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The Dynamic Behavior of the Electrically Charged Cloud of the Ice Crystals

Artur Wirowski*

Department of Structural Mechanic, Technical University of Lodz, Lodz, Poland
*Corresponding author: artur.wirowski@p.lodz.pl

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Abstract The paper includes the derivation of the equation of the two-dimensional, dynamic behavior of electrically charged cloud of ice crystals. A large crystal rotation angles and a continuous distribution of charges on the surface of the crystals are included in deliberations. Finally, possible solutions of model equation are discussed and compared with solutions available in the literature. The resulting model can be used as a mechanical basis for optic models of the atmospheric phenomenon called the "miracle of the sun".

Keywords: *the miracle of the Sun, finite difference method, non-linear vibrations, mathematical modeling*

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1. Introduction

1.1. The Motivation to Take the Research Topic

The topic of the vibrations of the electrically charged cloud of the ice crystals, have been already the subject of the author's research in the work [1]. The non-linear one-dimensional model of rotational vibration has been presented in that paper. It has been proposed as the mechanical basis of dynamic, atmospheric optical phenomenon commonly called the "miracle of the sun". The first simply model of this phenomenon was presented in [2].

However, one-dimensional model is not sufficient to a comprehensive consideration of phenomenon called the "miracle of the sun", because it allows for only the analysis of the reflection of light from a single crystal of ice. The real cloud which is a three-dimensional body, is causing multiple reflection of sunlight. Hence the need to define a two-dimensional mechanical model, which in the future could be used to better and more precise modeling of the phenomenon called the "miracle of the sun", and thus for better understand it.

1.2. Aim and the Methodology of the Paper

The aim of the paper is derivation of the differential equation which describes dynamic behavior of the cloud of electrically charged ice crystals. The considerations are limited to plate crystals, that assumes that the the horizontal dimension of the crystals is significantly larger than their thickness. A large crystal rotation angles and a continuous distribution of charges on the surface of the crystals are included in deliberations.

The methodology of the derivations of the model equation is analogous to that used in [1]. However, unlike to that considerations, four adjacent crystals distributed in two-dimensional space is taken into account. The starting point for the considerations are the electrostatic forces acting on a single crystal from adjacent. Next we calculate the rotating moments acting on the crystal. The resulting very complex equation is simplify to a short form that is usefull for further considerations. Ultimately by using the finite difference method formulas, we transform equation to the continuous form.

2. The Modelling

2.1. The Assumptions

Let us introduce the following modeling assumptions (Figure 1):

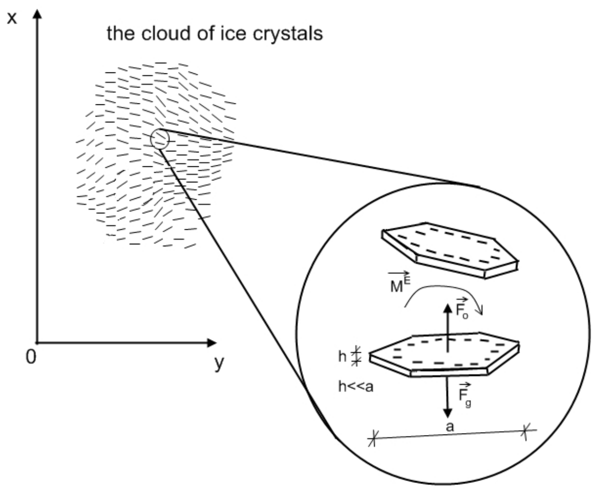

Figure 1. The schematic representation of the modeling assumptions. The individual regular hexagonal plate crystals are shown in enlargement

- A homogeneous cloud of identical, electrically charged ice crystals is given,
- The crystals are evenly distributed in the cloud both vertically and horizontally,
- The crystals have the shape of regular hexagonal plates, i.e. their height is negligibly small in comparison to other dimensions,
- The gravitational force acting on every crystal is balanced by the force of air resistance acting on a crystal falling down uniformly,
- The air resistance forces influencing the crystal rotation are linearly proportional to the linear velocity in the rotational motion,
- The electric charge of the crystal is uniformly distributed on its surface,
- We assume that the synchronization occurs between the crystals, i.e. the difference between the rotation of adjacent crystals is negligible.

2.2. The Equation of the Rotational Motion of an Individual Crystal

$$\vec{M}=I\vec{\varepsilon} \qquad (1)$$

where:

$I=\frac{5\sqrt{3}}{256}a^4 h\rho$ - the moment of inertia for the axis of crystal's rotation,

$\vec{M}$ - the total resultant moment acting on the crystal,

$\vec{\varepsilon}$ - the angular acceleration in the rotational motion,

ρ - the density of ice.

Due to the continuous charge distribution on the crystals' surfaces, in the following modeling we consider the moments derived from the interaction between the appropriate surfaces of adjacent crystals. Then we can expand the resultant moment as (Figure 2):

$$\vec{M}=\sum_{\gamma}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\vec{r}(\eta)\times\vec{F}_{\gamma}\left(\eta,\xi_{\gamma}\right)d\xi_{\gamma}d\eta + \int_{-\frac{a}{2}}^{\frac{a}{2}}\vec{r}(\eta)\times\vec{F}_{R}(\eta)d\eta+\vec{M}^{E},\gamma=1..4 \qquad (2)$$

where: $\vec{r}(\eta)=[\eta,0]$ is a conductive vector of an infinitely small segment of the surface of the analyzed crystal, $\vec{F}_{\gamma}\left(\eta,\xi_{\gamma}\right)$ are the forces of the electrostatic interaction between the infinitely small segments of the surface of adjacent crystals respectively $d\xi_{\gamma}$ and $d\eta$, $\vec{F}_{\gamma}\left(\eta,\xi_{\gamma}\right)$ is the force of air resistance acting on an infinitely small segment of the crystal's surface. The coordinates η, ξ_{γ} are local coordinates, running respectively on the surfaces of the center and adjacent crystals and adopting the values from -½ a to ½ a for the extreme edges of the crystal and 0 in the midst of each surface (Figure 2). $\vec{M}^{E}$ is the external forcing vibrations moment.

