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ON THE MODELLING OF VIBRATIONS OF FUNCTIONALLY GRADED PLATES

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In this paper there are considered functionally graded plates. To describe vibrations of these plates and take into account the effect of the microstructure it is applied the tolerance method, cf. [10, 11]. There are formulated governing equations of three presented models: the tolerance model, the asymptotic model and the combined asymptotic-tolerance model.

1. Introduction

There are considered thin plates with functionally graded macrostructure in planes parallel to the plate midplane, cf. Fig. 1. These plates have a tolerance-periodic microstructure along two directions on the microlevel. Hence, they are consisted of many small elements. It can be observed that distant elements can be very different, however adjacent elements are nearly identical. It is assumed that every element is treated as a thin plate with spans l_1 and l_2 along the x_1- and the x_2-axis, respectively. In various problems of these plates *the effect of the microstructure* cannot be neglected.

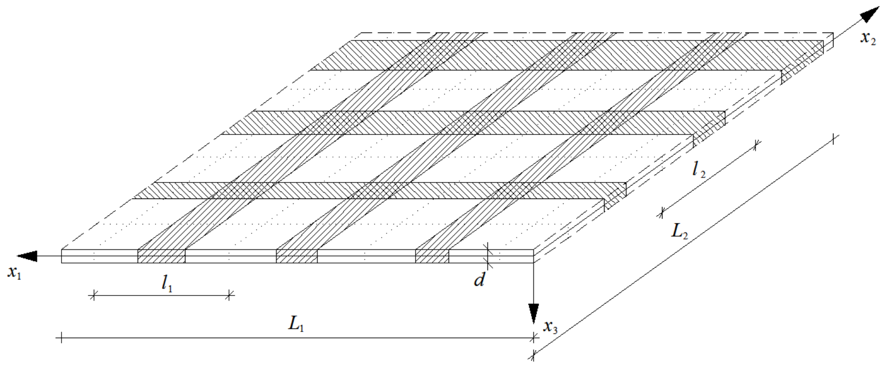

Fig. 1. A fragment of a functionally graded plate

These plates are described by partial differential equations, with highly oscillating, tolerance-periodic, non-continuous coefficients. To obtain averaged equations with continuous, functional coefficients, various simplified models are proposed, which replace tolerance-periodic plates by plates with averaged properties, being smooth, slowly-varying functions. Such plates are treated as made of *functionally graded materials*, cf. Suresh and Mortensen [9], and called *thin functionally graded plates*.

FGM-type structures are often analysed using averaging approaches for macroscopically homogeneous structures, e.g. periodic. Some of these methods are presented by Suresh and Mortensen [9]. We can distinguish models based on the asymptotic homogenization, cf. Jikov, Kozlov and Oleinik [4]. Unfortunately, the effect of the microstructure size is neglected in the governing equations of them.

This effect can be taken into account using *the tolerance averaging technique* (cf. Woźniak, Michalak and Jędrysiak (eds.) [11] and Woźniak et al. (eds.) [10]). Some applications of this method to the modelling of various periodic structures are shown in a series of papers, e.g. Michalak [8], Baron [1], Jędrysiak [2]. In the last years the tolerance modelling was adopted to functionally graded structures, e.g. for tolerance-periodic plates in Jędrysiak [3], Kaźmierczak and Jędrysiak [5, 6] and Kaźmierczak, Jędrysiak and Wirowski [7]. A summary can be found in Woźniak, Michalak and Jędrysiak (eds.) [9] and Woźniak et al. [10], and also in Jędrysiak [3], Michalak [8].

Here, modified governing equations of the tolerance and the asymptotic models for thin functionally graded plates are proposed and discussed, cf. Jędrysiak [3], Kaźmierczak and Jędrysiak [5, 6]. Moreover, a certain new combined asymptotic-tolerance model for these plates is derived.

