

INTERACTION OF GLOBAL AND LOCAL MODES AT FREE OSCILLATIONS OF COMPRESSED THIN-WALLED BEAMS

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The influence of a compressive force on free nonlinear oscillation of thin-walled members is studied. The case of internal autoparametric resonance is considered when the ratio of natural frequencies of global and local modes is close to 2:1. It is shown that in a vicinity of this resonance the oscillations become amplitude-frequency modulated due to the energy exchange between linear modes. Two coupled stationary oscillation modes exist (which can be regarded as synchronized vibrations in overall and local mode). A bifurcational value of the energy is established after exceeding of which the uncoupled oscillation in overall mode becomes unstable, and stationary modes can be only coupled.

1. INTRODUCTION

Dynamic behavior of thin-walled members under action of a compressive force was studied in a few works from various points of view. In [1] the response of a thin-walled column to the suddenly applied end compression was studied, with particular attention to a decrease in the load carrying capacity due to the sudden application of the load. The equations of motion included cubic nonlinearities, and modulated synchronized oscillations in global and local modes with a single frequency were observed at not very large loads. In [2] the influence of in-plane forces on natural frequencies and natural modes of thin-walled members was studied, in the framework of the linear theory.

In this report we focus our attention on another aspect of the problem. Application of the axial force changes the ratio of natural frequencies for various modes - overall and local. Usually the ratio ω_1/ω_2 (ω_1 and ω_2 are the natural frequencies for global and local mode, respectively) is less than unity when the force is absent. If the critical stress σ_2 for local mode is below than the critical stress for global mode σ_1 , then at gradually increasing compressive stress σ the frequencies ratio ω_1/ω_2 rises and tends to infinity when σ approaches to σ_2 . So this ratio equals to 2:1 for a certain σ (and close to this value in some interval of σ). If the equations of motion include quadratic nonlinearities (f.e., in the case of thin-walled beams with one axis of symmetry) then the internal autoparametric resonance occurs leading to specific features in the dynamic behavior.

In this paper free nonlinear oscillations of thin-walled beams under compression are studied with account of interaction of global and local modes in the case of autoparametric resonance. Note that in paper [3] an autoparametric resonance in thin-walled open-cross section beams has been studied which is caused by the interaction of another modes— global flexural and flexural-torsional modes.

2. GOVERNING EQUATIONS

Let us obtain the equations of motion of a compressed beam assuming that the natural modes of vibration coincide with the buckling modes (it is the case, in particular, for simply supported beams). Let λ is a load factor, U_j ($j=1-N$) are the linear buckling modes with critical load factor values λ_j close to the minimal critical value λ_{\min} . We assume the following expansion for the dynamic displacements field (Koiter's type expansion for the buckling problem):

$$U \equiv (u, v, w) = \lambda U_0 + \zeta_j(t) U_j + \zeta_j(t) \zeta_k(t) U_{jk} + \dots \quad (1)$$

where ζ_j is the amplitude of j^{th} mode (normalized, in given case, by the condition of equality of the maximal deflection to wall thickness t), U_{jk} are the second order displacement fields; summation is supposed on repeated indexes. Then, similarly to the Koiter's theory for the buckling problem, the potential energy can be written in the form

$$P = \frac{1}{2} c_0 \lambda^2 + \frac{1}{2} \sum_s c_s \zeta_s^2(t) \left(1 - \frac{\lambda}{\lambda_s}\right) + c_{jkl} \zeta_j(t) \zeta_k(t) \zeta_l(t) + \dots \quad (2)$$

where coefficients c_0, c_j, c_{jkl} are determined with known formulae [4,5].

The kinetic energy with account of expansion (1) and conditions of orthogonality for U_j and U_{jk} is as follows:

$$T = \frac{\rho}{2} \int_z \int_A (u_{,t}^2 + v_{,t}^2 + w_{,t}^2) dA dz = \frac{1}{2} \sum_s m_s \zeta_{s,t}^2 \quad (3)$$

where A is the cross-section area, z is the longitudinal coordinate, ρ is the density,

$$m_j = \rho \int_z \int_A (u_j^2 + v_j^2 + w_j^2) dA dz \quad (4)$$

Then the Lagrange's equations are as follows:

$$m_j \zeta_{j,tt} + c_j \left(1 - \frac{\lambda}{\lambda_j}\right) \zeta_j + c_{klj} \zeta_k \zeta_l = 0 \quad (j=1, \dots, N) \quad (5)$$

