Bogdan Balcerzak

Linear connections and secondary characteristic classes of Lie algebroids

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Preface

Lie algebroids appear in many structures related to geometry. Although the motivations for defining the concept of Lie algebroid come from Lie groupoids [74], on the one hand, we can view them as some generalizations of a tangent bundle or integrable distribution on a differential manifold, and on the other hand, as a generalization of Lie algebra. Further, the structures of the Lie algebroid can be generalized to, for example, structures in which the Lie bracket does not satisfy the Jacobi condition or at all the structure without the Lie bracket, however, equipped with a morphism acting from a given vector bundle into a tangent bundle (called an anchor). A vector bundle equipped with an anchor allows us to introduce the concept of connection. Our considerations focus on linear connections and their properties, and on the existence of a connection in a given vector bundle compatible with an existing metric structure.

The first part contains examples of Lie algebroids necessary to describe the discussed concepts. In the second part, we examine linear connections on Lie algebroids. We remark that linear connections can be considered even on anchored vector bundles. We define an exterior derivative operator related to a given connection in the case of an anchored structure equipped with a skew-symmetric bracket. We also note that the torsion of a connection is closely related to the exterior derivative operator, the square of which, in turn, is related to the curvature of the connection. This leads immediately to the first Bianchi identity. As the Bianchi identity, we interpret the Jacobi identity for the skew-symmetric bracket being simultaneously the skew-symmetric part of the connection. The considerations regarding the characteristic classes are related to metric structures. Therefore, the subject of our research are also connections compatible with metric structures. In particular, we present a generalization of the fundamental theorem of Riemannian geometry, which shows the importance of determining the torsion tensor for the uniqueness of such connections as, for example, the Levi-Civita connection. The final goal of our considerations is to define the secondary characteristic classes as elements from the image of some characteristic homomorphism. The given construction of the characteristic homomophisms generalizes some known approaches to pairs of Lie algebroids equipped with flat connections to connections with the specific curvature tensor with values in the kernel of given reduction.

We study the properties of the secondary characteristic homomorphism. In particular, we also study a certain universal homomorphism for pairs of Lie algebroids and connections of Lie algebroids. The importance of the considered characteristic classes consists in providing obstructions to the compatibility of the connection with the structure of the subalgebroid. In the case of the Lie algebroid of vector bundle algebroid and its Riemannian reduction, the non-triviality of a secondary characteristic homomorphism is simply an obstruction to compatibility of the connection with a given Riemannian metric.

Łódź, Autumn 2021 *Bogdan Balcerzak*

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Part I

Lie algebroids

1. Lie algebroids. Definitions. Examples

This section discusses the concept of a Lie algebroid. We will give basic examples of Lie algebroids and their properties. Simultaneously, we will also discuss more general structures such as an anchored vector bundle or an almost Lie algebroid structure. We show that some geometric objects can be considered on these generalized structures.

1.1 Lie algebroid. The Atiyah sequence

Definition 1.1.1. An *anchored vector bundle* (A, ϱ_A) over a manifold M is a vector bundle A over M equipped with a homomorphism of vector bundles

$$\varrho_A : A \longrightarrow TM$$

over the identity, which is called an *anchor.*

Definition 1.1.2. Let (A, ϱ_A) be an anchored vector bundle over a manifold M. If in the space $\Gamma(A)$ of smooth sections of A we have $\mathbb{R}$-bilinear skew-symmetric mapping $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ associated with the anchor with the following Leibniz type derivation law

$$[X, f \cdot Y] = f \cdot [X, Y] + (\varrho_A \circ X)(f) \cdot Y \tag{1.1}$$

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, we say that $(A, \varrho_A, [\cdot,\cdot])$ is a *skew-symmetric algebroid* over M.

Definition 1.1.3. A skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$ over M with the anchor preserving $[\cdot,\cdot]$ and the Lie bracket $[\cdot,\cdot]_{TM}$ of vector fields on M, i.e.,

$$\varrho_A \circ [X, Y] = [\varrho_A \circ X, \varrho_A \circ Y]_{TM}$$

for $X, Y \in \Gamma(A)$, is called an *almost Lie algebroid.*

Definition 1.1.4. The *Jacobiator* of the bracket $[\cdot,\cdot]$ in a skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$ is a map $\mathrm{Jac}_{[\cdot,\cdot]} : \Gamma(A) \times \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ given by

$$\mathrm{Jac}_{[\cdot,\cdot]}(X, Y, Z) = [[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X]$$

for $X, Y, Z \in \Gamma(A)$. We say that $[\cdot,\cdot]$ satisfies the *Jacobi identity* if its Jacobiator is identically zero.

Definition 1.1.5. Any skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$ over a manifold M in which $[\cdot,\cdot]$ satisfies the Jacobi identity is called a *Lie algebroid* over M.

Definition 1.1.6. By a *homomorphism of Lie algebroids* $(A, \varrho_A, [\cdot,\cdot]_A)$ and $(B, \varrho_B, [\cdot,\cdot]_B)$, both over the same manifold M, we mean a homomorphism of vector bundles $H : A \to B$ satisfying

$$H \circ [X, Y]_A = [H \circ X, H \circ Y]_B$$

for any $X, Y \in \Gamma(A)$, and

$$\varrho_B \circ H = \varrho_A,$$

i.e., the following diagram commutes

$$\begin{tikzcd} A \arrow[rr, "H"] \arrow[dr, "\varrho_A"'] & & B \arrow[dl, "\varrho_B"] \\ & TM. & \end{tikzcd}$$

Definition 1.1.7. We say that a homomorphism H of Lie algebroids $(A, \varrho_A, [\cdot,\cdot]_A)$ and $(B, \varrho_B, [\cdot,\cdot]_B)$, both over the same manifold M, is their *isomorphism* if H is simultaneously isomorphism of vector bundles A and B. If there exists an isomorphism of Lie algebroids A and B, we say that these Lie algebroids are *isomorphic*.

Using the Jacobi and the Leibniz identities, Herz, cf. [38], [39], showed that the representation

$$\lambda : C^\infty(M) \longrightarrow \mathrm{End}_{C^\infty(M)}(\Gamma(A))$$

given by

$$\lambda(f)(X) = f \cdot X$$

for $f \in C^\infty(M), X \in \Gamma(A)$, is faithful, and, in consequence, the anchor induces a homomorphism of Lie algebras

$$\operatorname{Sec} \varrho_A : \Gamma(A) \longrightarrow \Gamma(TM), \quad X \longmapsto \varrho_A \circ X$$

(cf. [30]). Therefore, ϱ_A is a homomorphism of Lie algebroids $(A, \varrho_A, [\cdot,\cdot]_A)$ and TM with the identity as an anchor and the Lie bracket of vector fields on M.

If a Lie algebroid $(A, \varrho_A, [\cdot,\cdot]_A)$ over a manifold M is regular, then $\ker \varrho_A$ and $\operatorname{Im} \varrho_A$ are vector bundles. It follows that

$$0 \longrightarrow \ker \varrho_A \longrightarrow A \xrightarrow{\varrho_A} \operatorname{Im} \varrho_A \longrightarrow 0 \tag{1.2}$$

is a short exact sequence.

Definition 1.1.8. A Lie algebroid $(A, \varrho_A, [\cdot,\cdot]_A)$ over a manifold M is called *regular* over $(M, \operatorname{Im} \varrho_A)$ if the anchor is a constant rank.

Definition 1.1.9. The short exact sequence (1.2) for a given regular Lie algebroid is called its *Atiyah sequence.*

Definition 1.1.10. A Lie algebroid is called *transitive* if its anchor is an epimorphism of the vector bundles.

Lemma 1.1.1. *Let $(A, \varrho, [\cdot,\cdot])$ be a Lie algebroid over a manifold M. If $X, X', Y \in \Gamma(A)$ and $X_{|U} = X'_{|U}$ for some open subset $U \subset M$, then*

$$[X, Y]_{|U} = [X', Y]_{|U} .$$

Proof. Let $x \in U$. Take $f \in C^\infty(M)$ that separates x in U, i.e., there is an open set $B \subset U$ such that $x \in B$, $f \geq 0$, $f|B = 1$, and $f|(M \setminus U) = 0$. Then,

$$\begin{aligned} 0 = [0, Y](x) &= [f \cdot (X - X'), Y]\,(x) \\ &= f(x) \cdot [X - X', Y]\,(x) - (\varrho \circ Y)_x(f) \cdot (X - X')(x) \\ &= [X - X', Y]\,(x) \\ &= [X, Y]\,(x) - [X', Y]\,(x). \end{aligned}$$

■

Corollary 1.1.1. *Let $(A, \varrho, [\cdot,\cdot])$ be a Lie algebroid over a manifold M. If $X, X', Y, Y' \in \Gamma(A)$, $X_{|U} = X'_{|U}$ and $Y_{|U} = Y'_{|U}$ for some open subset $U \subset M$, then*

$$[X, Y]_{|U} = [X', Y']_{|U} .$$

Lemma 1.1.2. *Let* $(A, \varrho, [\cdot,\cdot])$ *be a Lie algebroid over a manifold* M*,* $x \in M$*, and let* $\ker \varrho_x$ *denote the kernel of* $\varrho_x : A_x \longrightarrow T_xM$*. In* $\ker \varrho_x$ *there exist a Lie algebra structure with a Lie bracket* $[\cdot,\cdot]_x$ *defined in such a way that*

$$[\xi, \eta]_x = [X, Y]\,(x)$$

if $\xi, \eta \in \ker \varrho_x$ *satisfy* $\xi = X(x)$*,* $\eta = Y(x)$ *for some* $X, Y \in \Gamma(A)$*.*

Proof. Let $x \in M$, $\xi, \eta \in \ker \varrho_x \subset A_x$. Take sections X, X', Y of A such that

$$\xi = X(x) = X'(x) \quad \text{and} \quad Y(x) \in \ker \varrho_x.$$

There are $f_1, \ldots, f_k \in C^\infty(M)$, $X_1, \ldots, X_k \in \Gamma(A)$ satisfying

$$(X - X')_{|U} = \left(\sum_{j=1}^{k} f_j \cdot X_j\right)_{|U}$$

for some open subset $U \subset M$ such that $x \in U$ and $f_j(x) = 0 = g_j(x)$ for any $j = 1, \ldots, k$. Lemma 1.1.1 implies

$$\begin{aligned}[X - X', Y](x) &= \left[\sum_{j=1}^{k} f_j \cdot X_j, Y\right](x) \\ &= \sum_{j=1}^{k} f_j(x) \cdot [X_j, Z]\,(x) - \sum_{j=1}^{k} (\varrho \circ Y)_x\,(f_j) \cdot X_j(x) = 0\end{aligned}$$

since $f_j(x) = 0$ for $j = 1, \ldots, k$ and $Y(x) \in \ker \varrho_x$. Hence,

$$[X, Y](x) = [X', Y](x)$$

for any $Y \in \Gamma(A)$ such that $Y(x) \in \ker \varrho$. Consequently, if $Y, Y' \in \Gamma(A)$ and additionally $Y(x) = Y'(x) = \eta$, we have

$$[X, Y](x) = [X', Y](x) = -[Y', X'](x) = [X', Y'](x).$$

■

Definition 1.1.11. Let $(A, \varrho, [\cdot,\cdot]_A)$ be a Lie algebroid over a manifold M, $x \in M$. The Lie algebra $(\ker \varrho_x, [\cdot,\cdot]_x)$ is called the *isotropy Lie algebra of* $(A, \varrho, [\cdot,\cdot]_A)$ *at* x.

