

## **ASYMPTOTIC STUDY OF ELASTIC HALF-PLANE WITH EMBEDDED PUNCH**

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A contact problem for an elastic half-plane and an embedded rigid punch is studied. The employed mathematical model describes the behavior of a soil with embedded foundation. The analytical solution governing the stress field behaviour is derived. Singular perturbation and complex analysis techniques are used.

*Key words:* mixed boundary value problem, singular asymptotics, complex variable theory

### **1. Introduction**

The problem under consideration is important for engineers, especially from the point of view of earthquake engineering (Zeng and Cakmak, 1984; Tyapin, 1990). Usually, engineers and designers face with the dilemma whether to use numerical or analytical methods. The use of standard procedures of the finite or boundary element method (Zeng and Cakmak, 1984; Tyapin, 1990) results in the need for further processing of numerical files used to extract the required information. Analysis of available analytical approaches, for example, the method of reduction to a system of integral equations (Glushkov and Glushkova, 1990; Vorovich *et al.*, 1974) has indicated, however, that none of them makes it possible to solve the problem in an exact or, at least, in a simple approximate

way. As a rule, one must use complicated numerical procedures and, in addition, deal with ill-posed problems. These reasons stimulated our choice of the perturbation procedure earlier proposed in Manevitch *et al.* (1970), Manevitch *et al.* (1979), Manevitch and Pavlenko (1991), Shamrovskii (1997).

The remaining part of the paper is organized as follows. In Section 2 we present the governing relations. A significant simplification of the input boundary value problem through a singular perturbation technique is proposed in Section 3. In Section 4, by using a complex variable technique, we compute an analytical solution to the problem. Finally, we discuss and comment on the results obtained (Section 5).

## 2. Statement of the problem

We study an elastic half-plane with an embedded punch (see Fig. 1). According to the linear theory of elasticity, the equilibrium equations gain the following form (Muskhelishvili, 1953)

$$\begin{aligned} eU_{xx} + GU_{yy} + (e\mu + G)V_{xy} &= 0 \\ GV_{xx} + eV_{yy} + (e\mu + G)U_{xy} &= 0 \end{aligned} \quad (2.1)$$

where:  $e = E/(1 - \mu^2)$ ,  $E$  is Young's modulus,  $G$  is the shear modulus,  $\mu$  is Poisson's coefficient.

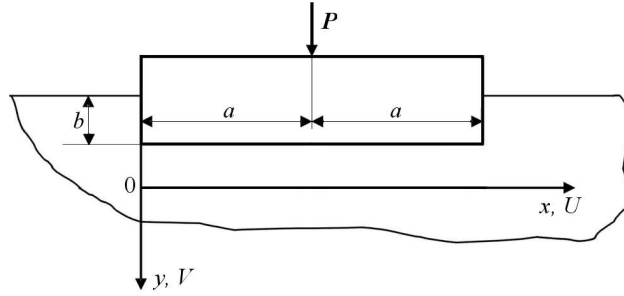


Fig. 1. Half-plane with embedded punch

In Cartesian coordinates, the components of the stress tensor have the following form (Muskhelishvili, 1953)

$$\sigma_y = e(V_y + \mu U_x) \quad \sigma_x = e(U_x + \mu V_y) \quad \tau_{xy} = G(V_x + U_y) \quad (2.2)$$

where:  $\sigma_x$ ,  $\sigma_y$  is the stress in the  $x$ ,  $y$  direction, respectively and  $\tau_{xy}$  is the shear stress.

Now let us consider an interaction between the punch and the elastic half-plane, assuming that the punch is rigidly coupled with the half-plane. Owing to the axial symmetry with respect to the line  $x = a$ , we can restrict our considerations to the zone  $x \leq a$ . The conditions of symmetry give

$$\left. \begin{array}{l} U = 0 \\ \tau_{xy} = 0 \end{array} \right\} \quad \text{at} \quad x = a \quad \wedge \quad y > 0 \quad (2.3)$$

The boundaries of the half-plane ( $y = -b$ ,  $x < 0$  and  $x > 2a$ ) are free from loading, hence

$$\sigma_y = \tau_{xy} = 0 \quad \text{at} \quad \left\{ \begin{array}{l} x < 0 \quad \wedge \quad y = -b \\ x > 2a \quad \wedge \quad y = -b \end{array} \right. \quad (2.4)$$

The conditions of the punch and the halfplane contact follow

$$\left. \begin{array}{l} U = 0 \\ V = d \end{array} \right\} \quad \text{at} \quad \left\{ \begin{array}{l} x = 0 \quad \wedge \quad -b < y \leq 0 \\ 0 < x < a \quad \wedge \quad y = 0 \end{array} \right. \quad (2.5)$$

where  $d$  is the displacement of the punch.