In the case of two interactive modes – global ($j=1$) and local ($j=2$) the equations of motion with account of symmetry and periodicity conditions for the local modes take the form (all terms in the potential energy (2) depending on odd degrees of the local mode amplitude ζ_2 should vanish):

$$\begin{aligned} \frac{1}{\omega_{01}^2} \zeta_{1,tt} + \left(1 - \frac{\lambda}{\lambda_1}\right) \zeta_1 - b_{111} \zeta_1^2 - b_{221} \zeta_2^2 &= 0 \\ \frac{1}{\omega_{02}^2} \zeta_{2,tt} + \left(1 - \frac{\lambda}{\lambda_2}\right) \zeta_2 - 2b_{122} \zeta_1 \zeta_2 &= 0 \end{aligned} \quad (6)$$

where

$$\omega_{0j}^2 = \frac{c_j}{m_j}, \quad b_{klj} = -\frac{c_{klj}}{c_j} \quad (7)$$

Denoting

$$\omega_j^2 = \omega_{0j}^2 \left(1 - \frac{\lambda}{\lambda_j}\right) \quad (8)$$

and introducing nondimensional time $\tau = \omega_2 t$, rewrite equations (6) in the nondimensional form

$$\begin{aligned} \zeta_{1,\tau\tau} + \frac{\omega_1^2}{\omega_2^2} \zeta_1 - d_{111} \zeta_1^2 - d_{221} \zeta_2^2 &= 0 \\ \zeta_{2,\tau\tau} + \zeta_2 - 2d_{122} \zeta_1 \zeta_2 &= 0 \end{aligned} \quad (9)$$

where

$$d_{klj} = \frac{b_{klj} \omega_{0j}^2}{\omega_{02}^2 \left(1 - \frac{\lambda}{\lambda_2}\right)} \quad (10)$$

3. SOLUTION BY THE MULTIPLE SCALES METHOD

We consider the case when the frequencies ratio is close to 2:1:

$$\omega_1 = 2 \omega_2 - \varepsilon \delta \quad (11)$$

where δ is a the detuning parameter, ε is a small parameter. The set of equations (9) is solved by the multiple scale method [6]. Introducing «slow» and «fast» times $T_n = \varepsilon^n \tau$ ($n = 0, 1, \dots$), we seek the solution in the form of asymptotic series

$$\zeta_1 = \varepsilon \zeta_{11} + \varepsilon^2 \zeta_{12} + \dots, \quad \zeta_2 = \varepsilon \zeta_{21} + \varepsilon^2 \zeta_{22} + \dots \quad (12)$$

The standard procedure of the method gives the following first and second order sets of equations:

$$\mathcal{E}^1: \quad D_0^2 \zeta_{11} + 4 \zeta_{11} = 0, \quad D_0^2 \zeta_{21} + \zeta_{21} = 0 \quad (13)$$

$$\mathcal{E}^2: \quad D_0^2 \zeta_{12} + 4 \zeta_{12} = -2 D_0 D_1 \zeta_{11} + d_{111} \zeta_{11}^2 + d_{221} \zeta_{21}^2 + \delta^* \zeta_{11} \quad (14)$$

$$D_0^2 \zeta_{22} + \zeta_{22} = -2 D_0 D_1 \zeta_{21} + 2 d_{122} \zeta_{11} \zeta_{21}$$

where $D_n = \partial(\dots)/\partial T_n$, $\delta^* = 4\delta/\omega_2$. Solution of Eqns (13) can be written as

$$\zeta_{11} = \frac{1}{2} \left(A_1(T_1) e^{i2T_0} + \bar{A}_1(T_1) e^{-i2T_0} \right), \quad (15)$$

$$\zeta_{21} = \frac{1}{2} \left(A_2(T_1) e^{iT_0} + \bar{A}_2(T_1) e^{-iT_0} \right)$$

Substituting (15) into the right hand sides of (14), one obtains from the condition of absence of secular terms following two complex equations:

$$8i \frac{dA_1}{dT_1} - d_{221} A_2^2 - 2\delta^* A_1 = 0, \quad 2i \frac{dA_2}{dT_1} - d_{122} A_1 \bar{A}_2 = 0 \quad (16)$$