2. Modelling foundations

Denote by $Ox_1x_2x_3$ the orthogonal Cartesian coordinate system and by t – the time coordinate. Let subscripts i,k,l run over 1,2,3 and α,β,γ run over 1,2. Set $\mathbf{x}\equiv(x_1,x_2)$ and $z\equiv x_3$. The region of the undeformed plate is denoted by $\Omega\equiv\{(\mathbf{x},z):-d(\mathbf{x})/2\le z\le d(\mathbf{x})/2, \mathbf{x}\in\Pi\}$, where Π is the midplane and $d(\cdot)$ is the plate thickness. Let ∂_α be derivatives of x_α, and also $\partial_{\alpha\ldots\delta}\equiv\partial_\alpha\ldots\partial_\delta$. The "basic cell" on Ox_1x_2 is denoted by $\square\equiv[-l_1/2,l_1/2]\times[-l_2/2,l_2/2]$, where l_1, l_2 are cell length dimensions along the x_1-, the x_2-axis, respectively, and the diameter of cell $\square$ – by $l\equiv[(l_1)^2+(l_2)^2]^{1/2}$. This diameter is called *the microstructure parameter* and satisfies condition $d_{\max}<<l<<\min(L_1,L_2)$. Thickness $d(\cdot)$ can be a tolerance-periodic function in $\mathbf{x}$ and all material and inertial properties of the plate, as mass density $\rho=\rho(\cdot,z)$ and elastic moduli $a_{ijkl}=a_{ijkl}(\cdot,z)$, can be also tolerance-periodic functions in $\mathbf{x}$ and even functions in z. Denote by $w(\mathbf{x},t)$ ($\mathbf{x}\in\overline{\Pi}$, $t\in(t_0,t_1)$) a plate deflection and by p total loadings in the z-axis direction. Let $a_{\alpha\beta\gamma\delta}$, $a_{\alpha\beta33}$, a_{3333} be the non-zero components of the elastic moduli tensor. Denote $c_{\alpha\beta\gamma\delta}\equiv a_{\alpha\beta\gamma\delta}-a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$.

Define the mean plate properties, being tolerance-periodic functions in $\mathbf{x}$, i.e. mass density μ, rotational inertia ϑ and bending stiffnesses $b_{\alpha\beta\gamma\delta}$, in the form:

$$\mu(\mathbf{x})\equiv\int_{-d/2}^{d/2}\rho(\mathbf{x},z)dz,\qquad \vartheta(\mathbf{x})\equiv\int_{-d/2}^{d/2}\rho(\mathbf{x},z)z^2dz,\qquad b_{\alpha\beta\gamma\delta}(\mathbf{x})\equiv\int_{-d/2}^{d/2}c_{\alpha\beta\gamma\delta}(\mathbf{x},z)z^2dz\,. \quad (1)$$

Using the Kirchhoff-type plates theory assumptions to functionally graded plates, the fourth order partial differential equation for deflection w is derived

$$\partial_{\alpha\beta}(b_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w)+\mu\ddot{w}-\partial_\alpha(\vartheta\partial_\alpha\ddot{w})=p\,. \quad (2)$$

This equation has highly oscillating, non-continuous, tolerance-periodic functional coefficients.

3. Modelling concepts and assumptions

Averaged equations for functionally graded plates will be obtained using the tolerance modelling, cf. Woźniak et al. (eds.) [10], Woźniak, Michalak and Jędrysiak (eds.) [11], where basic concepts of the modelling procedure are defined and explained, e.g.: an averaging operator, a tolerance-periodic function, a slowly-varying function. Below some of them are reminded.

Let $\square(\mathbf{x})\equiv\mathbf{x}+\square$, $\Pi_\square=\{\mathbf{x}\in\Pi:\ \square(\mathbf{x})\subset\Pi\}$, be a cell at $\mathbf{x}\in\Pi_\square$. The known *averaging operator* for an arbitrary integrable function f is defined by

$$<f>(\mathbf{x})=\frac{1}{l_1l_2}\int_{\square(\mathbf{x})}f(y_1,y_2)dy_1dy_2,\quad \mathbf{x}\in\Pi_\square. \quad (3)$$

If function f is tolerance-periodic in $\mathbf{x}$, then its averaged value by (3) is a slowly-varying function in $\mathbf{x}$.