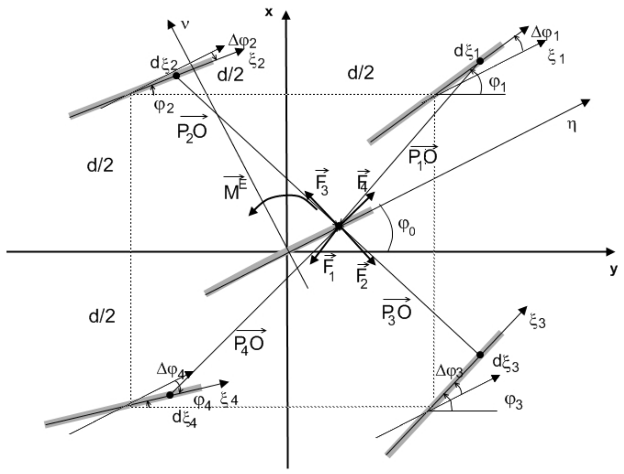

Figure 2. The electrostatic forces between the infinitely small segments of the surfaces of the analyzed crystals

The resistance force, in accordance to the assumptions, is proportional to the linear velocity of segment $d\eta$, forces $\vec{F}_{\gamma}\left(\eta,\xi_{\gamma}\right)$ are Coulomb forces and therefore:

$$\vec{F}_{\gamma}\left(\eta,\xi_{\gamma}\right)=\frac{dq\cdot dq}{4\pi\varepsilon_{r}\varepsilon_{0}}\cdot\frac{\overline{\overline{P_{\gamma}O}}}{\left|\overline{\overline{P_{\gamma}O}}\right|^{3}},\vec{F}_{R}(\eta)=R\frac{\partial\vec{\varphi}}{\partial t}\eta \qquad (3)$$

where:

R is the coefficient of air resistance,

$$\overline{P_{\gamma}O}=\begin{bmatrix}-d\left(\beta\sin\varphi_{0}+\alpha\cos\varphi_{0}\right)-\xi_{\gamma}\cos\Delta\varphi_{\gamma}+\eta, \\ d\left(\alpha\sin\varphi_{0}-\beta\cos\varphi_{0}\right)-\xi_{\gamma}\sin\Delta\varphi_{\gamma}\end{bmatrix}$$

are vectors of beginnings and ends at the points P_{γ} and O respectively (Figure 2), which are the points of the analyzed segments of crystals' surfaces $d\eta$ and $d\xi_{\gamma}$.

$$\gamma=1\rightarrow\alpha=\tfrac{1}{2}\wedge\beta=\tfrac{1}{2}$$
$$\gamma=2\rightarrow\alpha=-\tfrac{1}{2}\wedge\beta=\tfrac{1}{2}$$
$$\gamma=3\rightarrow\alpha=\tfrac{1}{2}\wedge\beta=-\tfrac{1}{2}$$
$$\gamma=4\rightarrow\alpha=-\tfrac{1}{2}\wedge\beta=-\tfrac{1}{2}$$

$\Delta\varphi_{\gamma}=\varphi_{\gamma}-\varphi_{0}$ are the differences between the angles of crystals' rotations respectively for *γ–th* and central crystal.

dq is the charge of the analyzed segment of the crystal's surface,

ε_{r} is the relative dielectric constant of air,

ε_{0} is the dielectric constant of vacuum.

By substituting these relations into equation (2) and expanding the vector product we obtain:

$$\vec{M}=\frac{1}{4\pi\varepsilon_{r}\varepsilon_{0}}\cdot$$

$$\sum_{\gamma}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{\eta\begin{pmatrix}d\left(\alpha\sin\varphi_0-\beta\cos\varphi_0\right)\\-\xi_\gamma\sin\Delta\varphi_\gamma\end{pmatrix}(dq)^2\,d\xi_\gamma d\eta}{\begin{pmatrix}\begin{pmatrix}-d\left(\beta\sin\varphi_0+\alpha\cos\varphi_0\right)\\-\xi_\gamma\cos\Delta\varphi_\gamma+\eta\end{pmatrix}^2\\+\begin{pmatrix}d\left(\alpha\sin\varphi_0-\beta\cos\varphi_0\right)\\-\xi_\gamma\sin\Delta\varphi_\gamma\end{pmatrix}^2\end{pmatrix}^{\frac{3}{2}}}\tag{4}$$

$$-R\frac{\partial\vec{\varphi}_0}{\partial t}\int_{-\frac{a}{2}}^{\frac{a}{2}}\eta^2 d\eta+\vec{M}^E$$

The equation (4) may be subject to certain simplifications. Let us note, that due to the uniform charge distribution on the surface of each crystal, we have:

$$Q=\int_{-\frac{a}{2}}^{\frac{a}{2}}dqd\xi_\gamma=\int_{-\frac{a}{2}}^{\frac{a}{2}}dqd\eta,\gamma=1,2,3,4\tag{5}$$