Passing to polar coordinates $A_s = a_s e^{i\theta_s}$ ($s=1, 2$) we obtain four ordinary differential equations with respect to amplitudes a_j and phases θ_j ($j=1,2$):

$$8 a_1' = d_{221} a_2^2 \sin \gamma, \quad 2 a_2' = -d_{122} a_1 a_2 \sin \gamma \quad (17)$$

$$8 a_1 \theta_1' = -d_{221} a_2^2 \cos \gamma - 2\delta^* a_1, \quad 2 a_2 \theta_2' = -d_{122} a_1 a_2 \cos \gamma$$

where $\gamma = 2\theta_2 - \theta_1$ and $(\dots)' = d(\dots)/dT_1$. We see that only coefficients d_{122} d_{221} (and since only coefficients $c_{122} = c_{212} = c_{221}$ in the potential energy (2)) influence the nonlinear mode interaction in the quadratic system considered. From first two Eqns (17) we get the energy integral (e is a constant proportional to the energy of oscillation):

$$4a_1^2 + \nu a_2^2 = e \quad (\nu = d_{221}/d_{122}) \quad (18)$$

From the last two Eqns (17) one has the following equation for γ' :

$$8 a_1 \gamma' = -d_{122} (8a_1^2 - \nu a_2^2) \cos \gamma + 2\delta^* a_1 \quad (19)$$

Let us introduce the new variable ξ changing in the interval (0,1):

$$\xi = \frac{2a_1}{\sqrt{e}} \quad (a_1^2 = \frac{\xi^2 e}{4}, \quad a_2^2 = \frac{e}{\nu}(1 - \xi^2)) \quad (20)$$

Then from (17)–(19) we have following set of two Eqns in ξ, γ :

$$\begin{aligned} 4 \frac{d\xi}{dT_1} &= d_{122} \sqrt{e} (1 - \xi^2) \sin \gamma \\ 4 \xi \frac{d\gamma}{dT_1} &= d_{122} \sqrt{e} \left[(1 - 3\xi^2) \cos \gamma + 2 \frac{\xi}{\psi} \right] \end{aligned} \quad (21)$$

where

$$\psi = 2 d_{122} \sqrt{e} / \delta^* \quad (22)$$

The set of equations (21) has the integral («integral of amplitude-frequency modulation»)

$$\psi \xi (1 - \xi^2) \cos \gamma + \xi^2 = C \quad (23)$$

This integral enables us to describe nonstationary oscillations with exchange of energy between the linear modes (15). The integral curves (23) constitute an «amplitude–frequency portrait» (AFP) in the plane ξ, γ . Due to periodicity it is sufficient to construct this AFP in the rectangle $0 \leq \xi \leq 1, 0 \leq \gamma \leq \pi$. The AFP depends on the single parameter ψ (22). In Fig. 1 there are presented AFP for three values of parameter ψ (for $\gamma > \pi$ these portraits should be continued symmetrically).

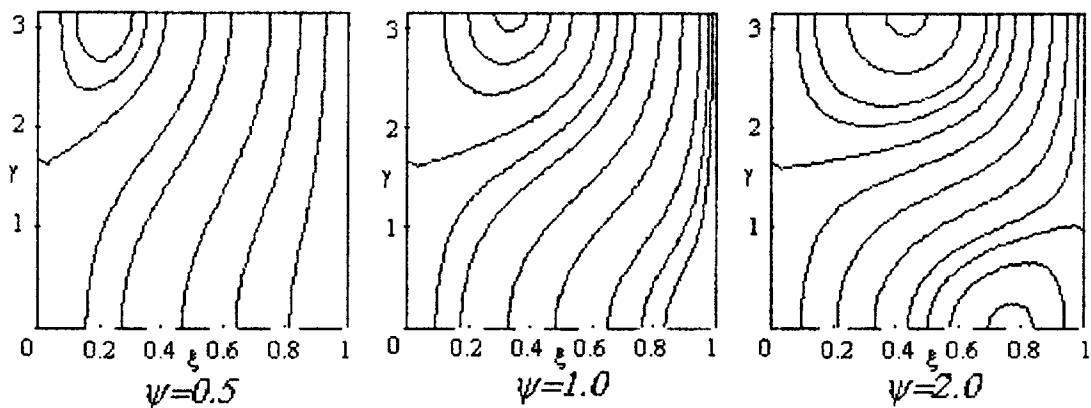


Fig. 1. Amplitude frequency portraits for three values of parameter ψ (22).