Following the aforementioned books let us denote a set of tolerance-periodic functions by $TP^{\alpha}_{\delta}(\Pi,\Box)$, a set of slowly-varying functions by $SV^{\alpha}_{\delta}(\Pi,\Box)$, a set of highly oscillating functions by $HO^{\alpha}_{\delta}(\Pi,\Box)$, where $\alpha \geq 0$, and δ is a tolerance parameter. Let us introduce a highly oscillating function $h(\cdot)$, $h \in HO^{2}_{\delta}(\Pi,\Box)$, defined on $\overline{\Pi}$, continuous together with gradient $\partial^1 h$ and having a piecewise continuous and bounded gradient $\partial^2 h$. Function $h(\cdot)$ is called *the fluctuation shape function* of the 2-nd kind, if it depends on l as a parameter and satisfies conditions: $\partial^k h \in O(l^{\alpha-k})$ for $k=0,1,\ldots,\alpha$, $\alpha=2$, $\partial^0 h \equiv h$; and $<\mu h>(\mathbf{x}) \approx 0$ for every $\mathbf{x} \in \Pi_{\Box}$, $\mu>0$, $\mu \in TP^{1}_{\delta}(\Pi,\Box)$. Set of all fluctuation shape functions of the 2-nd kind is denoted by $FS^{2}_{\delta}(\Pi,\Box)$.

Using the abovementioned concepts the modelling assumptions are introduced, cf. Jędrysiak [3].

The first of them is *the micro-macro decomposition*:

$$w(\mathbf{x},t) = U(\mathbf{x},t) + h^A(\mathbf{x}) Q^A(\mathbf{x},t), \qquad A = 1,\ldots,N, \quad \mathbf{x} \in \Pi, \tag{4}$$

where $U(\cdot,t), Q^A(\cdot,t) \in SV^{2}_{\delta}(\Pi,\Box)$ for every t. Functions $U(\cdot,t)$ and $Q^A(\cdot,t)$ are kinematic unknowns, called *the macrodeflection* and *the fluctuation amplitudes*, respectively; $h^A(\cdot)$ are the known fluctuation shape functions.

In *the tolerance averaging approximation* terms $O(\delta)$ are negligibly small, e.g. for $f \in TP^{2}_{\delta}(\Pi,\Box)$, $F \in SV^{2}_{\delta}(\Pi,\Box)$, $h^A \in FS^{2}_{\delta}(\Pi,\Box)$, in: $< f >(\mathbf{x}) = < \bar{f} >(\mathbf{x}) + O(\delta)$, $< fF >(\mathbf{x}) = < f >(\mathbf{x}) F(\mathbf{x}) + O(\delta)$, $< f \partial_\alpha (h^A F) >(\mathbf{x}) = < f \partial_\alpha h^A >(\mathbf{x}) F(\mathbf{x}) + O(\delta)$.

4. Tolerance modelling

Following the book Jędrysiak [3] the modelling procedure is outlined. The starting point is the formulation of the action functional in the form

$$\mathcal{A}(w(\cdot)) = \int_{\Pi} \int_{t_0}^{t_1} \mathcal{L}(\mathbf{y}, \partial_{\alpha\beta} w(\mathbf{y},t), \partial_\alpha w(\mathbf{y},t), \dot{w}(\mathbf{y},t), w(\mathbf{y},t))\, dt\, d\mathbf{y}, \tag{5}$$

where the lagrangean $\mathcal{L}$ is given by

$$\mathcal{L} = \tfrac{1}{2}(\mu \dot{w}\dot{w} + \vartheta \partial_\alpha \dot{w} \partial_\beta \dot{w} \delta_{\alpha\beta} - b_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} w \partial_{\gamma\delta} w) + pw. \tag{6}$$

From the principle of stationary action applied to the lagrangean $\mathcal{L}$, (6), we obtain the Euler-Lagrange equation in the form (2).