Let us recall the definition of Lie algebra bundle.

Definition 1.1.12. A *Lie algebra bundle* (LAB) is a vector bundle $L \xrightarrow{p} M$ over a manifold M together with a skew-symmetric map $[\cdot,\cdot] : \Gamma(L) \times \Gamma(L) \longrightarrow \Gamma(L)$ such that each $[\cdot,\cdot]_x : L_x \times L_x \longrightarrow L_x$ is a Lie algebra bracket and L admits a set of local trivializations $\{\psi_i : U_i \times \mathfrak{g} \longrightarrow L_{U_i}\}$ in which each $\psi_{i,x} : \mathfrak{g} \to L_x$ is a Lie algebra isomorphism.

Theorem 1.1.1. (cf. [61, Theorem 1.4]) *Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a transitive Lie algebroid over a manifold M, i.e., a Lie algebroid with the Atiyah sequence*

$$0 \longrightarrow \boldsymbol{g} \longrightarrow A \xrightarrow{\varrho_A} TM \longrightarrow 0,$$

then $\boldsymbol{g}$ is a Lie algebra bundle.

The notion of Lie algebroid was discovered as the infinitesimal part of a differentiable groupoid by Pradines [74]. Note that Lie algebroids are simultaneous generalizations of integrable distributions (in particular, tangent bundles) on the one hand, and Lie algebras on the other. We will present some Lie algebroids in the following sections.

Some geometric objects, such as connections, can be defined on vector bundles with an attached anchor. Hence some considerations will be carried out in anchored bundles. Anchored bundles equipped with additional structures such as a bracket in the module of sections of a given bundle have been the subject of studies by Marcela Popescu and Paul Popescu, [69], [70], [71], [73] (these are only some of them). In the latest, cf. [72], they also propose Chern classes for almost Lie algebroids. Let us recall that a skew-symmetric bracket in almost Lie algebroids does not necessarily satisfy the Jacobi identity, which is the main difficulty in defining the appropriate classes.

The idea of skew-symmetric algebroids was presented by Kosmann-Schwarzbach and Magri in [46] in the case of finitely generated projective modules over commutative and associative algebras with unit under the name *pre-Lie algebroids*. Skew-symmetric algebroids (under the name pre-Lie algebroids) were examined by Grabowski and Urbański in [33], [34], where a concept of generalized algebroids, which play an important role in analytical mechanics, was also introduced. Using general algebroids instead of Lie algebroids, one can describe a larger family of systems, both in the Lagrangian and Hamiltonian formalisms [29], [28].

There are several equivalent approaches to Lie algebroids in the sense that the existence of one structure determines the existence of the other (cf. [33], [34], [35], [31]). In particular, the structure of the Lie algebroid $(A, \varrho_A, [\cdot,\cdot]_A)$ is equivalent to the existence of a linear Poisson structure on A^* [16] (see [57] for the linear Poisson structure on $\mathfrak{g}^*$ where $\mathfrak{g}$ is a finite dimensional Lie algebra).

1.2 Lie algebroid of a principal bundle

The structure of the algebroid described here was actually introduced by Atiyah in his a study on the existence theory of complex analytic connections [2]. This structure in the context of Lie algebroids was initially studied independently by Mackenzie [61] and Kubarski [49], and next by Grabowski, Kotov, and Poncin [32].

Let (P, π, M, G, R) be a smooth principal G-bundle over a smooth manifold M with the projection $\pi : P \longrightarrow M$ and the right action $R : P \times G \longrightarrow P$ of G on P. Denote the Lie algebra of G by $\mathfrak{g}$. Let R_g denotes the right multiplication by $g \in G$. Then

$$R^T : TP \times G \longrightarrow TP, \quad (\mathrm{v}, g) \longmapsto (R_g)_* \mathrm{v}$$

is a right action of G on TP. Let us denote by

$$A(P) = TP/G$$

the space of all orbits of this action. Let [v] denote the orbit of R^T through v$\in TP$. In the space $A(P)$ we have a structure of a vector bundle with the projection

$$p : A(P) \longrightarrow M, \quad [\mathrm{v}] \longmapsto \pi(z), \quad \mathrm{v} \in T_z P.$$

Let $\pi_P : TP \longrightarrow P$ be the tangent bundle projection. For any $x \in M$, in the fibre $p^{-1}(x)$ there is a unique real vector space structure satisfying

$$[\mathrm{v}] + [\mathrm{w}] = [\mathrm{v} + \mathrm{w}]$$

and

$$r \cdot [\mathrm{v}] = [r \cdot \mathrm{v}]$$

if v,w $\in TP$, $\pi_P(\mathrm{v}) = \pi_P(\mathrm{w})$, $r \in \mathbb{R}$. Let

$$\pi^A : TP \longrightarrow A(P), \quad \mathrm{v} \longmapsto [\mathrm{v}]$$

be the mapping which to v $\in TP$ assigns the orbit [v] of the action R^T. For any $z \in P$ the mapping

$$\pi^A_{|z} : T_zP \longrightarrow A(P)_{\pi(z)}, \quad \mathrm{v} \longmapsto [\mathrm{v}]$$

is an isomorphism of vector spaces [49].

Lemma 1.2.1. [49] *A local trivialization $\varphi : U \times G \to \pi^{-1}(U)$ of P determines a dyfomorphism $\varphi^A : TU \times \mathfrak{g} \longrightarrow p^{-1}(U)$ given by*

$$(v,a) \longmapsto \left[\varphi_{*\,(x,e)}(v,a)\right] \quad \text{for} \quad (v,a) \in T_xU \times \mathfrak{g}. \tag{1.3}$$

Lemma 1.2.2. [49] *The following diagram commutes*

$$\begin{tikzcd} TU \times TG \arrow[r, "\mathrm{id}_{TU} \times \Theta^R"] \arrow[d, "\varphi_*"'] & TU \times \mathfrak{g} \arrow[d, "\varphi^A"] \\ TP \arrow[r, "\pi^A"'] & A(P) \end{tikzcd}$$

where $\Theta^R \in \Omega^1(G;\mathfrak{g})$ is the cannonical right-invariant 1-form on G, i.e., $\Theta^R_g : T_gG \longrightarrow T_eG$ ($g \in G$, $T_eG = \mathfrak{g}$) is given by

$$\Theta^R_g(v) = \left(r_g^{-1}\right)_{*e}(v) \quad \text{for} \quad v \in T_gG,$$

where $r_g : G \longrightarrow G$ is the right translation by $g \in G$ in the group G.

Lemma 1.2.3. [49] *Let $\varphi_j : U_j \times G \longrightarrow \pi^{-1}(U_j)$ $(j = 1,2)$ be local trivializations in the bundle P. Then $\left(\varphi_1^A\right)^{-1}(p^{-1}(U_1 \cap U_2))$ and $\left(\varphi_2^A\right)^{-1}(p^{-1}(U_1 \cap U_2))$ are open subsets of $T(U_1 \cap U_2) \times \mathfrak{g}$. Moreover,*

$$\left(\left(\varphi_1^A\right)^{-1} \circ \varphi_2^A\right)(v,a) = \left(v, \Theta^R(h_*v) + \mathrm{Ad}_G(h(x))(a)\right),$$

where $x \in U_1 \cap U_2$, $(v,a) \in \left(\varphi_2^A\right)^{-1}(p^{-1}(U_1 \cap U_2))$, $h : U_1 \cap U_2 \longrightarrow \pi^{-1}(U_1 \cap U_2)$ is a smooth mapping such that że $\varphi_2(x,e) = \varphi_1(x,e)\cdot h(x)$ and $\mathrm{Ad}_G : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ is defined by $\mathrm{Ad}_G(g) = (\tau_g)_{*e}$ ($\tau_g : G \longrightarrow G$ is given by $x \longmapsto g \cdot x \cdot g^{-1}$).

Corollary 1.2.1. *The above lemma shows that the transition functions for φ_j^A are smooth. From the above considerations we conclude that there is exactly one differential manifold structure in $A(P)$ and $(A(P), p, M)$ is a vector bundle with a system of local trivializations $\varphi^A : TU \times \mathfrak{g} \longrightarrow p^{-1}(U)$ defined by* (1.3).

Next, consider

$$\varrho_{A(P)} : A(P) \longrightarrow TM, \quad [\mathrm{v}] \longmapsto \pi_*(\mathrm{v}), \quad \mathrm{v} \in TP.$$

For every section η of the vector bundle $A(P)$ there exists exactly one right-invariant vector field $\eta' \in \mathfrak{X}^R(P)$ such that

$$[\eta'(z)] = \eta_{\pi(z)}.$$

The map

$$H : \Gamma(A(P)) \longrightarrow \mathfrak{X}^R(P), \quad \eta \longmapsto \eta'$$

is an isomorphism of $\mathcal{C}^\infty(M)$-modules. The inverse mapping to it is

$$H^{-1} : \mathfrak{X}^R(P) \longrightarrow \Gamma(A(P)), \quad X \longmapsto X_0,$$

where $X_0(x) = [X(z)]$ for $x \in M$, $z \in P_x$ [49].

