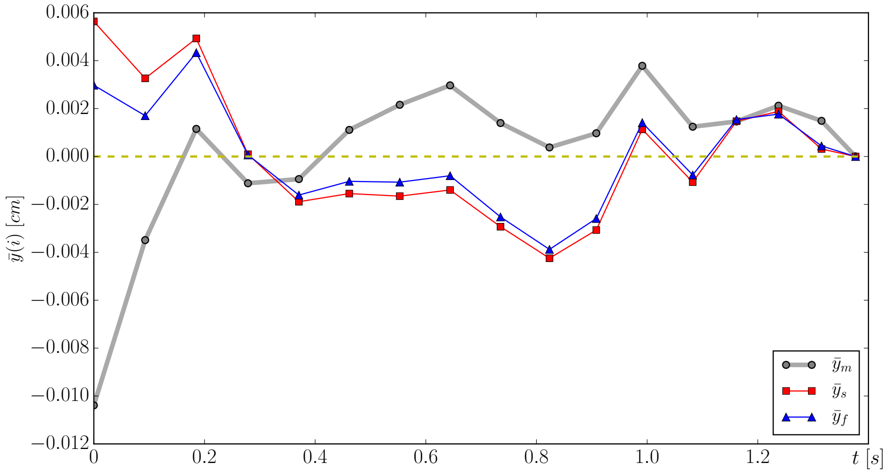


Figure 9. Average displacements of the pendulum in i-th cycle period of oscillations for the measurement $y_m(t)$ in relation to the approximates $y_s(t)$ and $y_f(t)$.

If we take a look at series $\bar{y}_s(i)$ and $\bar{y}_f(i)$ drawn in Fig. 9, then interchangeably, an average of $\bar{y}_s(i)$ is closer to $\bar{y}_m(i)$, but sometimes an average of $\bar{y}_f(i)$. As it is seen, our measurement series $y_m(t)$ is irregular in the duration of each time period of its oscillations. One would find the best result of our identification if $\bar{y}_s(i)$ or $\bar{y}_f(i)$ could as much as possible coincide with $\bar{y}_m(i)$. Hereby, a qualitative method of assessment of the presented identification a single pendulum's parameters has been proposed.

4. Conclusions

The high degree of coverage of trajectories $y_m(t)$ and $y_f(t)$ presented in Fig. 8 proves, that the two system parameters such as damping c and stiffness k introduced at the beginning in Eq. 1 have to be made dependent on the pendulum's state variables. For that requirement, the two implicitly state-dependent parameters $\tilde{c}_\varphi(t)$ and $\tilde{k}(t)$ are proposed.

The non-linear function of $\tilde{c}_\varphi(t)$ drawn in Fig. 7 has an important property. Up to about 1 second of the free decay response, damping of the pendulum depends almost linearly on time. In rough approximation it can be assumed as constant. In Fig. 4, the moment of time points to the turning point amplitude of about 0.05 [cm]. One can use the linear piece of function $\tilde{c}_{\varphi}(t)$ while the oscillator's angular velocity is high enough. When the vibration body exhibits an irregular dynamics by reaching low velocity regimes of motion, then the whole non-linear characteristics visible in Fig. 7 has to be used.

The identified parameters of irregularly damped single pendulum's motion have significant influence on dynamics of the entire dynamical system, which also includes the block-on-belt model. For the practical use of the identified functions of both system parameters, it is necessary to use a transition from the phase space of the free decay to the phase space of full system dynamics including self-excited vibrations of the block on the moving belt. It will regard to much deeper analysis supported by a dedicated numerical methods and will be presented by the authors in further extension of the work.

Acknowledgements

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