Using the tolerance modelling to action functional (5) with the lagrangean (6) we have two steps. Firstly, we substitute micro-macro decomposition (4) to (6), and secondly we use (3) to the action functional. It leads to the tolerance averaging of functional (5) with the averaged lagrangean $< \mathcal{L}_h >$ in the form

$$\begin{aligned} < \mathcal{L}_h > = & -\tfrac{1}{2}\{(< b_{\alpha\beta\gamma\delta} > \partial_{\alpha\beta} U + 2 < b_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} h^B > Q^B) \partial_{\gamma\delta} U + < \vartheta > \partial_\alpha \dot{U} \partial_\beta \dot{U} \delta_{\alpha\beta} - \\ & - < \mu > \dot{U}\dot{U} + < b_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} h^A \partial_{\gamma\delta} h^B > Q^A Q^B + \\ & + < \vartheta \partial_\alpha h^A \partial_\beta h^B > \dot{Q}^A \dot{Q}^B \delta_{\alpha\beta} - < \mu h^A h^B > \dot{Q}^A \dot{Q}^B \} + < p > U + < p h^A > Q^A. \end{aligned} \tag{7}$$

From the principle of stationary action applied to (7) we arrive at the Euler-Lagrange equations for $U(\cdot,t)$ and $Q^A(\cdot,t)$:

$$
\begin{aligned}
&\partial_{\alpha\beta}(<b_{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}U+<b_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^B>Q^B)+<\mu>\ddot{U}-<\vartheta>\partial_{\alpha\alpha}\ddot{U}=<p>,\\
&<b_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}h^A>\partial_{\alpha\beta}U+\\
&+<b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}h^A\partial_{\gamma\delta}h^B>Q^B+(\underline{<\mu h^A h^B>}+\underline{<\vartheta\partial_\alpha h^A\partial_\alpha h^B>})\ddot{Q}^B=\underline{<ph^A>},
\end{aligned}
\tag{8}
$$

being the system of partial differential equations. In equations (8) the underlined terms depend on the microstructure parameter l. All coefficients are slowly-varying functions in $\mathbf{x}$ in contrast to equation (2), which has non-continuous, highly oscillating and tolerance-periodic functional coefficients. Equations (8) and micro-macro decomposition (4) represent *the tolerance model of thin functionally graded plates*, which describes the effect of the microstructure size on dynamic problems of these plates. This model is an extension of the model presented in the book by Jędrysiak [3]. The basic unknowns U, Q^A, A=1,…,N, are slowly-varying functions in $\mathbf{x}$. Boundary conditions have to be formulated only for *the macrodeflection* U, but not for *the fluctuation amplitudes* Q^A.

5. Asymptotic modelling

The asymptotic modelling procedure is outlined here following Woźniak et al. (eds.) [10] and Jędrysiak [3]. The starting point of this procedure is equation (2). We introduce a parameter $\varepsilon\in(0,1]$, an interval $\Box_\varepsilon\equiv[-\varepsilon l_1/2,\varepsilon l_1/2]\times[-\varepsilon l_2/2,\varepsilon l_2/2]$ and ε-cell $\Box_\varepsilon(\mathbf{x})\equiv\mathbf{x}+\Box_\varepsilon$, $\mathbf{x}\in\overline{\Pi}$. For function $\tilde{f}(\mathbf{x},\cdot)\in H^1(\Box)$, $\forall\mathbf{x}\in\overline{\Pi}$, we define $\tilde{f}_\varepsilon(\mathbf{x},\mathbf{y})\equiv\tilde{f}(\mathbf{x},\mathbf{y}/\varepsilon)$, $\tilde{f}_\varepsilon(\mathbf{x},\cdot)\in H^1(\Box_\varepsilon)\subset H^1(\Box)$, $\mathbf{y}\in\Box_\varepsilon(\mathbf{x})$, $\mathbf{x}\in\overline{\Pi}$. Let us also introduce independent functions $h^A(\cdot)$, $h^A(\cdot)\in HO^2_\delta(\Pi,\Box)$, A=1,…,N, with their periodic approximations $\tilde{h}^A(\mathbf{x},\cdot)$, given by $\tilde{h}^A_\varepsilon(\mathbf{x},\mathbf{y})\equiv\tilde{h}^A(\mathbf{x},\mathbf{y}/\varepsilon),\mathbf{y}\in\Box_\varepsilon(\mathbf{x})$, for every $\mathbf{x}\in\overline{\Pi}$.