Equation (2.1) and conditions (2.2)-(2.4) give us a biharmonic mixed boundary value problem. Since the exact solution to this problem is unknown, we are going to present how the asymptotic technique can be used.

### 3. Reduction to the Laplace equations

Muskhelishvili (1953) successfully solved the plane elasticity problem for an isotropic body through the application of complex analysis. Meanwhile, the transition to the anisotropic case generally involves much higher complexity (Lekhnitskii, 1968; Ting, 1996). For a slight deviation from the isotropic case, it is possible to introduce a small parameter  $\gamma = (B_a - B_i)/B_i \ll 1$ , where  $B_a$  and  $B_i$  are corresponding properties of the anisotropic and isotropic media, respectively. Then an asymptotic solution (with respect to  $\gamma$ ) can be developed (Lekhnitskii, 1968). On the contrary, for a strong anisotropy, one can take into account the smallness of the parameter  $1/\gamma$ . It is obvious that loading in the  $OX$  ( $OY$ ) direction causes mainly displacement  $U$  ( $V$ ). These reasonable approximations have been used for a long time in the aircraft (Kuhn, 1956) and rocket (Balabukh *et al.*, 1969) engineering. Furthermore, they were successfully applied to the theory of composite and nonhomogeneous materials (Everstine and Pipkin, 1971, 1973; Spencer, 1974; Christensen, 1979; Kosmodamianskii, 1975, 1976). Most of the authors referred above used only the zero

order approximation of the procedure. However, further development of this partially empirical engineering approach was restricted by the evident drawbacks: boundary conditions were not satisfied, the choice of an appropriate approximation was not unique and clear, there was a lack of error estimations, etc.

Beginning from the paper by Manevitch *et al.* (1970), a special asymptotic technique using expansions with respect to  $\gamma$  was developed. A singular character of the asymptotic solution was detected, and the input biharmonic equation was reduced to two Laplace equations. It gave possibility to use the theory of potential. In Manevitch *et al.* (1979), Manevitch and Pavlenko (1991), Shamrovskii (1997), higher order approximations were derived, and it was also shown that even in the isotropic case (when the small parameter has the maximum value) the error of the first approximation is rather low. Mathematical foundations of the described asymptotic technique were also studied by Bogan (1983, 1987).

Now let us briefly describe the asymptotic procedure used (for technical details we refer the readers to references: Awrejcewicz *et al.* (1998), Bauer *et al.* (1994), Shamrovskii (1979)). We introduce dimensionless parameters  $\varepsilon = G/e$ ,  $\bar{\mu} = \varepsilon^{-1}\mu$ . Then input equations (2.1) can be rewritten as follows

$$\begin{aligned} U_{xx} + \varepsilon U_{yy} + \varepsilon(\bar{\mu} + 1)V_{xy} &= 0 \\ \varepsilon V_{xx} + V_{yy} + \varepsilon(\bar{\mu} + 1)U_{xy} &= 0 \end{aligned} \quad (3.1)$$

As soon as  $\varepsilon < 1$ ,  $\bar{\mu} \approx 1$ , we can use the parameter  $\varepsilon$  for asymptotic splitting of system (3.1). We pose expansions for the functions  $V$ ,  $U$  of the forms

$$\begin{aligned} U &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \\ V &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \end{aligned} \quad (3.2)$$

The derivatives of the displacements are estimated as follows

$$\frac{\partial(U; V)}{\partial x} \approx \varepsilon^{\alpha_1}(U; V) \quad \frac{\partial(U; V)}{\partial y} \approx \varepsilon^{\alpha_2}(U; V) \quad (3.3)$$

The order of the function  $U$  with respect to the function  $V$  can be estimated in the following manner

$$U \approx \varepsilon^{\alpha_3} V \quad (3.4)$$

Substituting formulas (3.2) and (3.3),(3.4) into equations (3.1), and comparing the coefficients, one may conclude that asymptotics strongly depend on the parameters  $\alpha_k$ . Therefore, all possible values of  $\alpha_k$  are sought, when the asymptotics ( $\varepsilon \rightarrow 0$ ) have both mathematical (well-posedness) and physical

meanings. It is remarkable that by this simple (but laborious) procedure we obtain only two systems which are analyzed below:

—  $\alpha_1 = 0.5$ ,  $\alpha_2 = 0$ ,  $\alpha_3 = 1.5$

$$u_{0xx}^{(1)} + \varepsilon u_{0yy}^{(1)} = 0 \quad v_{0y}^{(1)} = -\varepsilon(\bar{\mu} + 1)u_{0x}^{(1)} \quad (3.5)$$

—  $\alpha_1 = -0.5$ ,  $\alpha_2 = 0$ ,  $\alpha_3 = 1.5$

$$\varepsilon v_{0xx}^{(2)} + v_{0yy}^{(2)} = 0 \quad u_{0x}^{(2)} = -\varepsilon(\bar{\mu} + 1)v_{0y}^{(2)} \quad (3.6)$$