As parameter ξ^2 characterizes the ratio of energy oscillation in the first (global) mode to the total energy, these curves visually show magnitude of energy exchange between global and local modes. The energy exchange depends on the nondimensional parameter ψ , which is determined by the energy, detuning parameter δ^* and coefficient

d_{122} . The AFP presented show that the energy exchange rises with increasing ψ , i.e. with increasing energy, and that it depends on the phase difference γ between the linear modes.

4. STATIONARY COUPLED OSCILLATIONS. BIFURCATIONAL VALUE OF THE ENERGY

Stationary points on the amplitude–frequency portraits (Fig.1) correspond to stationary oscillations, i.e. to synchronized dynamic regimes. These points are determined by condition of vanishing the right hand sides in Eqns (21):

$$(1 - \xi^2) \sin \gamma = 0, \quad \psi (1 - 3\xi^2) \cos \gamma + 2\xi = 0 \quad (24)$$

It follows from the first equation that either $\xi = 1$ or $\gamma = n\pi$. The point $\xi = 1, \gamma = (-1)^k \arccos(1/\psi) + k\pi, k = 0, 1, \dots$ correspond to uncoupled oscillation in the first (overall) mode. Another roots of (24) are

$$\begin{aligned} \gamma = 0, \quad \xi = \xi_1 &= \frac{1}{3\psi} \left(1 \pm \sqrt{1 + 3\psi^2} \right) \\ \gamma = \pi, \quad \xi = \xi_2 &= \frac{1}{3\psi} \left(-1 \pm \sqrt{1 + 3\psi^2} \right) \end{aligned} \quad (25)$$

These are coupled stationary oscillations which can be «in phase» ($\gamma = 0$) or «antiphase» ($\gamma = \pi$). Values ξ should lie in the interval (0,1). It is easily seen that only for one sign (before the radicals in (25)) values ξ_1 and ξ_2 are positive (if $\psi > 0$ – for «+», if $\psi < 0$ – for «-»). It is also easily seen that it is sufficient to consider only the case $\psi > 0$, because at change of the ψ sign the root ξ_1 commutes with ξ_2 . If $\psi > 0$ then the root $\xi_2 < 1$ for any ψ , and therefore this stationary point exists at any energy of oscillation. But the root $\xi_1 < 1$ only for $\psi > 1$. The value $\psi = 1$ determines a bifurcational value of the energy e_* . From (22) we have for this value:

$$e_* = \left(\frac{\delta^*}{2d_{122}} \right)^2 = \left(\frac{2\delta}{\omega_2 d_{122}} \right)^2 \quad (26)$$

It can be proved with the use of integral (23) that stationary coupled modes (25) are stable. When $e < e_*$ only one coupled stationary mode exists (see the first graph in Fig. 1). It is seen from this graph (and can be easily proved) that uncoupled linear

oscillation in local mode ($\xi = 0$) is unstable – any perturbation results in excitation of the global mode, and the periodical exchange of energy between two modes takes place. When $e = e^*$ the second stationary point appears (point $\gamma = 0, \xi = 1$ at the second graph in Fig.1), and at $e > e^*$ two coupled stationary modes exist (the third graph in Fig.1). Simultaneously the uncoupled global mode becomes unstable – any oscillation in this mode is accompanied by oscillation in the local mode.

Thus the autoparametric resonance yields to following consequences at free oscillations: a) the linear local mode becomes unstable (it exchanges the energy with the global mode); b) when energy of oscillations is small there exist single coupled stationary mode (with prevailing deflection in local mode); c) there exist a bifurcational value of the energy e^* , corresponding to appearance of another coupled stationary mode (with prevailing deflection in global mode), and simultaneously the linear overall mode becomes unstable (so at large amplitudes both linear modes – global and local – are unstable, and both coupled modes are stable).