The fundamental assumption of the asymptotic modelling is *the asymptotic decomposition* for the deflection $w(\mathbf{x},t)$, which takes the form:

$$
w_\varepsilon(\mathbf{x},\mathbf{y},t)=U(\mathbf{y},t)+\varepsilon^2\tilde{h}^A_\varepsilon(\mathbf{x},\mathbf{y})Q^A(\mathbf{y},t),
\tag{9}
$$

with $\mathbf{y}\in\Box_\varepsilon(\mathbf{x}), t\in(t_0,t_1)$; functions w, U, Q^A (A=1,…,N) are continuous and bonded in $\overline{\Pi}$ with their derivatives. Introduce denotation $\hat{\partial}_\alpha\tilde{h}^A_\varepsilon(\mathbf{x},\mathbf{y})\equiv\varepsilon\partial_\alpha\tilde{h}^A(\mathbf{x},\bar{\mathbf{y}})|_{\bar{\mathbf{y}}=\mathbf{y}/\varepsilon}$, $\hat{\partial}_{\alpha\beta}\tilde{h}^A_\varepsilon(\mathbf{x},\mathbf{y})\equiv\varepsilon^2\partial_{\alpha\beta}\tilde{h}^A(\mathbf{x},\bar{\mathbf{y}})|_{\bar{\mathbf{y}}=\mathbf{y}/\varepsilon}$. Taking into account ε→0, since $\mathbf{y}\in\Box_\varepsilon(\mathbf{x})$, $\mathbf{x}\in\overline{\Pi}$, formula (9) of the deflection and their derivatives take the form:

$$
\begin{aligned}
&w_\varepsilon(\mathbf{x},\mathbf{y},t)=U(\mathbf{x},t)+O(\varepsilon), \qquad \partial_\alpha w_\varepsilon(\mathbf{x},\mathbf{y},t)=\partial_\alpha U(\mathbf{y},t)+O(\varepsilon),\\
&\partial_{\alpha\beta}w_\varepsilon(\mathbf{x},\mathbf{y},t)=\partial_{\alpha\beta}U(\mathbf{y},t)+\hat{\partial}_{\alpha\beta}\tilde{h}^A_\varepsilon(\mathbf{x},\mathbf{y})Q^A(\mathbf{y},t)+O(\varepsilon).
\end{aligned}
\tag{10}
$$

Under the limit passage $\varepsilon \to 0$ in the above relations we neglect terms $O(\varepsilon)$. Then lagrangeans $\tilde{\mathcal{L}}_\varepsilon = \tilde{\mathcal{L}}(\mathbf{x}, \mathbf{y}/\varepsilon, \partial_\alpha U, \dot{U}, U, Q^A)$, in the action functionals $\mathcal{A}_h^\varepsilon(U, Q^A)$, $\mathbf{y} \in \square_\varepsilon(\mathbf{x}), \mathbf{x} \in \overline{\Pi}, t \in (t_0, t_1)$, are introduced. In the asymptotic procedure for $\varepsilon \to 0$ functions $\tilde{\mathcal{L}}_\varepsilon$ of $\mathbf{y}/\varepsilon$, $\mathbf{y} \in \square_\varepsilon(\mathbf{x})$, tend to the averaged function $\mathcal{L}_0$. Using formulas (9) and (10) we arrive at the lagrangean $\mathcal{L}_0$

$$\begin{aligned}\mathcal{L}_0 = -\tfrac{1}{2}\{(< b_{\alpha\beta\gamma\delta} > \partial_{\alpha\beta} U + 2 < b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} h^B > Q^B)\partial_{\gamma\delta} U + < \vartheta > \partial_\alpha \dot{U} \partial_\beta \dot{U} \delta_{\alpha\beta} - \\ - < \mu > \dot{U}\dot{U} + < b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} h^A \partial_{\gamma\delta} h^B > Q^A Q^B\} + < p > U.\end{aligned} \tag{11}$$