Due to the assumption of the synchronization of the crystals, which means that the adjacent crystals are rotated by almost the same angles, we have:

$$\Delta\varphi_\gamma<<1\Rightarrow\sin\Delta\varphi_\gamma=\Delta\varphi_\gamma,\cos\Delta\varphi_\gamma=1\tag{6}$$

Due to these relations, the equation (4) can be simplified after some transformations to the form:

$$\vec{M}$$

$$=\frac{Q^2}{4\pi\varepsilon_r\varepsilon_0}\cdot$$

$$\sum_{\gamma}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{\eta\left(d\left(\alpha\sin\varphi_0-\beta\cos\varphi_0\right)-\xi_\gamma\Delta\varphi_\gamma\right)d\xi_\gamma d\eta}{\begin{pmatrix}\frac{d^2}{2}+\left(\xi_\gamma-\eta\right)^2\\+2d\left(\xi_\gamma-\eta\right)\left(\beta\sin\varphi_0+\alpha\cos\varphi_0\right)\\-2\xi_\gamma d\Delta\varphi\left(\alpha\sin\varphi_0-\beta\cos\varphi_0\right)\end{pmatrix}^{\frac{3}{2}}}\tag{7}$$

$$-R\frac{\partial\vec{\varphi}_0}{\partial t}\int_{-\frac{a}{2}}^{\frac{a}{2}}\eta^2 d\eta+\vec{M}^E$$

2.3. The Derivation of the Discrete Equation of the Rotational Motion of the Crystal

Although the solution of double integrals (7) exists in the general case, it is very complicated. In view of the fact that it is also a discrete equation of the rotation angle, which subsequently will be subject of conversion into a continuous form, the use of the exact solutions of the equation (7) for this purpose is impossible. To solve the given problem we can find the approximate solution of double integrals (7) with a reasonable accuracy. We use the assumption of the synchronization of crystals, i.e. the fact that the differences between the rotation angles of adjacent crystals are relatively small. In this case, we may assume that the moment induced by the crystals' rotation by identical angles is independent of the moment induced by the components of the differences of the appropriate angles. Small differences between the angles of rotation crystals 1-4 in two directions are divided on the symmetric and antisymmetric parts. The result are a 5 component of the moment acting on the central crystal: principal moment of rotation deriving from identical angles of rotation and four components (symmetric-symmetric, antisymmetric-symmetric, symmetric-antisymmetric and antisymmetric-antisymmetric) deriving from different of angles of rotation of crystals 1-4. (Figure 3):

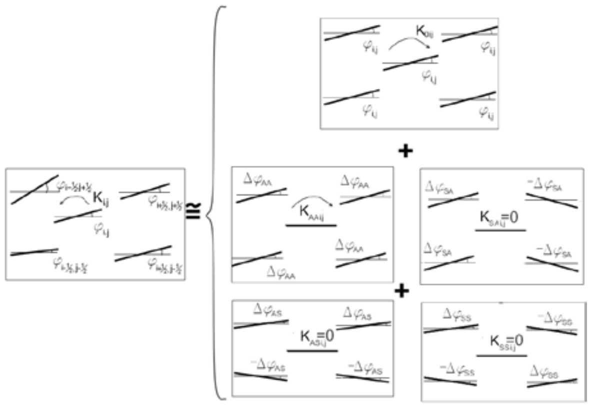

Figure 3. The assumption of superposition of the moments caused by crystals' rotation by identical angles and moments caused by the symmetric and antisymmetric components of the differences of the appropriate angles, which is described by (9)

$$\Delta\varphi_{SS}=\frac{1}{4}\left(-\varphi_1+\varphi_2+\varphi_3-\varphi_4\right)$$

$$\Delta\varphi_{AS}=\frac{1}{4}\left(\varphi_1+\varphi_2-\varphi_3-\varphi_4\right)$$

$$\Delta\varphi_{SA}=\frac{1}{4}\left(-\varphi_1+\varphi_2-\varphi_3+\varphi_4\right)$$

$$\Delta\varphi_{AA}=\frac{1}{4}\left(\varphi_1+\varphi_2+\varphi_3+\varphi_4\right)$$

Let us denote the integrals from equation (7) by $K\left(d,a,\varphi_0,\varphi_1,\varphi_2,\varphi_3,\varphi_4\right)$:

$$M=\frac{Q^2}{4\pi\varepsilon_r\varepsilon_0}K+\frac{Ra^3}{12}\frac{\partial\varphi_0}{\partial t}+M^E\tag{8}$$

We assume, that the following superposition of the moments is true:

$$K\cong K_0+K_{SS}+K_{AS}+K_{SA}+K_{AA}\tag{9}$$

$$K_0=K\Big|_{\varphi_1=\varphi_2=\varphi_3=\varphi_4=0},\;K_{SS}=K\Big|_{\left\{\begin{array}{l}\varphi_0=0\\\varphi_2=\varphi_3=\Delta\varphi_{SS}\\\varphi_1=\varphi_4=-\Delta\varphi_{SS}\end{array}\right.}$$

$$K_{AS}=K\Big|_{\left\{\begin{array}{l}\varphi_0=0\\\varphi_1=\varphi_2=\Delta\varphi_{AS}\\\varphi_3=\varphi_4=-\Delta\varphi_{AS}\end{array}\right.},\;K_{SA}=K\Big|_{\left\{\begin{array}{l}\varphi_0=0\\\varphi_1=\varphi_3=-\Delta\varphi_{SA}\\\varphi_2=\varphi_4=\Delta\varphi_{SA}\end{array}\right.}$$

$$K_{AA}=K\Big|_{\left\{\begin{array}{l}\varphi_0=0\\\varphi_2=\varphi_3=\Delta\varphi_{AA}\\\varphi_1=\varphi_4=\Delta\varphi_{AA}\end{array}\right.}$$