In $\Gamma(A(P))$ we have the structure of a real Lie algebra with the bracket $[\![\cdot,\cdot]\!]$ given in such a way that

$$[\![\eta_1, \eta_2]\!]' = [\eta_1', \eta_2'] \text{ for } \eta_1, \eta_2 \in \Gamma(A(P)),$$

i.e.,

$$[\![\eta_1, \eta_2]\!] = H^{-1}([\eta_1', \eta_2']) \text{ for } \eta_1, \eta_2 \in \Gamma(A(P)).$$

The triple $\left(A(P), \varrho_{A(P)}, [\![\cdot,\cdot]\!]\right)$ is a Lie algebroid over M called a **Lie algebroid of the principal bundle**, and the Atiyah sequence of $\left(A(P), \varrho_{A(P)}, [\![\cdot,\cdot]\!]\right)$ is

$$0 \longrightarrow \ker \varrho_A \longrightarrow A(P) \xrightarrow{\varrho_{A(P)}} TP \longrightarrow 0$$

where $\ker \varrho_A$ is isomorphic to the adjoint bundle $P \times_G \mathfrak{g}$ associated to the principal G-bundle P with the adjoint action of G on P via the mapping

$$\tau : P \times_G \mathfrak{g} \longrightarrow \ker \varrho_A, \quad [z, \mathrm{v}] \longmapsto [A_{z*}(\mathrm{v})],$$

and where $A_z : G \to P$, $A_z(a) = za$ (cf. [61], [49]).

1.3 Lie algebroid of a vector bundle

In this section we will discuss the Lie algebroid of a vector bundle. We present a construction of the appropriate vector bundle indicating local trivializations [49], [51]. Moreover, we give other equivalent approaches to the Lie algebroid of a vector bundle.

Let E be any real vector bundle of rank r over a base manifold M of dimension m with the standard fibre V and the bundle projection $p : E \longrightarrow M$. For $x \in M$, let $A(E)_x$ be the space of all linear homomorphism $l : \Gamma(E) \longrightarrow E_x$ such that there exists a vector $u \in T_xM$ with the property

$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u(f) \cdot \nu_x \tag{1.4}$$

for any $\nu \in \Gamma(E)$ and $f \in C^\infty(M)$. Remark that the vector $u \in T_xM$ is uniquely determined by $l \in A(E)_x$. In fact: Let $u_1, u_2 \in T_xM$ satisfy

$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u_1(f) \cdot \nu_x,$$
$$l(f \cdot \nu) = f(x) \cdot l(\nu) + u_2(f) \cdot \nu_x$$

for any $\nu \in \Gamma(E)$ and $f \in C^\infty(M)$. Thus,

$$0 = l(f \cdot \nu) - l(f \cdot \nu) = u_1(f) \cdot \nu_x - u_2(f) \cdot \nu_x = (u_1(f) - u_2(f)) \cdot \nu_x$$

for any $\nu \in \Gamma(E)$ and $f \in C^\infty(M)$. Taking $\nu \in \Gamma(E)$ satisfying $\nu_x \neq 0$ we obtain $u_1 = u_2$.

Take the disjoint union

$$A(E) = \bigsqcup_{x \in M} A(E)_x$$

and

$$\pi : A(E) \longrightarrow M$$

such that $\pi(l) = x$ if and only if $l \in A(E)_x$.

Let $\psi : U \times V \longrightarrow p^{-1}(U)$ be a local trivialization of the bundle E. For $\nu \in \Gamma(E)$, let ν_ψ denote the mapping

$$\nu_\psi : U \longrightarrow V, \quad x \longmapsto \psi_x^{-1}(\nu_x). \tag{1.5}$$

Let $x \in M$. Remark that if sections $\nu, \eta \in \Gamma(E)$ are equal on some open set $O \subset M$, and $x \in O$, then for any $l \in A(E)_x$, we have the equality $l(\nu) = l(\eta)$. For each $u \in T_xU$ and $a \in \mathrm{End}(V)$, define the mapping

$$\overline{\psi}(u,a) : \Gamma(E) \longrightarrow E_x, \ \ \overline{\psi}(u,a)(\nu) = \psi_x(u(\nu_\psi) + (a \circ \nu_\psi)(x)). \quad (1.6)$$

One can check that $\overline{\psi}(u,a) \in A(E)_x$. Thus we obtain a well-defined map

$$\overline{\psi} : TU \times \mathrm{End}(V) \longrightarrow A(E)_U \ \text{ where } \ A(E)_U := \pi^{-1}(U).$$

Using the identification $TU \cong U \times \mathbb{R}^m$, we will show that $\overline{\psi}$ is a local trivialization of $A(E)$.

It is showed in [50] that for any $x \in M$ the mapping $\overline{\psi}_x : T_xU \times \mathrm{End}(V) \longrightarrow A(E)_x$ given by

$$\overline{\psi}_x(u,a) = \overline{\psi}(u,a) \text{ for } u \in T_xU, a \in \mathrm{End}(V),$$

is a linear isomorphism. We recall the arguments: Let $(u,a) \in T_xU \times \mathrm{End}(V)$ and $\overline{\psi}_x(u,a) = 0$. Then

$$0 = \overline{\psi}_x(u,a)(f \cdot \nu) = f(x) \cdot \overline{\psi}_x(u,a)(\nu) + u(f) \cdot \nu_x = u(f) \cdot \nu_x$$

for any $f \in C^\infty(M)$ and $\nu \in \Gamma(E)$. If we take $\nu \in \Gamma(E)$ such that $\nu_x \neq 0$, then we deduce immediately that u is the zero tangent vector. Consider the family

$$\{\xi^\omega \in \Gamma(E) \mid \ \xi^\omega(x) = \psi_x(\omega)\}_{\omega \in V}$$

of global sections of E. On account of (1.6), for each $\omega \in V$, we have $a(\omega) = \psi_x^{-1}(\overline{\psi}(0,a)(\xi^\omega))$. Hence $a = 0$. Therefore the map $\overline{\psi}_x$ is a monomorphism.

Now take $l \in A(E)_x$, and let $u \in T_xM$ be the tangent vector which satisfies (1.4) for l. The element $\psi_x^{-1}(l(\sigma)) - u(\sigma_\psi)$ of V depends only on the value of $\sigma \in \Gamma(E)$ at x. In fact, let $\varepsilon \in \Gamma(E)$ and $\varepsilon_x = \sigma_x$. There exist functions $f^1, \ldots, f^k \in C^\infty(M)$ and sections $\eta_1, \ldots, \eta_k \in \Gamma(E)$ such that $f^j(x) = 0$ for any $j = 1, \ldots, k$ and

$$(\varepsilon - \sigma)|_O = \left(\sum_{j=1}^{k} f^j \cdot \eta_j \right) \Bigg|_O$$

for a certain neighborhood $O \subset M$ of x. Therefore

$$\psi_x^{-1}(l(\varepsilon)) - u(\varepsilon_\psi) = \psi_x^{-1}(l(\sigma)) - u(\sigma_\psi).$$

For $\omega \in V$, we will denote by $a(\omega)$ the element of the form $\psi_x^{-1}(l(\tau)) - u(\tau_\psi)$ where $\tau \in \Gamma(E)$ is an arbitrary taken section such that $\tau(x) = \psi_x(\omega)$. Now define a mapping $a : V \longrightarrow V$ by $\omega \longmapsto a(\omega)$. Clearly, a is linear. According to the obvious equality $\nu_x = \psi_x(\nu_\psi(x))$, we see that

$$\overline{\psi}_x(u, a)(\nu) = \psi_x(u(\nu_\psi) + \psi_x^{-1}(l(\nu)) - u(\nu_\psi)) = l(\nu).$$

Thus we have proved that the map $\overline{\psi}_x$ is an epimorphism.

Theorem 1.3.1. [4] *Let $\psi_1 : U_1 \times V \longrightarrow E_{U_1}$, $\psi_2 : U_2 \times V \longrightarrow E_{U_2}$ be local trivializations of the bundle E with $U_1 \cap U_2 \neq \varnothing$, and let $\overline{\psi}_1 : TU_1 \times \mathrm{End}(V) \longrightarrow A(E)_{U_1}$, $\overline{\psi}_2 : TU_2 \times \mathrm{End}(V) \longrightarrow A(E)_{U_2}$ be determined by ψ_1 and ψ_2, respectively, via* (1.6)*. Moreover, let $\lambda : U_1 \cap U_2 \longrightarrow GL(V)$ be the mapping given by $y \mapsto (\psi_1)_y^{-1} \circ (\psi_2)_y$. Then,*

$$\Big((\overline{\psi}_2)_x^{-1} \circ (\overline{\psi}_1)_x \Big)(u, a) = \big(u, \lambda(x)^{-1} \circ (\lambda_{*x}(u) + a \circ \lambda(x))\big) \tag{1.7}$$

for $u \in T_xU$, $a \in \mathrm{End}(V)$.

Let $\psi_1 : U_1 \times V \longrightarrow E_{U_1}$, $\psi_2 : U_2 \times V \longrightarrow E_{U_2}$ be local trivializations of the bundle E with $U_1 \cap U_2 \neq \varnothing$. From (1.7) in Theorem 1.3.1 we conclude that the map

$$\overline{\psi}_2^{-1} \circ \overline{\psi}_1 : T(U_1 \cap U_2) \times \mathrm{End}(V) \longrightarrow T(U_1 \cap U_2) \times \mathrm{End}(V)$$

is smooth. From the theorem on the construction of a vector bundle, we have that $A(E)$ is a vector bundle with the standard fibre $\mathbb{R}^m \oplus \mathrm{End}(V)$ and the projection π. Moreover, $\overline{\psi} : TU \times \mathrm{End}(V) \longrightarrow A(E)_U$ is an isomorphism of vector bundles.

It remains to introduce in $A(E)$ the structure of a Lie algebroid.