5. NUMERICAL EXPERIMENT

In the numerical experiment we considered a beam of channel cross section with parameters: length $L = 400$ mm, web width $b_1 = 50$ mm, flange width $b_2 = 25$ mm, thickness $t = 1$ mm, Young's modulus $E = 2 \cdot 10^5$ MPa. As a load factor λ the nondimensional stress $\sigma^* = \sigma \cdot 10^3 / E$ was assumed. The critical stresses for global and local modes and postcritical coefficients were calculated by the general program [5] and are as follows: $\sigma_1^* = 3.563$, $\sigma_2^* = 1.0507$ (for number of halfwaves $m = 6$), $b_{111} = 0.00457$, $b_{221} = 0.3358$, $b_{122} = 0.1103$. The natural frequencies (without compressive force) are: $\omega_{01} = 2422$ (rad/s), $\omega_{02} = 7900$ (rad/s). Here only some results of the numerical experiment are presented.

There were assumed various values of compressive stresses. For two values – $\sigma^* = 1.0$ and 1.03 the natural frequencies, their ratio, detuning parameter δ^* and bifurcational value of energy are:

	ω_1	ω_2	ω_1/ω_2	δ^*	e^*
$\sigma^* = 1.00$	2054.2	1735.4	1.1837	0.6082	0.3232
$\sigma^* = 1.03$	2042.1	1108.8	1.8416	2.5988	0.0029

In the second case ($\sigma^* = 1.03$) the ratio ω_1/ω_2 is close to 2:1. In Fig. 2 a, b results of integration of set (9) for this case at energy $e = 0.02$ are presented (the global and local oscillation, respectively). Initially the global mode was excited, with very small perturbation in local mode ($\xi = 0.99999$). Energy is periodically transferred from the global mode to local mode and inversely; oscillations are synchronized with frequencies ratio 2:1. For comparison in Fig. 3 results are given for $\sigma^* = 1.00$ (with

more pronounced initial excitation of the local mode). Oscillations are not synchronized, energy exchange is not periodical.

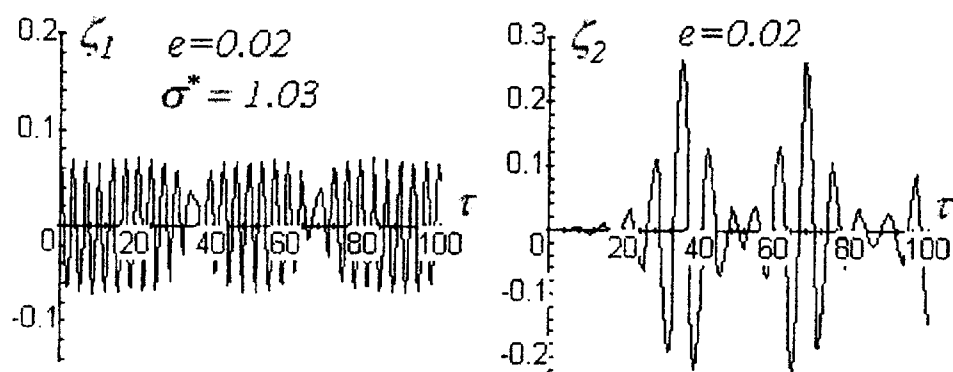


Fig. 2 Nonstationary oscillation with periodical energy exchange between modes.

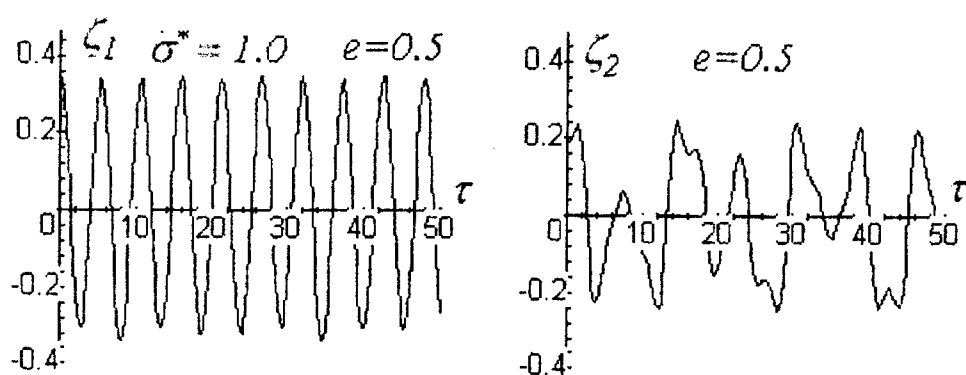


Fig. 3 Global and local oscillations far from autoparametrical resonance.

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