Let us calculate the moment K_0. In this case we have:

$$\varphi_\gamma = \varphi_0 \text{ and } \Delta\varphi_\gamma = 0$$

Both integrals (7) give the same solution for the couple of the crystals $\gamma = 1$ and $\gamma = 4$ and the second couple $\gamma = 2$ and $\gamma = 3$ due to the symmetry of the task (Figure 3), we get:

$$K_0 = 2\sum_{\gamma=1,2}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{\left(\eta d\left(\alpha\sin\varphi_0-\beta\cos\varphi_0\right)\right)d\xi_\gamma d\eta}{\left(\begin{array}{l}\frac{d^2}{2}+\left(\xi_\gamma-\eta\right)^2\\+2d\left(\xi_\gamma-\eta\right)\\\left(\beta\sin\varphi_0+\alpha\cos\varphi_0\right)\end{array}\right)^{\frac{3}{2}}} \quad (10)$$

It can be proved that:

$$K_{SA} = K_{AS} = K_{SS} = 0 \quad (11)$$

due to the symmetry of the moments induced by the electrostatic forces from corresponding pairs of crystals, as it is shown in Figure 3. For the remaining K_{AA} we have:

$$\varphi_0 = 0, \varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \Delta\varphi_{AA}$$

After the transformations and the substitutions, which are analogous to the ones performed in the calculation of K_0, we obtain:

$$K_{AA} = 2\sum_{\gamma=1,2}\int_{-\frac{a}{2}}^{\frac{a}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}}\frac{\eta\left(-\beta d-\xi_\gamma\Delta\varphi_{AA}\right)d\xi_\gamma d\eta}{\left(\begin{array}{l}\frac{d^2}{2}+\left(\xi_\gamma-\eta\right)^2\\+2\alpha d\left(\xi_\gamma-\eta\right)+2\beta\xi_\gamma d\Delta\varphi_{AA}\end{array}\right)^{\frac{3}{2}}} \quad (12)$$

We can calculate the integrals (10) and (12) in an accurate manner. By denoting respectively (in symbols "μ","$\pm$" the upper operator corresponds to $k = 1$, the lower one to $k = 2$)

$$s_\pm = \sin(\varphi_0) \pm \cos(\varphi_0), s_\mu = \sin(\varphi_0)\,\mu\cos(\varphi_0)$$

$$\hat{d} = \frac{d}{a}, d^{(\pm)}{}_\pm = \sqrt{(\pm)s_\pm + 1 + \frac{\hat{d}^2}{2}}, \Delta\varphi_{AA} = \Delta$$

we get:

$$\begin{aligned}K_0 = {} & \frac{\hat{d}s_\mu}{4a}\ln\left(\frac{\left(\hat{d}s_\pm - 2 + d^{(-)}{}_+\right)\left(\hat{d}s_\pm + 2 + d^{(+)}{}_+\right)}{\left(ds_\pm + \sqrt{2}d\right)^2}\right)\\ & + \frac{s_\pm}{2as_\mu}\left(d^{(-)}{}_+ + d^{(+)}{}_+ - \frac{\sqrt{2}}{2}\right)\\ & - \frac{\hat{d}s_\pm}{4a}\ln\left(\frac{\left(\hat{d}s_\mu - 2 + d^{(-)}{}_-\right)\left(\hat{d}s_\mu + 2 + d^{(+)}{}_-\right)}{\left(ds_\mu + \sqrt{2}d\right)^2}\right)\\ & - \frac{s_\mu}{2as_\pm}\left(d^{(-)}{}_- + d^{(+)}{}_- - \frac{\sqrt{2}}{2}\right)\end{aligned} \quad (13)$$

and by denoting:

$$d_{\Delta kl} = \sqrt{2(-1)^k\hat{d}\left((-1)^k\hat{d}+\Delta\right) + 4\delta_{kl}\left(1+(-1)^k\hat{d}\right)}$$

for $k, l = 1, 2$

$$d_{\Delta\Delta 1} = -\frac{\left(2\Delta+\hat{d}\right)\left(\begin{array}{l}-3\hat{d}^2\Delta - 2\hat{d} + \hat{d}^2 + 6\hat{d}\Delta - 4\Delta\\+2\hat{d}\Delta^2 - \left(2\hat{d}\Delta - \hat{d} - 2\Delta\right)d_{\Delta 11}\end{array}\right)}{\left(2\Delta-\hat{d}\right)\left(\begin{array}{l}3\hat{d}^2\Delta - \hat{d}^2 - 2\hat{d}\Delta + 2\hat{d}\Delta^2\\+\left(2\hat{d}\Delta - \hat{d} - 2\Delta\right)d_{\Delta 21}\end{array}\right)}$$

$$d_{\Delta\Delta 2} = -\frac{\left(2\Delta+\hat{d}\right)\left(\begin{array}{l}-3\hat{d}^2\Delta + \hat{d}^2 - 2\hat{d}\Delta + 2\hat{d}\Delta^2\\-\left(2\hat{d}\Delta - \hat{d} + 2\Delta\right)d_{\Delta 22}\end{array}\right)}{\left(2\Delta-\hat{d}\right)\left(\begin{array}{l}3\hat{d}^2\Delta - 2\hat{d} - \hat{d}^2 + 6\hat{d}\Delta + 4\Delta\\+2\hat{d}\Delta^2 + \left(2\hat{d}\Delta - \hat{d} + 2\Delta\right)d_{\Delta 12}\end{array}\right)}$$