Let $l \in A(E)$. There exists a point $x \in M$ such that a linear homomorphism $l : \Gamma(E) \longrightarrow E_x$ is an element of the vector space $A(E)_x$. Let us denote by u^l (determined by l uniquely) a tangent vector to the manifold M at x, satisfying (1.4). We define a mapping

$$\varrho : A(E) \longrightarrow TM, \quad l \longmapsto u^l. \tag{1.8}$$

Certainly, the maps $\varrho_x : A(E)_x \longrightarrow T_xM$ ($x \in M$) are linear. Moreover, since the diagrams

$$\begin{tikzcd} A(E) \arrow[r, "\varrho"] \arrow[d, "\pi"'] & TM \arrow[d, "\pi_M"] \\ M \arrow[r, "\mathrm{id}_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} TU \times \mathrm{End}(V) \arrow[r, "\overline{\psi}"] \arrow[d, "\mathrm{pr}_1"'] & A(E)_U \arrow[d, "\varrho|_{A(E)_U}"] \\ TU \arrow[r, "\iota"'] & TM \end{tikzcd}$$

commute, ϱ is an epimorphism of vector bundles.

Lemma 1.3.1. [4] *Let $\mathcal{L} : M \longrightarrow A(E)$ be a function such that $\pi \circ \mathcal{L} = \mathrm{id}_M$. If, for each $\nu \in \Gamma(E)$, the mapping $\zeta_\nu : M \longrightarrow E$, $\zeta_\nu(x) = \mathcal{L}_x(\nu)$, is a smooth section of the vector bundle E, then $\mathcal{L}$ is a smooth section of the vector bundle $A(E)$.*

Proof. The problem is local. Therefore we need only consider a trivial vector bundle $E = M \times V$. Then $TM \times \mathrm{End}(V) \cong A(E)$ via the isomorphism $\overline{\psi}$ defined for the identical trivialization $\psi = \mathrm{id}_{M\times V}$. Let $\nu \in \Gamma(M \times V)$, $x \in M$, and take $\widetilde{\nu} = \mathrm{pr}_2 \circ \nu \in C^\infty(M; V)$. There exist functions $X : M \longrightarrow TM$ and $\sigma : M \to \mathrm{End}(V)$ (whose smoothness is to be proved) for which

$$\begin{aligned} \zeta_\nu(x) &= \mathcal{L}_x(\nu) \\ &= \overline{\mathrm{id}_{M\times V}}(X_x, \sigma(x)) \\ &= \big(x, X_x(\nu_{\mathrm{id}_{M\times V}}) + \sigma(x)(\nu_{\mathrm{id}_{M\times V}}(x))\big) \\ &= (x, X_x(\widetilde{\nu}) + \sigma(x)(\widetilde{\nu}(x))) \\ &= \Big(\mathrm{id}_M, X(\widetilde{\nu}) + \widetilde{T} \circ (\sigma, \widetilde{\nu})\Big)(x). \end{aligned}$$

Therefore, the map $X(\widetilde{\nu}) + \widetilde{T} \circ (\sigma, \widetilde{\nu})$ is smooth for arbitrary taken smooth function $\widetilde{\nu} \in C^\infty(M; V)$.

Let $(e_j)_{j=1}^r$ be a base of the linear space V and let $\nu_j \in \Gamma(M\times V)$ be the constant sections of the vector bundle $M \times V$, defined by $\nu_j(x) = (x, e_j)$. Observe that there exist functions $\beta^i_j \in C^\infty(M)$ such that

$$\Big(X(\widetilde{\nu}_j) + \widetilde{T} \circ (\sigma, \widetilde{\nu}_j)\Big)(x) = \sigma(x)(e_j) = \sum_{i=1}^r \beta^i_j(x) e_i. \tag{1.9}$$

From the above in (1.9) it follows that

$$\sigma(x) = \sum_{i,j=1}^r \beta^i_j(x)\gamma^j_i$$

where the maps γ^j_i form a base of the space $\mathrm{End}(V)$ determined by $(e_j)_{j=1}^r$ in such a way that $\gamma^j_i(e_k) = \delta^j_k e_i$. Hence it is appears that σ is smooth. Since the mappings $X(\widetilde{\nu}) + \widetilde{T} \circ (\sigma, \widetilde{\nu})$, $\widetilde{T}, \widetilde{\nu}$ and σ are smooth, it follows that $X(\widetilde{\nu}) \in C^\infty(M; V)$. From this we conclude that X is a smooth vector field on M.

A section $\mathcal{L}$ of the vector bundle $A(E)$ determines a covariant differential operator $L_{\mathcal{L}} : \Gamma(E) \longrightarrow \Gamma(E)$ by $L_{\mathcal{L}}(\nu)(x) = \mathcal{L}_x(\nu)$ for $x \in M$, $\nu \in \Gamma(E)$. Moreover, each covariant differential operator in E is of the form $L_{\mathcal{L}}$ for exactly one section $\mathcal{L}$ of the vector bundle $A(E)$. In fact, a covariant differential operator L is equal to $L_{\mathcal{L}}$ for $\mathcal{L}_x(\nu) = (L(\nu))(x)$, $x \in M$, $\nu \in \Gamma(E)$. The smoothness of $\mathcal{L}$ now follows from Lemma 1.3.1.

The Lie bracket $[\![\cdot,\cdot]\!]$ in $\Gamma(A(E))$ is defined in the classical way, like that for differential operators. For $\mathcal{K}, \mathcal{L} \in \Gamma(A(E))$ we define $[\![\mathcal{K}, \mathcal{L}]\!] \in \Gamma(A(E))$ in such a way that $L_{[\![\mathcal{K},\mathcal{L}]\!]} = L_{\mathcal{K}} \circ L_{\mathcal{L}} - L_{\mathcal{L}} \circ L_{\mathcal{K}}$ noticing that the right-hand side of the last formula is a covariant differential operator. This also shows that $\operatorname{Sec} \varrho : \Gamma(A(E)) \longrightarrow \Gamma(TM)$, $\mathcal{L} \mapsto \varrho \circ \mathcal{L}$, is a homomorphism of Lie algebras. Moreover,

$$[\![\mathcal{K}, f \cdot \mathcal{L}]\!]_x(\nu) = f(x) \cdot [\![\mathcal{K}, \mathcal{L}]\!]_x(\nu) + (\varrho \circ \mathcal{K})_x(f) \cdot \mathcal{L}_x(\nu)$$

for any $\mathcal{K}, \mathcal{L} \in \Gamma(A(E))$, $f \in C^\infty(M)$, $x \in M$, $\nu \in \Gamma(E)$.

Thus we have demonstrated that in the vector bundle $A(E)$ we have a structure of a transitive Lie algebroid with the introduced Lie bracket $[\![\cdot,\cdot]\!]$ on the space of global sections of $A(E)$ and the anchor ϱ defined by (1.8).

The Atiyah sequence of $A(E)$ is

$$0 \longrightarrow \operatorname{End}(E) \longrightarrow A(E) \xrightarrow{\ \varrho\ } TM \longrightarrow 0.$$

The approach presented here, described in [4], consists mainly in the construction of a vector bundle, which is additionally equipped with an anchor and a Lie bracket in the module of section of $A(E)$. Other approaches are related to the notion of the Lie groupoid or the covariant differential operator, jets bundles and the symbol [75], [56],[79], [61], [48], [55]. For the first time the structure of the Lie algebroid of a vector bundle was presented by Ngô Van Quê in [75]. Each Lie groupoid Φ on a manifold M has a vector bundle $A(\Phi)$ (called the Lie algebroid of Φ) of all α-vertical vectors on Φ tangent at units of Φ [74]. The construction of the vector bundle $A(\Phi)$ was based on some generalization of the fundamental relations between the Lie groupoid $\Pi^k M$ of all invertible k-jets of M and the vector bundle $J^k TM$ of all k-jets of the tangent bundle TM [59]. The functor $\Phi \longmapsto A(\Phi)$ is called *the Lie functor* for Lie groupoids. Let E be any vector bundle and $GL(E)$ the Lie groupoid of all isomorphisms between fibres of E. In the language of exponential mappings for Lie groupoids the Lie algebra of sections of the Lie algebroid $A(GL(E))$ was discovered by Ngô Van Quê [75] (cf. [54], [55], [48]) and can be briefly described as follows: Let $\theta(E)$ contain any differential operator $L : \Gamma(E) \longrightarrow \Gamma(E)$ for which there exists a vector field X on M such that $L(f\cdot\nu) = f\cdot L(\nu)+X(f)\cdot\nu$ for any $\nu \in \Gamma(E)$ and $f \in C^\infty(M)$. The rank of such an operator L is at most 1 and $\sigma(L) = X\otimes\mathrm{Id} \in \Gamma(TM\otimes E^*\otimes E)$ is its symbol. It is called, by Mackenzie [61], a *covariant differential operator*. The space of all of them forms an $\mathbb{R}$-Lie algebra with respect to the natural commutator of differential operators. For $\Theta \in \Gamma(A(GL(E)))$, the formula

$$L(\Theta)(\nu)(x) = \left.\frac{d}{dt}\right|_0 \left[(\operatorname{Exp} t\Theta)(x)\right]^{-1} \cdot \left[\nu \circ \exp(tX)(x)\right]$$

determines a covariant differential operator, and the mapping

$$L : \Gamma(A(GL(E))) \longrightarrow \theta(E), \qquad \Theta \longmapsto L(\Theta)$$

is a $C^\infty(M)$-linear isomorphism of real Lie algebras. Mackenzie gives an equivalent definition of the Lie algebroid $A(GL(E))$ as a subbundle $\mathrm{CDO}(E) \subset \mathrm{Hom}(J^1E, E)$ of the vector bundle of linear homomorphisms from the bundle of 1-jets of E to E containing elements $d \in \mathrm{Hom}(J^1E, E)_x$ such that the value $\sigma(d)$ of the symbol map

$$\sigma : \mathrm{Hom}(J^1E, E) \longrightarrow \mathrm{Hom}(T^*M, \mathrm{End}(E)) = TM \otimes \mathrm{End}(E)$$

is equal to $u \otimes \mathrm{Id}$ for some vector $u \in T_xM$ (cf. [75] for the symbol map). Thus, $\mathrm{CDO}(E) = \sigma^{-1}(TM)$ with TM considered as a subbundle of $\mathrm{Hom}(T^*M, \mathrm{End}(E))$. From the above interpretation one can show that the fibre $A(GL(E))_x$ over $x \in M$ may be identified with the space of linear homomorphisms $l : \Gamma(E) \longrightarrow E_x$ for which there exists a vector $u \in T_xM$ satisfying (1.4).