From the principle of stationary action for (11) we obtain the Euler-Lagrange equations:

$$\begin{aligned}&\partial_{\alpha\beta}((< b_{\alpha\beta\gamma\delta} > - < b_{\alpha\beta\gamma\delta}\partial_{\gamma\delta} h^B >< b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} h^A \partial_{\gamma\delta} h^B >^{-1} < b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} h^A >)\partial_{\gamma\delta} U) + \\ &\qquad + < \mu > \ddot{U} - < \vartheta > \partial_{\alpha\alpha} \ddot{U} = < p >, \\ &Q^B = - < b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta} h^A \partial_{\gamma\delta} h^B >^{-1} < b_{\alpha\beta\gamma\delta}\partial_{\gamma\delta} h^A > \partial_{\alpha\beta} U.\end{aligned} \tag{12}$$

Equations (12) together with asymptotic decomposition (9) describe *the asymptotic model of thin functionally graded plates*, neglecting the effect of the microstructure size. These equations have smooth, slowly-varying coefficients in the contrast to equation (2) with non-continuous, tolerance-periodic coefficients. It can be observed that we obtain one differential equation $(12)_1$ for the macrodeflection U and algebraic equations $(12)_2$ for the fluctuation amplitudes Q^A. Moreover, equations (12) can be obtained from equations (8) by neglecting the underlined terms.

6. Combined asymptotic-tolerance modelling

In the combined asymptotic-tolerance modelling we can distinguish two fundamental steps, cf. Woźniak et al. (eds.) [10]. In *the first of them* the asymptotic modelling procedure is applied, cf. Sec. 5. Because the macrodeflection U is the solution of equation $(12)_1$ and the fluctuation amplitudes Q^A are determined by relation $(12)_2$ we have the known following function:

$$w_0(\mathbf{x}, t) = U(\mathbf{x}, t) + h^A(\mathbf{x}) Q^A(\mathbf{x}, t). \tag{13}$$

In *the second step* of the combined modelling we apply the tolerance modelling procedure, cf. Sec. 4. Using the known function $w_0(\cdot, t) \in TP_\delta^2(\Pi, \square)$ and introducing the known fluctuation shape functions $g^K(\cdot) \in FS_\delta^2(\Pi, \square)$, $K=1,\dots,N$, satisfying the condition $< \tilde{\mu}(\mathbf{x}, \cdot)\tilde{g}^K(\mathbf{x}, \cdot) > = 0$, where $\tilde{g}^K(\mathbf{x}, \cdot)$ are periodic approximations of g^K, we assume *the additional decomposition posed on function* w_0 in the form:

$$\overline{w}(\mathbf{x}, t) = w_0(\mathbf{x}, t) + g^K(\mathbf{x}) V^K(\mathbf{x}, t), \tag{14}$$

where V^K are slowly-varying functions in $\mathbf{x}$. Tolerance-periodic function $\overline{w}(\cdot, t)$, $\overline{w}(\cdot, t) \in TP_\delta^2(\Pi, \square)$, has a periodic approximation in $\square(\mathbf{x})$. We can write:

$$
\begin{aligned}
&\widetilde{\overline{w}}(\mathbf{x},\mathbf{y},t) = w_0(\mathbf{x},t) + g^K(\mathbf{y})V^K(\mathbf{x},t),\\
&\partial_\alpha \widetilde{\overline{w}}(\mathbf{x},\mathbf{y},t) = \partial_\alpha w_0(\mathbf{x},t) + \partial_\alpha g^K(\mathbf{y})V^K(\mathbf{x},t),\\
&\partial_{\alpha\beta} \widetilde{\overline{w}}(\mathbf{x},\mathbf{y},t) = \partial_{\alpha\beta} w_0(\mathbf{x},t) + \partial_{\alpha\beta} g^K(\mathbf{y})V^K(\mathbf{x},t),\\
&\dot{\widetilde{\overline{w}}}(\mathbf{x},\mathbf{y},t) = \dot{w}_0(\mathbf{x},t) + g^K(\mathbf{y})\dot{V}^K(\mathbf{x},t),\\
&\mathbf{x}\in\overline{\Pi},\quad t\in(t_0,t_1),\quad \text{a.e.}\,\mathbf{y}\in\square(\mathbf{x}).
\end{aligned}
\tag{15}
$$