$$d_{\Delta\Delta\Delta} = \frac{\left(\begin{array}{l}\left(\hat{d}\Delta + 2 + \hat{d} + d_{\Delta 11}\right)\left(\hat{d}\Delta + \hat{d} - d_{\Delta 21}\right)\\\left(\hat{d}\Delta + \hat{d} - d_{\Delta 12}\right)\\\left(\begin{array}{l}-2\hat{d} + 3\hat{d}^2\Delta - \hat{d}^2 + 6\hat{d}\Delta + 4\Delta + 2\hat{d}\Delta^2\\+\left(2\hat{d}\Delta - \hat{d} + 2\Delta\right)d_{\Delta 11}\end{array}\right)\end{array}\right)}{\left(\begin{array}{l}2\left(2\Delta+\hat{d}\right)\left(\hat{d}\Delta + \hat{d} + d_{\Delta 12}\right)\left(\hat{d}\Delta + \hat{d} + d_{\Delta 21}\right)\\\left(\hat{d}\Delta - 2 + \hat{d} - d_{\Delta 22}\right)\left(\hat{d}\Delta + 2 + \hat{d} - d_{\Delta 11}\right)\end{array}\right)}$$

We get (in (14) summation convention is used):

$$\begin{aligned}K_{AA} & \\ = {} & \frac{1-\delta_{kl}}{a\hat{d}}(-1)^k d_{\Delta kl} + \left(-\frac{1}{2} - \frac{\Delta}{2} + \frac{3}{2\Delta}\right)(-1)^{k+l} d_{\Delta kl}\\ & + \frac{2\hat{d}\ln d_{\Delta\Delta k}}{a\left(2\hat{d}\Delta - \hat{d} + (-1)^k\Delta\right)}\left(\begin{array}{l}\frac{3}{2} + (-1)^k\frac{\hat{d}}{\Delta^2} + (-1)^k\frac{3\hat{d}}{2}\\-\frac{3}{4\Delta} + (-1)^k\frac{3\hat{d}}{2\Delta}\end{array}\right)\\ & + \frac{\hat{d}}{a}\ln\left(d_{\Delta\Delta\Delta}\right)\cdot\left(\frac{1}{2} - \frac{3}{8\Delta^2} + \frac{3}{4\Delta} + \frac{1}{2}\Delta + \frac{1}{4}\Delta^2\right)\end{aligned} \quad (14)$$

Finally, the equation of the crystals' rotation takes the form:

$$I\frac{\partial^2\varphi_0}{\partial t^2} - \frac{Q^2}{4\pi\varepsilon_r\varepsilon_0}\left(K_0 + K_{AA}\right) + \frac{Ra^3}{12}\frac{\partial\varphi_0}{\partial t} = -M^E \quad (15)$$

2.4. The Approximation of the Discrete Equation of the Motion

The obtained equation (15) is a very complicated formula of the parameters φ_0 and Δ. Let us notice that in order to be able to transform it to a continuous model using the finite difference method formulas, the coefficients K_0 and K_{AA} must depend on the parameters φ_0 and Δ in the simplest possible way.

Hence, the series of graphs illustrating the correlation between the coefficients K_0 and K_{AA} and the parameters φ_0 and Δ have been performed. The Figure 4 and Figure 5 have been made for the selected allowable parameters of the model:

$$a \in (0.5mm, 10mm), \hat{d} \in (0.5, 50), \varphi \in (-60°, 60°)$$

$$\Delta\varphi_1 \in (-0.2°, 0.2°), \Delta\varphi_2 \in (-0.2°, 0.2°)$$

In the case of K_0 we get good results in the range of angles $\varphi \in (-60°, 60°)$ after using a polynomial approximation of the 3rd order. It is worth noting that the polynomial approximating K_0 has the form (Figure 4):

$$K_0 = -\kappa_2(\hat{d}, a)\varphi_0{}^3 + \kappa_3(\hat{d}, a)\varphi_0 \tag{16}$$

The remaining coefficients of the 3-rd order development are equal nearly to zero due to the specific shape of the graph K_0 and they can be ignored, regardless of the combination of the other parameters of the model.

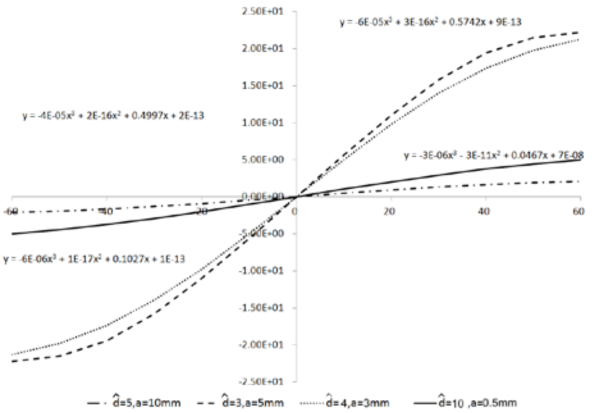

Figure 4. The dependency of the coefficients K_0 from the discrete parameter φ_0 for the selected combinations of the parameters *a* and $\hat{d}$

We apply the linear approximation of the coefficient K_{AA}, which is the exact approximation in the range of angles. $\Delta \in (-0.2°, 0.2°)$.