An interesting approach to the Lie algebroid of a vector bundle $E \xrightarrow{p} M$ from the point of view of a reduction to its subalgebroid was described by Teleman in [79], where this algebroid is understood as the Lie algebroid of the principal bundle L_E of all frames of E. A similar up to isomorphism interpretation of the corresponding Lie algebroids was also remarked by Mackenzie in [61] and by Kubarski in [50]. Namely, Lie algebroids $A(E)$ and $A(L_E)$ are isomorphic. The isomorphism considered in [50] is defined as follows: Since the frames of a vector bundle E with a standard fibre V can be treated as linear isomorphisms from V to E_x ($x \in M$), any section ν of E determines a smooth map

$$\widetilde{\nu} : L_E \longrightarrow V, \quad \widetilde{\nu}(u) = u^{-1}(\nu(\pi(u))),$$

where $\pi : L_E \longrightarrow M$ is the projection in L_E. Then, arbitrarily taken $\varphi \in \Gamma(A(L_E))$ and $\nu \in \Gamma(E)$ determine the mapping

$$L_\varphi(\nu) : M \longrightarrow E, \; L_\varphi(\nu)(x) = u(\varphi'_u(\widetilde{\nu})), \quad u \in (L_E)_x,$$

which is a section of the bundle E. Next, for any $\varphi \in \Gamma(A(L_E))$ the map

$$\hat{L}_\varphi : \Gamma(E) \longrightarrow \Gamma(E), \quad \hat{L}_\varphi(\nu) = L_\varphi(\nu)$$

is a covariant differential operator in the sense of [61] with

$$\hat{L}_{f\cdot\varphi} = f \cdot \hat{L}_\varphi, \quad \hat{L}_{[\![\varphi,\phi]\!]} = \hat{L}_\varphi \circ \hat{L}_\phi - \hat{L}_\phi \circ \hat{L}_\varphi$$

for $f \in C^\infty(M)$, $\varphi, \phi \in \Gamma(A(L_E))$. Thus, we have an isomorphism

$$\mathrm{Sec}\, \Phi_E : \Gamma(A(L_E)) \longrightarrow \Gamma(A(E)), \quad \varphi \longmapsto \hat{L}_\varphi$$

at the level of $C^\infty(M)$-modules, which is also an isomorphism of Lie algebroids, i.e., it commutes with the anchors and preserves the Lie brackets. Thus, the corresponding isomorphism

$$\Phi_E : A(L_E) \longrightarrow A(E)$$

of Lie algebroids is defined by

$$\Phi_E([\mathrm{w}])(\nu) = u(\mathrm{w}(\widetilde{\nu})), \qquad \mathrm{w} \in T_u(L_E), \; u \in L_E, \; \nu \in \Gamma(E).$$

1.4 Restriction of a Lie algebroid to an open subset

In this section we describe the Lie algebroid induced by a given algebroid and an open set (cf. [61], [63]) understood as the restriction of structures given in the starting algebroid.

Let $(A, \varrho_A, [\cdot,\cdot])$ be a Lie algebroid over a manifold M and let U be an open subset of M. Moreover, let A_U denote the restriction of a vector bundle to U.

Consider sections ξ_1, ξ_2 of A_U. For any $x \in M$ there exists an open set $B \subset U$ such that $x \in B$ and sections $\overline{\xi}_1$, $\overline{\xi}_2$ of A such that $\overline{\xi}_1|B = \xi_1$ and $\overline{\xi}_2|B = \xi_2$. To be more precise, let $f \in C^\infty(M)$ be a function that separates x in U. There is thus an open set $B \subset U$ satisfying $x \in B$, $f|B = 1$, and $f|(M \setminus U) = 0$. Then actually sections $\overline{\xi}_1$, $\overline{\xi}_2 \in \Gamma(A)$ defined by

$$\overline{\xi}_i(x) \triangleq \begin{cases} f(x) \cdot \xi_i(x) & \text{for } \ x \in U \\ \quad 0 & \text{for } \ x \in M \setminus U \end{cases} \qquad (i \in \{1,2\})$$

meet the required conditions. By Corollary 1.1.1, if sections $X, X', Y, Y' \in \Gamma(A)$ satisfy $X_{|O} = X'_{|O}$ and $Y_{|O} = Y'_{|O}$ for an open subset $O \subset M$, then $[X,Y]_{|O} = [X',Y']_{|O}$. Thus, we can correctly define the skew-symmetric bracket in $\Gamma(A_U)$ in such a way that

$$[\![\xi_1, \xi_2]\!]_x = \left[\overline{\xi}_1, \overline{\xi}_2\right]_x \quad \text{for } \ x \in U.$$

Now, we define

$$\varrho \triangleq \varrho_A|\, A_U : A_U \longrightarrow TU$$

as an anchor in A_U. Let ξ, $\eta \in \Gamma(A_U)$, $f \in C^\infty(U)$ and $x \in U$. Then there are sections $\overline{\xi}$, $\overline{\eta} \in \Gamma(A)$, $g \in C^\infty(M)$ and an open set $B \subset U$ such that $x \in B$, $\overline{\xi}|B = \xi$, $\overline{\eta}|B = \eta$ and $g|B = f|B$. From the Leibniz identity for the Lie bracket in $\Gamma(A)$ we get

$$\begin{aligned} [\![\xi, f \cdot \eta]\!]_x &= \left[\overline{\xi}, g \cdot \overline{\eta}\right]_x = g(x) \cdot \left[\overline{\xi}, \overline{\eta}\right]_x + \left(\gamma \circ \overline{\xi}\right)_x (g) \cdot \overline{\eta}_x \\ &= f(x) \cdot [\![\xi, \eta]\!]_x + (\overline{\gamma} \circ \xi)_x\, (f) \cdot \eta_x \\ &= (f \cdot [\![\xi, \eta]\!] + (\overline{\gamma} \circ \xi)\,(f) \cdot \eta)(x). \end{aligned}$$

Thus, we remarked that the restriction A_U of the bundle A has a Lie algebroid structure $(A_U, \varrho, [\![\cdot,\cdot]\!])$ induced from the Lie algebroid $(A, \varrho_A, [\cdot,\cdot])$.

1.5 Trivial Lie algebroid

Let M be a smooth manifold and $\mathfrak{g}$ be a finitely dimensional real Lie algebra with a Lie bracket $[\cdot,\cdot]$ and $\dim \mathfrak{g} = p \in \mathbb{N}$. Let us take any basis $(\mathrm{v}_i)_{i=1}^p$ of the vector space $\mathfrak{g}$. We define

$$\mathcal{L}_X \eta = \sum_{i=1}^{p} X(f^i) \cdot \mathrm{v}_i$$

for $X \in \Gamma(TM)$, $\eta \in C^\infty(M;\mathfrak{g})$ such that $\eta(x) = \sum_{i=1}^{p} f^i(x) \cdot \mathrm{v}_i$ for $x \in M$, and where $f^i \in C^\infty(M)$. This definition is correct because the value of $\mathcal{L}_X \eta$ does not depend on the choice of the basis of the vector space $\mathfrak{g}$. Moreover, we have

Lemma 1.5.1. *For $\eta, \sigma \in C^\infty(M;\mathfrak{g})$, $f \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$ the following equalities hold:*

(*a*) $\mathcal{L}_{f \cdot X} \eta = f \cdot \mathcal{L}_X \eta$,

(*b*) $\mathcal{L}_X (f \cdot \eta) = f \cdot \mathcal{L}_X \eta + X(f) \cdot \eta$,

(*c*) $\mathcal{L}_{[X,Y]} \eta = \mathcal{L}_X (\mathcal{L}_Y \eta) - \mathcal{L}_Y (\mathcal{L}_X \eta)$,

(*d*) $\mathcal{L}_X ([\eta, \sigma]) = [\mathcal{L}_X \eta, \sigma] + [\eta, \mathcal{L}_X \sigma]$.

Define

$$[\![\, (X,\sigma) \,, (Y,\eta) \,]\!] = ([X,Y] \,, \mathcal{L}_X \eta - \mathcal{L}_Y \sigma + [\sigma, \eta])$$

for $X, Y \in \Gamma(TM)$, and $\sigma, \eta \in C^\infty(M;\mathfrak{g})$. The map $[\![\cdot,\cdot]\!]$ is bilinear over $\mathbb{R}$ and skew-symmetric. Moreover, applying the properties from Lemma 1.5.1, we have

$$[\![(X,\sigma), f \cdot (Y,\eta)]\!] = f \cdot [\![(X,\sigma),(Y,\eta)]\!] + (\gamma \circ (X,\sigma))(f) \cdot (Y,\eta),$$

where

$$\gamma = \mathrm{pr}_1 : TM \times \mathfrak{g} \longrightarrow TM, \quad (\mathrm{v},\mathrm{w}) \longmapsto \mathrm{v}$$

has the role of an anchor, and

$$\mathrm{Jac}_{[\![\cdot,\cdot]\!]}((X,\sigma),(Y,\eta),(Z,\tau)) = [\![(X,\sigma),[\![(Y,\eta),(Z,\tau)]\!]]\!] + cycl = 0$$

for $X, Y, Z \in \Gamma(TM)$, and $\sigma, \eta, \tau \in C^\infty(M;\mathfrak{g})$. Based on the obtained results, we conclude that $(TM \times \mathfrak{g},\ [\![\cdot,\cdot]\!],\ \mathrm{pr}_1)$ is a transitive Lie algebroid with the Atiyah sequence

$$0 \longrightarrow \mathfrak{g} \longrightarrow TM \times \mathfrak{g} \xrightarrow{\mathrm{pr}_1} TM \longrightarrow 0.$$

Remark 1.5.1. Let E be a vector bundle over M with a projection $p : E \longrightarrow M$, and let $\psi : U \times V \longrightarrow p^{-1}(U)$ be its local trivialization. We demonstrate that

$$\overline{\psi} : TU \times \operatorname{End}(V) \longrightarrow A(E)_U$$

defined in Section 1.3 by (1.6) is an isomprphism of the trivial Lie algebroid $TU \times \operatorname{End}(V)$ and the restriction $A(E)_U$ of $A(E)$ to U [50]. The map $\overline{\psi}$ is an isomorphism of vector bundles (cf. Section 1.3). It remains to show that $\overline{\psi}$ preserves the Lie brackets of considered algebroids.