Analysis of stress relations (2.2) leads to the following results. For the first and second type of stress-strain state (3.5) and (3.6), the main stresses read

$$\begin{aligned} \sigma_{0x}^{(1)} &= e u_{0x}^{(1)} & \tau_{0xy}^{(1)} &= G u_{0y}^{(1)} \\ \sigma_{0y}^{(2)} &= e v_{0y}^{(2)} & \tau_{0xy}^{(2)} &= G v_{0x}^{(2)} \end{aligned} \quad (3.7)$$

Simplified equations (3.5), (3.7)<sub>1,2</sub> and (3.6), (3.7)<sub>3,4</sub> describe all possible stress-strain states. Links between these states can be established after splitting the input boundary conditions, see Awrejcewicz *et al.* (1998), Bauer *et al.* (1994), Shamrovskii (1979). Equations (3.5)<sub>1</sub> and (3.6)<sub>1</sub> can be reduced to Laplace equations due to a simple affine transformation of the independent variables (see below). So, we can use the highly developed theory of complex variables.

#### 4. Analytical solution

We restrict our investigation only to the function  $v_0^{(2)}$ , since the punch pressure can be expressed by it. For equations (3.6)<sub>1</sub> and (3.7)<sub>2</sub>, the affine transformation  $x_1 = x\sqrt{\varepsilon}$  is used. Let us denote  $v_0^{(2)} \equiv v$ ,  $\sigma_{0y}^{(2)} \equiv \sigma_y$ ,  $\tau_{0xy}^{(2)} \equiv \tau_{xy}$ . Then one obtains the following Laplace equation

$$v_{x_1x_1} + v_{yy} = 0 \quad (4.1)$$

and the expressions for the stresses

$$\sigma_y = e v_y \quad \tau_{xy} = \sqrt{G} e v_x \quad (4.2)$$

Boundary conditions for equation (4.1) can be written as follows

$$\begin{aligned} v y &= 0 & \text{at } x_1 < 0 & \wedge y = -b \\ v_x &= 0 & \text{at } x = a_1 & \wedge 0 < y < \infty \\ v &= d & \text{at } \begin{cases} x_1 = 0 & \wedge -b < y < 0 \\ 0 < x_1 < a_1 & \wedge y = 0 \end{cases} \end{aligned} \quad (4.3)$$

where  $a_1 = a\sqrt{\varepsilon}$ .

We replace boundary conditions (4.3)<sub>3</sub> by the expressions

$$\begin{aligned} v_y = 0 & \quad \text{at } x_1 = 0 \quad \wedge \quad -b < y < 0 \\ v_x = 0 & \quad \text{at } 0 < x_1 < a_1 \quad \wedge \quad y = 0 \end{aligned} \quad (4.4)$$

Let us introduce a complex variable  $z = x_1 + iy$  and an analytical function  $\bar{\Phi}(z) = v_x - iv_y$ . One can write the following expressions for the boundary values of the complex variable function  $\bar{\Phi}$

$$\begin{aligned} \text{Im } \bar{\Phi}(z) = 0 & \quad \text{at } (x_1 < 0, y = -b) \cup (x_1 = 0, -b < y < 0) \\ \text{Re } \bar{\Phi}(z) = 0 & \quad \text{at } (0 < x_1 < a_1, y = 0) \cup (x_1 = a_1, 0 < y < \infty) \\ \bar{\Phi}(z) \rightarrow 0 & \quad \text{when } |z| \rightarrow \infty \end{aligned} \quad (4.5)$$

Now we map our governing area (see Fig. 1) onto the upper half-plane by the Schwarz-Christoffel transformation (Walker, 1964)

$$z = iC \int_0^\xi \sqrt{\frac{\varphi}{(\varphi - 1)(\varphi + B)}} d\varphi \quad (4.6)$$

where  $B, C$  are constants.

The correspondence of the points can be written as follows:

$z$	$\xi$
$-\infty - ib$	$-\infty - i0$
$-0 - ib$	$-B + i0$
$0 + i0$	$0 + i0$
$a_1 + i0$	$1 + i0$
$a_1 + i\infty$	$\infty + i0$

The boundary values for  $\bar{\Phi}(z(\xi))$  read

$$\begin{aligned} \text{Im } \bar{\Phi}(z(\xi)) = 0 & \quad \text{at } -\infty < \xi < 0 \\ \text{Re } \bar{\Phi}(z(\xi)) = 0 & \quad \text{at } 0 < \xi < \infty \\ \bar{\Phi}(z(\xi)) \rightarrow 0 & \quad \text{when } |\xi| \rightarrow 0 \end{aligned} \quad (4.7)$$

The solution to boundary value problem (4.7) has the form

$$\bar{\Phi}(z(\xi)) = \frac{Ai}{\sqrt{\xi}} \quad (4.8)$$

where  $A$  is a constant.