$$K_{AA} = -\kappa_1(\hat{d}, a)\Delta \tag{17}$$

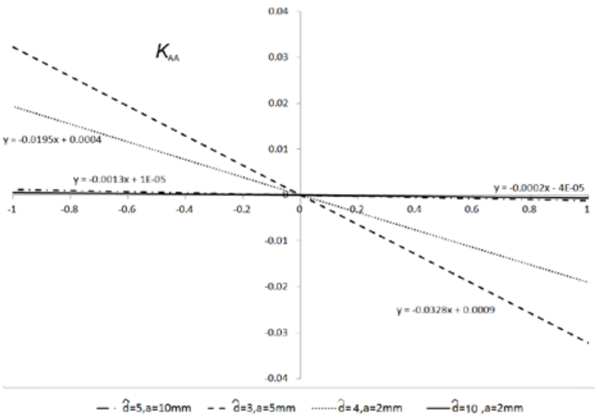

Figure 5. The dependency of the coefficients K_{AA} from the discrete parameter Δ for the selected combinations of the parameters *a* and $\hat{d}$

We can also note that the approximation coefficients κ_1, κ_2 and κ_3 are positive, regardless of the parameters of the model, and the following inequality is satisfied:

$$\kappa_3(\hat{d}, a) > \kappa_1(\hat{d}, a) > \kappa_2(\hat{d}, a) \tag{18}$$

As a result of the polynomial approximation of the coefficients K_0 and K_{AA} the following equation is obtained:

$$I\frac{\partial^2\varphi_0}{\partial t^2} + \frac{Q^2}{4\pi\varepsilon_r\varepsilon_0}\begin{pmatrix}\kappa_2(\hat{d}, a)\varphi_0{}^3 - \kappa_3(\hat{d}, a)\varphi_0 \\ +\kappa_1(\hat{d}, a)\Delta\end{pmatrix} + \frac{Ra^3}{12}\frac{\partial\varphi_0}{\partial t} = -M^E \tag{19}$$

where

$$\kappa_1(\hat{d}, a) = -\frac{180}{\pi}K_{AA}(\hat{d}, a, \Delta)\Big|_{\Delta=\frac{\pi}{180}}$$

$$\kappa_2(\hat{d}, a) = -\frac{108}{7\pi^3}\begin{pmatrix}K_0(\hat{d}, a, \varphi_0)\Big|_{\varphi=\frac{\pi}{3}} \\ -2K_0(\hat{d}, a, \varphi_0)\Big|_{\varphi=\frac{\pi}{6}}\end{pmatrix}$$

$$\kappa_3(\hat{d}, a) = \frac{1}{\pi}\left(K_0(\hat{d}, a, \varphi_0)\Big|_{\varphi=\frac{\pi}{3}} - 8K_0(\hat{d}, a, \varphi_0)\Big|_{\varphi=\frac{\pi}{6}}\right)$$

The equation (19) has coefficients, which are independent of the discrete parameters φ_0 and Δ and it can be used to construct the continuous model of the vibrations of the cloud of the ice crystals.

2.5. The Continuous Model of the Vibrating Cloud

Next we transform the discrete equation (19) to a continuous form in order to build the continuous model of the vibrating cloud. Let us rotate the coordinate system by an angle of Π/4 to obtain a regular finite difference method grid, which is shown in Figure 6:

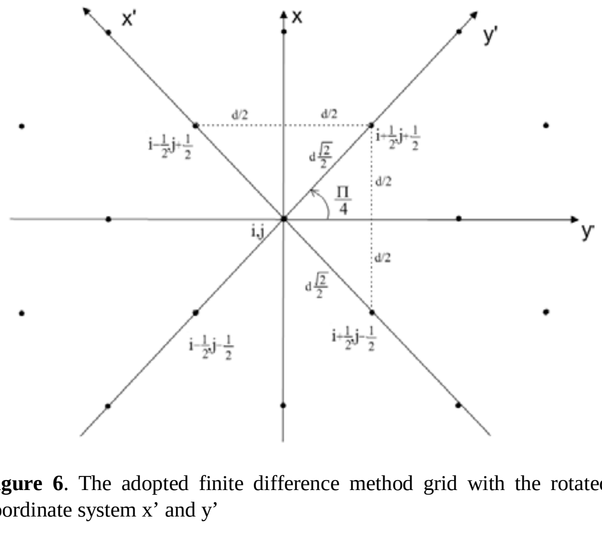

Figure 6. The adopted finite difference method grid with the rotated coordinate system x' and y'

We will use the finite difference method formulas [12]:

$$\varphi = \varphi_i\,, \frac{\partial^2\varphi}{\partial y'^2} = \frac{2}{d^2}\left(\varphi_{i+\frac{1}{2},j+\frac{1}{2}} - 2\varphi_{i,j} + \varphi_{i-\frac{1}{2},j-\frac{1}{2}}\right)$$

$$\frac{\partial^2\varphi}{\partial x'^2} = \frac{2}{d^2}\left(\varphi_{i-\frac{1}{2},j+\frac{1}{2}} - 2\varphi_{i,j} + \varphi_{i+\frac{1}{2},j-\frac{1}{2}}\right) \tag{20}$$

$$\varphi_1 = \varphi_{i+\frac{1}{2},j+\frac{1}{2}}\,, \varphi_2 = \varphi_{i-\frac{1}{2},j+\frac{1}{2}}\,,$$

$$\varphi_3 = \varphi_{i+\frac{1}{2},j-\frac{1}{2}}\,, \varphi_4 = \varphi_{i-\frac{1}{2},j-\frac{1}{2}}$$