Let $X, Y \in \mathfrak{X}(U)$, $\sigma, \eta \in C^\infty(U; \operatorname{End}(V))$, $x \in U$, and $\nu \in \Gamma(E)$. We first observe that

$$\begin{aligned}
\left(\left(\overline{\psi} \circ (X, \sigma)\right)(\nu)\right)_\psi &= X(\nu_\psi) + T \circ (\sigma, \nu_\psi)\,, \\
\left(\left(\overline{\psi} \circ (Y, \eta)\right)(\nu)\right)_\psi &= Y(\nu_\psi) + T \circ (\eta, \nu_\psi)\,,
\end{aligned}$$

where $\nu_\psi : U \longrightarrow V$ is defined in (1.5), and $T : \operatorname{End}(V) \times V \longrightarrow V$ is a bilinear map given by $T(L, z) = L(z)$ for $(L, z) \in \operatorname{End}(V) \times V$. Moreover,

$$\begin{aligned}
X_x(T \circ (\eta, \nu_\psi)) &= T(\eta_x, X_x(\nu_\psi)) + T(X_x(\eta), \nu_\psi(x)) \\
&= \eta_x(X_x(\nu_\psi)) + (X_x(\eta))\,(\nu_\psi(x))
\end{aligned}$$

and

$$\begin{aligned}
Y_x(T \circ (\sigma, \nu_\psi)) &= T(\sigma_x, Y_x(\nu_\psi)) + T(Y_x(\sigma), \nu_\psi(x)) \\
&= \sigma_x(Y_x(\nu_\psi)) + (Y_x(\sigma))(\nu_\psi(x)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&[\![\overline{\psi} \circ (X, \sigma)\,, \overline{\psi} \circ (Y, \eta)\,]\!]_x(\nu) \\
&= \left(\overline{\psi} \circ (X, \sigma)\right)_x \left(\left(\overline{\psi} \circ (Y, \eta)\right)(\nu)\right) - \left(\overline{\psi} \circ (Y, \eta)\right)_x \left(\left(\overline{\psi} \circ (X, \sigma)\right)(\nu)\right) \\
&= \overline{\psi}_x\,(X_x, \sigma_x) \left(\left(\overline{\psi} \circ (Y, \eta)\right)(\nu)\right) - \overline{\psi}_x\,(Y_x, \eta_x) \left(\left(\overline{\psi} \circ (X, \sigma)\right)(\nu)\right) \\
&= \psi_x \left(X_x \left(\left(\left(\overline{\psi} \circ (Y, \eta)\right)(\nu)\right)_\psi\right) + \sigma_x \left(\left(\left(\overline{\psi} \circ (Y, \eta)\right)(\nu)\right)_\psi (x)\right)\right) \\
&\quad - \psi_x \left(Y_x \left(\left(\left(\overline{\psi} \circ (X, \sigma)\right)(\nu)\right)_\psi\right) + \eta_x \left(\left(\left(\overline{\psi} \circ (X, \sigma)\right)(\nu)\right)_\psi (x)\right)\right) \\
&= \psi_x \left(X_x \left(Y(\nu_\psi) + T \circ (\eta, \nu_\psi)\right) + \sigma_x \left(Y_x(\nu_\psi) + \eta_x(\nu_\psi(x))\right)\right) \\
&\quad - \psi_x \left(Y_x \left(X(\nu_\psi) + T \circ (\sigma, \nu_\psi)\right) + \eta_x \left(X_x(\nu_\psi) + \sigma_x(\nu_\psi(x))\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \psi_x \left(X_x \left(Y \left(\nu_\psi \right) \right) - Y_x \left(X \left(\nu_\psi \right) \right) \right) \\
&\quad + \psi_x \left(X_x \left(T \circ (\eta, \nu_\psi) \right) - Y_x \left(T \circ (\sigma, \nu_\psi) \right) \right) \\
&\quad + \psi_x \left(\sigma_x (Y_x (\nu_\psi)) - \eta_x (X_x (\nu_\psi)) \right) \\
&\quad + \psi_x \left(\sigma_x (\eta_x (\nu_\psi (x))) - \eta_x (\sigma_x (\nu_\psi (x))) \right) \\
&= \psi_x ([X,Y]_x (\nu_\psi) + X_x(\eta)(\nu_\psi(x)) - Y_x(\sigma)(\nu_\psi(x)) + [\sigma_x, \eta_x](\nu_\psi\,(x))) \\
&= \psi_x \left([X,Y]_x (\nu_\psi) + \left(\mathcal{L}_X(\eta) - \mathcal{L}_Y(\sigma) + [\sigma,\eta] \right)_x (\nu_\psi(x)) \right) \\
&= \left(\overline{\psi} \circ \left([X,Y], \mathcal{L}_X(\eta) - \mathcal{L}_Y(\sigma) + [\sigma,\eta] \right) \right)_x (\nu) \\
&= \left(\overline{\psi} \circ [\![\, (X,\sigma)\,, (Y,\eta) \,]\!] \right)_x (\nu).
\end{aligned}
$$

So, indeed $\operatorname{Sec} \overline{\psi} : \Gamma(TU \times \operatorname{End}(V)) \longrightarrow \Gamma(A(E)_U)$ preserves the Lie brackets. Moreover, from the definition of the mapping $\overline{\psi}$ it follows that $\varrho_{A(E)_U}(\overline{\psi}(u,a)) = u$ for $(u,a) \in TU \times \operatorname{End}(V)$, which means that the diagram

$$
\begin{tikzcd}
TU \times \operatorname{End}(V) \arrow[rr, "\overline{\psi}"] \arrow[dr, "\mathrm{pr}_1"'] & & A(E)_U \arrow[dl, "\varrho_{A(E)_U}"] \\
& TU &
\end{tikzcd}
$$

is commutative. Thus, $\overline{\psi}$ as a bundle isomorphism is also an isomorphism of Lie algebroids $TU \times \operatorname{End}(V)$ and $A\,(E)_U$.

In the special case of the trivial bundle $E = M \times V$, we have the following isomorphism of Lie algebroids:

$$A(M \times V) \cong TM \times \operatorname{End}(V).$$

1.6 Cartesian product of Lie algebroids

Let $(A, \varrho_A, [\![\cdot,\cdot]\!])$ and $(A', \varrho_{A'}, [\![\cdot,\cdot]\!]')$ be Lie algebroids over manifolds M and M', respectively. By the Cartesian product of Lie algebroids A and A' we mean the Lie algebroid

$$\left(A \times A', \varrho_A \times \varrho_{A'}, [\![\cdot,\cdot]\!]^{\times}\right)$$

over the manifold $M \times M'$, and where the Lie bracket $[\![\cdot,\cdot]\!]^{\times}$ is defined in such a way that for any $\widetilde{\xi} = (\widetilde{\xi}^1, \widetilde{\xi}^2)$, $\widetilde{\eta} = (\widetilde{\eta}^1, \widetilde{\eta}^2) \in \Gamma(A \times A')$, $(x, x') \in M \times M'$ we have

$$[\![\widetilde{\xi}, \widetilde{\eta}]\!]^{\times} = \left([\![\widetilde{\xi}, \widetilde{\eta}]\!]^1, [\![\widetilde{\xi}, \widetilde{\eta}]\!]^2\right),$$

where

$$\begin{aligned} &[\![\widetilde{\xi}, \widetilde{\eta}]\!]^1_{(x,x')} \\ &= [\![\widetilde{\xi}^1(\cdot, x'), \widetilde{\eta}^1(\cdot, x')]\!]_x + (\varrho_{A'} \circ \widetilde{\xi}^2)_{(x,x')}(\widetilde{\eta}^1(x, \cdot))) - (\varrho_{A'} \circ \widetilde{\eta}^2)_{(x,x')}(\widetilde{\xi}^1(x, \cdot)), \end{aligned}$$

and

$$\begin{aligned} &[\![\widetilde{\xi}, \widetilde{\eta}]\!]^2_{(x,x')} \\ &= [\![\widetilde{\xi}^2(x, \cdot), \widetilde{\eta}^2(x, \cdot)]\!]'_{x'} + (\varrho_A \circ \widetilde{\xi}^1)_{(x,x')}(\widetilde{\eta}^2(\cdot, x')) - (\varrho_A \circ \widetilde{\eta}^1)_{(x,x')}(\widetilde{\xi}^2(\cdot, x')). \end{aligned}$$

For the Cartesian product of Lie algebroids we refer to [51].

1.7 The inverse-image of a Lie algebroid

Let $(A, \varrho_A, [\![\cdot,\cdot]\!])$ be a regular Lie algebroid over a manifold M, $\operatorname{Im} \varrho_A = F$, and let $f : (M', F') \longrightarrow (M, F)$ be a smooth function between foliated manifolds, i.e., $F' \subset TM'$ is such a subbundle of TM' that $f_*[F'] \subset F$. Then, there is a regular Lie algebroid, cf. [50],

$$(f^{\wedge}A, [\![\cdot,\cdot]\!]^{\wedge}, \mathrm{pr}_1)$$

called the *inverse-image* of A *by* f, in which:

1. $f^{\wedge}A = F' \times_{(f_*, \varrho_A)} A = \{(\mathrm{v}, \mathrm{w}) \in F' \times A : f_*(\mathrm{v}) = \varrho_A(\mathrm{w})\}$
   $\subset F' \oplus f^*A$,
2. an anchor is the projection $\mathrm{pr}_1 : F' \times_{(f_*, \varrho_A)} A \longrightarrow F'$ given by $(\mathrm{v}, \mathrm{w}) \longmapsto \mathrm{v}$,
3. the bracket $[\![\cdot,\cdot]\!]^{\wedge}$ in $\Gamma(f^{\wedge}A)$ is defined as follows: Let $(X_1, \overline{\xi}_1)$, $(X_2, \overline{\xi}_2) \in \Gamma(f^{\wedge}A)$, where $X_1, X_2 \in \Gamma(F')$, $\overline{\xi}_1, \overline{\xi}_2 \in \Gamma(f^*A)$. Then, for any $x \in M'$ there is an open set $U \subset M'$ such that $x \in U$ and

$$\overline{\xi}_1|U = \sum_j g_1^j \cdot (\xi_1^j \circ f)\bigg|_U, \qquad \overline{\xi}_2|U = \sum_j g_1^j \cdot (\xi_1^j \circ f)\bigg|_U$$

for some $g_i^j \in C^\infty(M')$ and $\xi_i^j \in \Gamma(A)$. Set

$$\begin{aligned}&[\![(X_1, \overline{\xi}_1), (X_2, \overline{\xi}_2)]\!]^{\wedge}_{|U} \\ &= \left([X_1, X_2], \sum_{j,k} g_1^j \cdot g_2^k \cdot \left([\![\xi_1^j, \xi_2^k]\!] \circ f \right) + \sum_k X_1 \left(g_2^k \right) \cdot \left(\xi_2^k \circ f \right) \right. \\ &\qquad \left. - \sum_j X_2 \left(g_1^j \right) \cdot \left(\xi_1^j \circ f \right) \right)_{|U} .\end{aligned}$$