For the shear stress  $\sigma_{xy}$ , one obtains

$$\tau_{xy} = \sqrt{Ge}v_{x_1} = -\sqrt{Ge} \operatorname{Re} \Phi(z) = \frac{\sqrt{Ge}A}{\sqrt{-\xi}} \quad (4.9)$$

at  $x_1 = 0$ ,  $-b < y < 0$ , where

$$\begin{aligned} y &= -\frac{2C}{k'} [E(\delta, k) - k'^2 F(\delta, k)] + 2C \sqrt{\frac{\xi(1-\xi)'}{\xi+B}} \\ -\beta < \xi < 0 \quad \delta &= \arcsin \sqrt{\frac{B-\xi}{1+\xi}} \\ k &= \sqrt{\frac{B}{1+B}} \quad k' = \sqrt{1-k^2} = \frac{1}{\sqrt{1+B}} \end{aligned} \quad (4.10)$$

The stress  $\sigma_y$  has the form

$$\begin{aligned} \sigma_y = ev_y = -e \operatorname{Im} \Phi(z) &= -\frac{eA}{\sqrt{\xi}} \quad \text{at } 0 < x_1 < a_1 \\ x_1 = \frac{2C}{k} [E(\delta', k') - k^2 F(\delta', k')] - 2C \sqrt{\frac{\xi(1-\xi)'}{\xi+B}} & \quad 0 < \xi < 1 \end{aligned} \quad (4.11)$$

where

$$\delta' = \arcsin \sqrt{\frac{\xi(1+B)}{\xi+B}}$$

and  $F(\cdot, \cdot)$ ,  $E(\cdot, \cdot)$  are elliptic integrals of the first and second kind, respectively.

Schwarz-Christoffel integral (4.6) is calculated through formulas (3.141.9)-(3.141.11) taken from the handbook by Gradshteyn and Ryzhik (1965).

For the determination of the unknown constants  $A$ ,  $B$ ,  $C$ , one can use the following conditions:

— the equilibrium condition

$$\begin{aligned} \frac{1}{2}P &= \int_{-b}^0 \tau_{xy} dy - \int_0^a \sigma_y dx = G \left[ \int_{-b}^0 \operatorname{Re} \Phi(z) dy + \int_0^{a_1} \operatorname{Im} \Phi(z) dx_1 \right] = \\ &= GAC \int_B^1 \frac{d\xi}{\sqrt{(1-\xi)(\xi+B)}} = \pi AGC \end{aligned} \quad (4.12)$$

— conditions of the  $x_1$  and  $y$  correspondence

$$\begin{aligned} a_1 = x_1|_{\xi=1} &= \frac{2C}{k'} \left[ E\left(\frac{\pi}{2}, k'\right) - k^2 F\left(\frac{\pi}{2}, k'\right) \right] \\ -b = y|_{\xi=-B} &= \frac{-2C}{k} \left[ E\left(\frac{\pi}{2}, k\right) - k'^2 F\left(\frac{\pi}{2}, k\right) \right] \end{aligned} \quad (4.13)$$

and, therefore, one gets

$$A = \frac{\frac{1}{2}P}{\pi GC} \quad C = \frac{bk'}{2\left[E\left(\frac{\pi}{2}, k\right) - k'^2 F\left(\frac{\pi}{2}, k\right)\right]} \quad (4.14)$$

Equations (4.13) yield a transcendental equation for the constant  $B$

$$\frac{b}{a_1} = \frac{E\left(\frac{\pi}{2}, k\right) - k'^2 F\left(\frac{\pi}{2}, k\right)}{E\left(\frac{\pi}{2}, k'\right) - k^2 F\left(\frac{\pi}{2}, k\right)} \quad (4.15)$$

Since (4.15) can be solved numerically by application of a routine procedure, formulas (4.9)-(4.11) and (4.14) yield an analytical solution.

## 5. Conclusion

In this paper, a novel approximate solution to the contact problem for the elastic half-plane with an embedded punch is proposed. Although our approach is based on the approximate method, it seems that the obtained accuracy is acceptable in the engineering practice. This question is going to be investigated numerically in our future research.

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### **Analiza asymptotyczna zagadnienia kontaktowego stempla i półprzestrzeni sprężystej**

#### Streszczenie

W pracy rozważono zagadnienie kontaktowe związane z oddziaływaniem sztywnego stempla i półprzestrzeni sprężystej. Analizowany problem może np. modelować fundamenty budowli usytuowane w podatnym gruncie. Sformułowano w postaci analitycznej pole naprężeń przy użyciu metody perturbacji osobliwych i techniki zmiennych zespolonych.

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