Let us note, that $\Delta = \Delta\varphi_{AA} = \frac{1}{4}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)$ (8) (Figure 3). By substituting the relations (20) into the equation (19) and after appropriate transforming and grouping the components, we get:

$$\mu^2 \frac{\partial^2 \phi(x',y',t)}{\partial t^2} + \psi^2 \frac{\partial \varphi(x',y',t)}{\partial t} + \alpha^2 \left(\frac{\partial^2 \varphi(x',y',t)}{\partial x'^2} + \frac{\partial^2 \varphi(x',y',t)}{\partial y'^2} \right) - \beta^2 \varphi(x',y',t) + \gamma^2 \left(\varphi(x',y',t)\right)^3 = -\tilde{M}^E(x',y',t) \quad (21)$$

where

$$\mu^2 = \frac{5\sqrt{3}\pi\varepsilon_r\varepsilon_0 h\rho a^4}{64Q^2}, \psi^2 = \frac{\pi\varepsilon_r\varepsilon_0 R a^3}{3Q^2}$$

$$\alpha^2 = \frac{d^2}{8}\kappa_1(\hat{d},a), \beta^2 = \kappa_3(\hat{d},a) - \kappa_1(\hat{d},a)$$

$$\gamma^2 = \kappa_2(\hat{d},a), \tilde{M}^E = \frac{4\pi\varepsilon_r\varepsilon_0}{Q^2} M^E$$

Then we go back with the turnover of the coordinate system by an angle of Π/4 and we transform the coordinate system according to patterns:

$$x = -x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2}, y = x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2} \quad (22)$$

The corresponding derivatives are as follows:

$$\frac{\partial^2 \varphi(x'(x,y),y'(x,y),t)}{\partial y'^2} = \frac{1}{2}\frac{\partial^2 \varphi(x,y,t)}{\partial x^2} + \frac{\partial^2 \varphi(x,y,t)}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 \varphi(x,y,t)}{\partial y^2} \quad (23)$$

$$\frac{\partial^2 \varphi(x'(x,y),y'(x,y),t)}{\partial x'^2} = \frac{1}{2}\frac{\partial^2 \varphi(x,y,t)}{\partial x^2} - \frac{\partial^2 \varphi(x,y,t)}{\partial x \partial y} + \frac{1}{2}\frac{\partial^2 \varphi(x,y,t)}{\partial y^2} \quad (24)$$

By substituting (23) and (24) into (21) we obtain the final equation in x and y coordinates:

$$\mu^2 \frac{\partial^2 \varphi(x,y,t)}{\partial t^2} + \psi^2 \frac{\partial \varphi(x,y,t)}{\partial t} + \alpha^2 \left(\frac{\partial^2 \varphi(x,y,t)}{\partial x^2} + \frac{\partial^2 \varphi(x,y,t)}{\partial y^2} \right) - \beta^2 \varphi(x,y,t) + \gamma\left(\varphi(x,y,t)\right)^3 = -M^E(x,y,t) \quad (25)$$

The equation (25) describes the non-linear vibrations of the cloud of the electrically charged ice crystals in the continuous form in two dimentions. It takes into account the continuous distribution of the electric charge on the surface of the crystals, their rotation by large angles, the external forcing and the air resistance.

2.6. The Discussion of the Possible Solutions

The equation (25) is the second degree differential equation with respect to time and space. It is a generalization in the two-dimensional space equation called in the literature as the equation φ^4 [3,4] presented in [1]. It has analytical solutions only for some very specific boundary and initial conditions, and specific parameters of the equation. General analytical solutions of the equation (25) does not exist, so it should be solve numerically.

However, some solutions of special cases of simplified problems can suggest us, how they might look like potential solutions of equation (25). In [1] it has been shown the selected solution of this equation with the simplification of assuming a total lack of wave propagation in the spatial directions.

$$\alpha = 0$$

Then the equation (25) is converted into a simpler form which is called Duffing equation [5]. It has been shown, that the oscillations described by this equation may be of a harmonic, quasi-harmonic or chaotic, in dependency on the parameters of the equation and the initial-boundary conditions.

In the case of the equation (25) it can be expected that the disturbance of any kind, will be transmitted in the directions x and y, depending on the initial-boundary conditions, external torque which inducement vibration, damping parameters, and the model parameters corresponding to the propagation of the vibration between the crystals. We will have to deal in some areas with chaotic vibrations in other with harmonics, and these areas will be vary in time and space. The computer simulations will allow us for the analysis and visualization of the solutions of equation (25). It will be the subject of further research of the author.

2.7. The Comparison with Data from Motivation of Paper

As previously mentioned, in the general case, the equation (25) can be solved only numerically. However, in certain very specific cases, we can find analytical solutions of this equation and compare them with the solutions derived from linear and one-dimensional model proposed in [2] and use them to model the phenomenon commonly called "miracle of the sun". Below we present a very simple example of the use of equation (25) for modeling this phenomenon.