The Atiyah sequence of $f^\wedge A$ is

$$0 \longrightarrow f^*(\ker \varrho_A) \longrightarrow f^\wedge A \xrightarrow{\mathrm{pr}_1} F' \longrightarrow 0.$$

1.8 Transformation Lie algebroid

Let $\mathfrak{g}$ be a finite real Lie algebra with a Lie bracket $[\cdot, \cdot]$, which acts on a manifold M. Let

$$\lambda : \mathfrak{g} \longrightarrow \Gamma(TM)$$

be a homomorphism of Lie algebras $\mathfrak{g}$ and the Lie algebra of vector fields on M equipped with the classical Lie bracket of vector fields. Set

$$A = M \times \mathfrak{g}$$

as the trivial bundle with the anchor

$$\varrho : M \times \mathfrak{g} \longrightarrow TM,$$

$$\varrho(x, \nu) = \lambda(\nu)(x)$$

Taking the identification $\Gamma(M \times \mathfrak{g}) \cong C^\infty(M; \mathfrak{g})$, we define the bracket

$$[\![\cdot, \cdot]\!] : \Gamma(M \times \mathfrak{g}) \times \Gamma(M \times \mathfrak{g}) \longrightarrow \Gamma(M \times \mathfrak{g})$$

in $\Gamma(M \times \mathfrak{g})$ by

$$[\![f, g]\!](x) = [f(x), g(x)] + ((\lambda \circ f)(g))(x) - ((\lambda \circ g)(f))(x)$$

for $f, g \in C^\infty(M; \mathfrak{g})$, $x \in M$. The triple $(M \times \mathfrak{g}, \varrho, [\![\cdot, \cdot]\!])$ is a transitive Lie algebroid over M called the *transformation Lie algebroid* induced by λ (cf. [62]).

1.9 Lie algebroid of a Poisson manifold

In this section we describe the structure of the Lie algebroid on the cotangent bundle that any Poisson structure induces, in particular any symplectic manifold determines such an algebroid. In the literature, this structure appears in articles of several authors independently. For the first time this structure was introduced by Fuchssteiner in [26], and by Magri and Morosi in [64], by Dazord and Sondaz in [21], and by Weinstein in [81] (cf. [46]).

Let M be a Poisson manifold with a Poisson bracket

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M).$$

Recall that $\{\cdot,\cdot\}$ yields the structure of a real Lie algebra on $\mathcal{C}^\infty(M)$ and satisfies the following the Leibniz rule of derivation:

$$\{f, g\cdot h\} = g\cdot\{f,h\} + \{f,g\}\cdot h$$

for any $f,g,h \in \mathcal{C}^\infty(M)$.

There exists a unique smooth section $\Pi \in \Gamma(\bigwedge^2 TM)$, called the **Poisson bivector**, such that

$$\{f,g\} = \Pi(df, dg) \tag{1.10}$$

and

$$[\Pi,\Pi]_{S\text{-}N} = 0, \tag{1.11}$$

where $[\cdot,\cdot]_{S\text{-}N}$ is the Schouten-Nijenhuis bracket ([60]). Let us recall here that the Schouten-Nijenhuis bracket is defined on the multivector fields, i.e., on sections of vector bundles $\bigwedge^p TM$. Let

$$\mathcal{V}(M) = \bigoplus_{p\geq 0} \mathcal{V}^p(M) \;\text{ where } \mathcal{V}^p(M) = \Gamma(\textstyle\bigwedge^p TM).$$

Then $[\cdot,\cdot]_{S\text{-}N} : \mathcal{V}(M)\times\mathcal{V}(M) \to \mathcal{V}(M)$ is an $\mathbb{R}$-bilinear map of a degree -1 defined in such a way that

$$[P,Q]_{S\text{-}N} \in \mathcal{V}^{p+q-1}(M) \text{ for } P\in\mathcal{V}^p(M),\ Q\in\mathcal{V}^q(M),$$

$$[P,Q]_{S\text{-}N} = (-1)^{pq}\,[Q,P]_{S\text{-}N} \text{ for } P\in\mathcal{V}^p(M),\ Q\in\mathcal{V}^q(M),$$

$$[X,Y]_{S\text{-}N} = [X,Y] \text{ for } X,Y\in\mathcal{V}^1(M),$$

$$[X, f]_{S\text{-}N} = X(f) \text{ for any } X \in \mathcal{V}^1(M) \text{ and } f \in \mathcal{V}^0(M) = \mathcal{C}^\infty(M),$$

$$[X_1 \wedge \ldots \wedge X_p, Y]_{S\text{-}N} = \sum_{i=1}^{p} (-1)^{i+1} X_1 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge X_p \wedge [X_i, Y],$$

$$[P, Q \wedge R]_{S\text{-}N} = [P, Q]_{S\text{-}N} \wedge R + (-1)^{(p+1)q} Q \wedge [P, R]_{S\text{-}N}$$

for $X_1, \ldots, X_p, Y \in \mathcal{V}^1(M)$, $P \in \mathcal{V}^p(M), Q \in \mathcal{V}^q(M)$, $R \in \mathcal{V}(M)$, and where the symbol $\widehat{X}_i$ means that X_i is omitted (cf. [82]). The Schouten-Nijenhuis bracket is therefore an extension of the Lie bracket of vector fields on M. Remark that (1.11) is equivalent to the Jacobi identity of the Poisson bracket $\{\cdot,\cdot\}$ because $[\Pi, \Pi]_{S\text{-}N}(df, dg, dh)$ is equal up to a constant to the Jacobiator $\mathrm{Jac}_{\{\cdot,\cdot\}}(f, g, h)$ for $f, g, h \in \mathcal{C}^\infty(M)$.

In a local coordinate system $(x_1, \ldots, x_n)$ we have

$$\Pi = \sum_{1 \le i < j \le n} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j},$$

and where $\Pi_{ij} = \{x_i, x_j\} = \Pi(dx_i, dx_j)$. Therefore, for $f, g \in \mathcal{C}^\infty(M)$ we have

$$\{f, g\} = \Pi(df, dg) = \sum_{i,j=1}^{n} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

The anchor in the cotangent bundle is the vector bundle map

$$\sharp : T^*M \longrightarrow TM$$

given by

$$\langle \sharp(\alpha), \beta \rangle = \Pi(\alpha, \beta)$$

for $\alpha, \beta \in \Omega^1(M) = \Gamma(T^*M)$. Therefore,

$$\sharp(df)(g) = \{f, g\}$$

for $f, g \in \mathcal{C}^\infty(M)$. The Lie algebra structure on $\Omega^1(M) = \Gamma(T^*M)$ is defined by the bracket

$$[\![\cdot,\cdot]\!] : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$$

which is uniquely given by

$$[\![df, dg]\!] = d\{f, g\}$$

and

$$[\![\alpha, f \cdot \beta]\!] = f \cdot [\![\alpha, \beta]\!] + \sharp(\alpha)(f) \cdot \beta$$

for all $f, g \in \mathcal{C}^\infty(M)$ and $\alpha, \beta \in \Omega^1(M)$. Explicitly, the skew-symmetric bracket $[\![\cdot, \cdot]\!]$ is given by the formula

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\sharp(\alpha)}(\beta) - \mathcal{L}_{\sharp(\beta)}(\alpha) - d\left(\Pi(\alpha, \beta)\right) \tag{1.12}$$

for $\alpha, \beta \in \Omega^1(M)$. Therefore, $\sharp$ defines the Lie algebra homomorphism

$$\operatorname{Sec}\sharp : \Omega^1(M) \to \Gamma(TM), \quad \alpha \mapsto \sharp\alpha,$$

i.e., $\sharp[\![\alpha, \beta]\!] = [\sharp(\alpha), \sharp(\beta)]$ for $\alpha, \beta \in \Omega^1(M)$. The Jacobi identity of $[\![\cdot, \cdot]\!]$ is equivalent to (1.11) because (cf. [46]) $2\,\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(df, dg, dh)$ is equal to $d([\Pi, \Pi]_{S\text{-}N}(df, dg, dh))$ for any $f, g, h \in \mathcal{C}^\infty(M)$. Summarizing, we have a vector bundle monomorphism $\sharp$ such that the following diagram commutes

$$\begin{tikzcd}
T^*M \arrow[r, "\sharp"] \arrow[d, "p_{T^*M}"'] & TM \arrow[d, "p_{TM}"] \\
M \arrow[r, "\mathrm{id}_M"'] & M
\end{tikzcd}$$

where $p_{TM} : TM \to M$ and $p_{T^*M} : T^*M \to M$ are projections of the tangent and the cotangent bundle, respectively. Moreover, we see at once that $\operatorname{Sec}\sharp$ is a homomorphism of Lie algebras.

The Lie algebroid $(T^*M, \sharp, [\![\cdot, \cdot]\!])$ defined in this way is called the **Lie algebroid of a Poisson manifold** M.

Every symplectic manifold (M, ω), where ω is a closed nondegenerate 2-form, has a Poisson structure with the bracket defined by

$$\{f, g\} = \omega(X_f, X_g)$$

where the Hamiltonian vector field X_f is given by

$$i_{X_f}\omega \equiv \omega(X_f, \cdot) = -df.$$

In particular, if $M = \mathbb{R}^{2m}$ with coordinates $(q_1, \dots, q_m, p_1, \dots, p_m)$ and $\omega = \sum_{i=1}^{m} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_j}$, then

$$\{f,g\} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

for $f, g \in \mathcal{C}^\infty(M)$.