Setting $\overline{w}\equiv w$ from (6) we obtain $\mathcal{L}_g(\mathbf{x},\partial_\alpha\overline{w},\overline{w})$, being tolerance-periodic function which has a periodic approximation $\widetilde{\mathcal{L}}_g(\mathbf{x},\partial_\alpha\widetilde{\overline{w}},\widetilde{\overline{w}})$. Substituting (15) into this lagrangean and using (3) we arrive at the averaged lagrangean $<\mathcal{L}_g>$ for (14):

$$
\begin{aligned}
<\mathcal{L}_g> = \tfrac{1}{2}(&<\mu\dot{w}_0\dot{w}_0> + \underline{<\mu g^K g^J>\dot{V}^K\dot{V}^J} -\\
&-<b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w_0\partial_{\gamma\delta}w_0> -2<b_{\gamma\delta\alpha\beta}\partial_{\alpha\beta}g^J\partial_{\gamma\delta}w_0>V^J +\\
&+\underline{<\vartheta\partial_\alpha g^K\partial_\beta g^J>\dot{V}^K\dot{V}^J\delta_{\alpha\beta}} - <b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g^K\partial_{\gamma\delta}g^J>V^K V^J) +\\
&+<pw_0> + \underline{<pg^K>V^K}.
\end{aligned}
\tag{16}
$$

From the principle of stationary action for (16) the Euler-Lagrange equations are:

$$
\begin{aligned}
<b_{\alpha\beta\gamma\delta}\partial_{\alpha\beta}g^K\partial_{\gamma\delta}g^J>V^J + &\underline{(<\mu g^K g^J> + <\vartheta\partial_\alpha g^K\partial_\beta g^J>)\ddot{V}^J} =\\
&= \underline{<pg^K>} - <b_{\gamma\delta\alpha\beta}\partial_{\alpha\beta}g^K\partial_{\gamma\delta}w_0>.
\end{aligned}
\tag{17}
$$

The above equations stand the system of differential equations for the fluctuation amplitudes V^J, $J=1,\dots,N$, with the known function w_0, calculated in the first step of the modelling by (13). Equations (17) together with $(12)_1$ and decompositions (13)-(14) represent *the combined asymptotic-tolerance model of thin functionally graded plates*.

It can be observed the this model makes it possible to analyse the effect of the microstructure size on vibrations of the plates under consideration, because equations (17) involve terms dependent of the microstructure parameter l.

7. Remarks

In this contribution there are shown three models of thin functionally graded plates:

- *the tolerance model*, which describes the effect of the microstructure size on vibrations of these plates;
- *the asymptotic model*, neglecting the aforementioned effect and describing only macrovibrations of the plates under consideration;
- *the combined asymptotic-tolerance model*, which takes into account the effect of the microstructure size on vibrations of these plates.

Two of these models – the tolerance and the combined asymptotic-tolerance model make it possible to investigate not only macrovibrations, but also microvibrations, which are related to the microstructure of the functionally graded plates under consideration. Some applications of these models will be presented separately.

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MODELOWANIE PŁYT O FUNKCYJNEJ GRADACJI WŁASNOŚCI

W pracy rozpatrywane są płyty o funkcyjnej gradacji własności. Aby opisać drgania tych płyt, wykorzystano technikę tolerancyjnego modelowania [10, 11]. Równania zostały wyprowadzone w ramach trzech zaproponowanych modeli: modelu tolerancyjnego, modelu asymptotycznego oraz modelu asymptotyczno-tolerancyjnego.