Suppose we are given the cloud of ice crystals satisfying the assumptions of the model with dimensions L_x to L_y. For simplicity, let's assume that we are dealing only with the issue of sinusoidal forced and linear vibration without damped, ie:

$$\gamma = 0, \psi = 0, \tilde{M}^E = \varphi_M \sin\left(\frac{\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_y}\right) \sin(\omega t) \quad (26)$$

The assumption of linearity of vibration is equivalent to the assumption of small deflection angles, smaller than

20-30°. (see Figure 4). Then the equation (25) simplifies greatly to the form:

$$\mu^2 \frac{\partial^2 \varphi(x,y,t)}{\partial t^2} + \alpha^2 \left(\frac{\partial^2 \varphi(x,y,t)}{\partial x^2} + \frac{\partial^2 \varphi(x,y,t)}{\partial y^2} \right) - \beta^2 \varphi(x,y,t) = -\varphi_M \sin\left(\frac{\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_y}\right) \sin(\omega t) \quad (27)$$

Suppose we are given the following boundary conditions:

$$\tilde{\varphi}(x,0,t) = \tilde{\varphi}(x,L_y,t) = \tilde{\varphi}(0,y,t) = \tilde{\varphi}(L_x,y,t) = 0 \quad (28)$$

and the initial conditions:

$$\tilde{\varphi}(x,y,0) = \frac{\partial \tilde{\varphi}}{\partial t}(x,y,0) = 0 \quad (29)$$

We get the following solution of equation (27) with boundary conditions (28) and initial conditions (29):

$$\varphi(x,y,t) = \varphi_M \frac{\sin\left(\frac{\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_y}\right) \sin(\omega t)}{\frac{\alpha^2 \pi^2 \left(L_x{}^2 + L_y{}^2\right)}{L_x{}^2 L_y{}^2} + \beta^2 + \mu^2 \omega^2} \quad (30)$$

Due to the nature of forcing vibration and the boundary-initial conditions, the solution does not depend on the model parameters, but they affect only on the amplitude of vibration. Thus, we can simplify solution (30) to the form:

$$\varphi(x,y,t) = A_\varphi \sin\left(\frac{\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_y}\right) \sin(\omega t) \quad (31)$$

where A_φ is a constant amplitude dependent from parameters of the model and amplitude of forced vibration.

When we have a simple form of field of rotation of crystals (31), we can take advantage of the optical conditions given in [1], which defining the relative brightening or darkening of the sun. Figure 7 shows an example resulting obtain from formula (31) in the spatial coordinates, that is, the distribution of crystals in the analyzed cloud. In addition, to illustrate the idea of the phenomenon "miracle of the sun", the situation of four observers is considered. The sun's rays fall from the right side on the vibrating cloud crystals. At time *t* shown in Figure 7, as a result of reflections from the ice crystals, a part of the rays is dispersed, a part is concentrated and others rays passed without encountering any a crystal. Therefore, depending on the position of the observer in relative to the crystal cloud, the visual effect will be very different. At the time *t* shown in the figure:

- the observer A will see a darkening of the image of the sun, because the cloud will disperse part of the sunlight that would reach him,

- the observer B is observed brightening the image of the sun, because the cloud will focus part of the rays,

- the observer C will see additional area of brightening in the sky above the real sun,

- the observer D will not notice any change in the appearance of the sun.

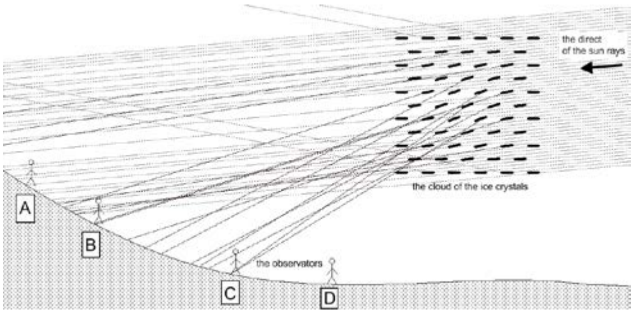

Figure 7. An example of the use of solutions of equation (31) for the analysis of the phenomenon called "miracle of the sun". For the simulation we used 66 crystals and 48 sun rays. The crystal system corresponds to the moment $t = \pi / 2\omega$ and amplitude 20°. The figure shows the situation of the four observers and their observations. By using the two dimensional model, we can explained the fact that their observations may be extremely different

As the solution obtain from the formula (31) is dependent on time, the situation which is shown on Figure 7 is dynamically changing. A similar dynamics, time and spatial diversity of observation was characteristic of the witnesses descriptions of the phenomenon called miracle of the sun in 1917 [6]. The previous attempts of the scientific explanation of the phenomenon of the "miracle of the sun" [7,8] did not explain such a large spatial variability at all.

As we can see, from the two-dimensional solutions of equation (25) we can provide much greater opportunities for modeling the phenomenon of the "miracle of the sun" than the one-dimensional solution shown in [2].

2.8. The Other Applications of the Model

The proposed model can be also used to modeling other physical phenomena. From the mechanics point of view is not an essential if we are dealing crystals of ice or other tiles made from different material. Just as the forces of interaction between them may not necessarily be associated with electrostatic forces. We can imagine that the role of the interaction forces act for example magnetic forces.

This model could be used for example to model the mechanics of the Kerr phenomena [9], or Cotton-Mouton phenomena [10]. Similarly, it could be used to model the behavior of some of the liquid crystals in the dynamically changing electric field [11].

3. The Conclusions

In this paper, the differential equation which describe the mechanical vibrations of the electrically charged cloud of plated ice crystals was obtained. It determines a mathematical description of the dynamic behavior of this media. The derived equation is a nonlinear, the second degree differential equation with respect to time and space. Although its general analytical solutions do not exist, it could be used to find numerical solutions of the problem of dynamic behavior of electrically charged cloud of ice crystals. That solutions could be used for further modeling of atmospheric phenomena such as optical phenomenon called "miracle of the sun".

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