In the cotangent bundle of a manifold M with a symplectic form ω, we can define the Lie algebroid structure with the bracket defined as in (1.12) and with an anchor which is the isomorphism of vector bundles $\sharp_\omega : T^*M \longrightarrow TM$ given by

$$\omega \langle \sharp_\omega(\alpha), Y \rangle = \alpha(Y)$$

for $\alpha \in \Gamma(T^*M)$, $Y \in \Gamma(TM)$. Consequently, in this case

$$0 \longrightarrow 0 \longrightarrow T^*M \xrightarrow{\sharp_\omega} TM \longrightarrow 0$$

is the Atiyah sequence of $(T^*M, \sharp_\omega, [\![\cdot,\cdot]\!])$, so it is a transitive Lie algebroid in which the anchor has a trivial kernel.

1.10 Structures of Lie algebroids on cotangent bundles determined by vector fields

In this section we present a family of Lie algebroids determined by specific vector fields on a given manifold, defined in [22] by Dobrogowska and Jakimowicz. In [22] the authors also note that the starting point can be replaced by any Lie algebroid and corresponding vector fields by sections.

Let M be a differential manifold and let $[\cdot,\cdot]$ denote the classical Lie bracket of vector fields on M. We choose the vector fields X, Y on M related to each other in such a way that

$$[X, Y] = cY \text{ for some } c \in \mathbb{R}.$$

Next, in the cotangent bundle of M we introduce the structure of the Lie algebroid in which the anchor is the homomorphism of vector bundles

$$\varrho_{X,Y} : T^*M \to TM$$

such that

$$\varrho_{X,Y}(\alpha) = -\alpha(Y)X \text{ for } \alpha \in \Gamma(T^*M),$$

while the Lie bracket $[\cdot,\cdot]_{X,Y}$ in the space of 1-differential forms $\Gamma(T^*M)$ is given by

$$[\alpha,\beta]_{X,Y} = \beta(Y)\mathcal{L}_X\alpha - \alpha(Y)\mathcal{L}_X\beta$$

for $\alpha,\beta \in \Gamma(T^*M)$. In particular, each vector field X on M defines in the cotangent bundle of M the structure of Lie algebroid $(T^*M, \varrho_{X,X}, [\cdot,\cdot]_{X,X})$.

In Section 2.4 below we observe that the defined Lie bracket on 1-differential forms is in fact a skew-symmetric part of some torsion-free and flat connection. This observation gives us the immediate fulfillment of the Jacobi identity by the given bracket of section in the cotangent bundle. Moreover, in Section 2.4, we remark that the introduced Lie bracket can be generalized to a pair of vector fields X, Y on M such that $[X,Y] = fY$ for some $f \in C^\infty(M)$.

Some linear combination of Lie algebroids defined in this way again creates a new Lie algebroid as shown by the following theorem.

Theorem 1.10.1. [22] *Let $\lambda \in \mathbb{R}$ and $X, Y \in \Gamma(TM)$ be such that $[X,Y] = 0$, then a structure $(T^*M, \varrho^\lambda_{X,Y}, [\cdot,\cdot]_{X,Y})$ is a Lie algebroid, where the anchor and the Lie bracket is given by*

$$[\cdot,\cdot]^\lambda_{X,Y} = [\cdot,\cdot]_{X,Y} + \lambda\,[\cdot,\cdot]_{Y,X}\,, \quad \varrho^\lambda_{X,Y} = \varrho_{X,Y} + \lambda\cdot\varrho_{Y,X};$$

explicitly,

$$[\alpha,\beta]^\lambda_{X,Y} = \beta(Y)\mathcal{L}_X\alpha - \alpha(Y)\mathcal{L}_X\beta + \lambda\cdot\left(\beta(X)\mathcal{L}_Y\alpha - \alpha(X)\mathcal{L}_Y\beta\right),$$

and

$$\varrho^\lambda_{X,Y}(\alpha) = -\alpha(Y)X - \lambda\cdot\alpha(X)Y$$

for any $\alpha,\beta \in \Gamma(T^*M)$.

In the special case when $\lambda = -1$ the bracket $[\cdot,\cdot]^\lambda_{X,Y}$ is related to the structure of Poisson manifold determined by the Poisson bivector $\Pi = X \wedge Y$. Namely, if $\{\cdot,\cdot\}$ is a Poisson bracket determined by Π,

$$[df, dg]^{-1}_{X,Y} = d\{f,g\}$$

for any $f, g \in C^\infty(M)$.

Part II

Linear connections

2. Linear connections on Lie algebroids

2.1 Linear connections in a vector bundle

Let $(A, \varrho_A, [\cdot,\cdot])$ be a Lie algebroid over a manifold M.

Definition 2.1.1. A *linear connection* in a vector bundle $E \to M$ is any an $\mathbb{R}$-bilinear map

$$\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$$

satisfying the following properties:

$$\nabla_{f \cdot X}(u) = f \cdot \nabla_X(u),$$

$$\nabla_X(f \cdot u) = f \cdot \nabla_X(u) + (\varrho_A \circ X)(f) \cdot u$$

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, $u \in \Gamma(E)$.

Definition 2.1.2. Let E be a vector bundle over a manifold M. The module of all sections of the Lie algebroid $A(E)$ is denoted by

$$\mathcal{CDO}(E).$$

Remark 2.1.1. We recall that $\mathcal{CDO}(E)$ is the space of all such $\mathbb{R}$-linear operators

$$\ell : \Gamma(E) \longrightarrow \Gamma(E)$$

that there exist unique $X_\ell \in \Gamma(TM)$ with the property

$$\ell(f \cdot u) = f \cdot \ell(u) + X_\ell(f) \cdot u$$

for $f \in C^\infty(M)$ and $u \in \Gamma(E)$.

Remark 2.1.2. Let E be a vector bundle over a manifold M and let

$$\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$$

be a connection in E. Set

$$\widehat{\nabla} : \Gamma(A) \longrightarrow \mathrm{End}_{C^\infty(M)}(\Gamma(E))$$

by

$$\widehat{\nabla}(X) = \nabla_X \quad \text{for} \quad X \in \Gamma(A).$$

Since

$$\widehat{\nabla}(X)(r \cdot u) = r \cdot \widehat{\nabla}(X)(u),$$

$$\widehat{\nabla}(X)(u_1 + u_2) = \widehat{\nabla}(X)(u_1) + \widehat{\nabla}(X)(u_2),$$

and

$$\widehat{\nabla}(X)(f \cdot u) = f \cdot \widehat{\nabla}(X)(u) + (\varrho_A \circ X)(f) \cdot u \tag{2.1}$$

for $X \in \Gamma(A)$, $r \in \mathbb{R}$, $u, u_1, u_2 \in \Gamma(E)$, $f \in C^\infty(M)$, we observe that

$$\widehat{\nabla}(X) \in \mathcal{CDO}(E) \text{ for any } X \in \Gamma(A).$$

Moreover, (2.1) implies that

$$\widehat{\nabla} : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

is a homomorphism of $C^\infty(M)$-modules $\Gamma(A)$ and $\mathcal{CDO}(E)$ such that the diagram

$$\begin{tikzcd}
\Gamma(A) \arrow[rr, "\widehat{\nabla}"] \arrow[dr, "\mathrm{Sec}\,\varrho_A"'] & & \mathcal{CDO}(E) \arrow[dl, "\mathrm{Sec}\,\varrho_{A(E)}"] \\
& \Gamma(TM) &
\end{tikzcd}$$

commutes. This observation leads to the generalization of the concept of a linear connection given in the definition below as vector bundles homomorphism acting from a Lie algebroid into a Lie algebroid that commutes with anchors (cf. [9]), or homomorphism of modules acting from the module of sections of the first Lie algebroid into the module of sections of the second Lie algebroid, which commutes with the anchors at the level of sections.

Definition 2.1.3. Let $(A, \varrho_A, [\cdot,\cdot]_A)$ and $(B, \varrho_B, [\cdot,\cdot]_B)$ be Lie algebroids, both over the same manifold M. By an *A-connection* in the Lie algebroid B we mean any homomorphism of $C^\infty(M)$-modules

$$\nabla : \Gamma(A) \longrightarrow \Gamma(B)$$

for which the diagram

$$\begin{tikzcd} \Gamma(A) \arrow[rr, "\nabla"] \arrow[dr, "\mathrm{Sec}\,\varrho_A"'] & & \Gamma(B) \arrow[dl, "\mathrm{Sec}\,\varrho_B"] \\ & \Gamma(TM) & \end{tikzcd}$$

is commutative.

Lemma 2.1.1. *Let $(A, \varrho_A, [\cdot,\cdot]_A)$ and $(B, \varrho_B, [\cdot,\cdot]_B)$ be Lie algebroids, both over the same manifold M, and let ϱ_B be a constant rank, i.e., $(B, \varrho_B, [\cdot,\cdot]_B)$ is a regular Lie algebroid. Then, for any A-connection $\nabla : \Gamma(A) \longrightarrow \Gamma(B)$ in B we have*

$$\nabla \circ [X, Y]_A - [\nabla \circ X, \nabla \circ Y]_A \in \Gamma(\ker \varrho_B)$$

for any $X, Y \in \Gamma(A)$.

Proof. Let $X, Y \in \Gamma(A)$. Then, since $\varrho_B \circ \nabla = \varrho_A$ and anchors preserves the Lie brackets, we have

$$\begin{aligned} &\varrho_B \circ (\nabla \circ [X, Y]_A - [\nabla \circ X, \nabla \circ Y]_B) \\ &= (\varrho_B \circ \nabla) \circ [X, Y]_A - [(\varrho_B \circ \nabla) \circ X, (\varrho_B \circ \nabla) \circ Y]_{TM} \\ &= \varrho_A \circ [X, Y]_A - [\varrho_A \circ X, \varrho_A \circ Y]_{TM} \\ &= \varrho_A \circ [X, Y]_A - \varrho_A \circ [X, Y]_A \\ &= 0. \end{aligned}$$

■

Definition 2.1.4. Let $(A, \varrho_A, [\cdot,\cdot]_A)$ and $(B, \varrho_B, [\cdot,\cdot]_B)$ be Lie algebroids over a manifold M. By the *curvature* of an A-connection $\nabla : \Gamma(A) \longrightarrow \Gamma(B)$ in B we mean the following skew-symmetric 2-form

$$R^\nabla \in \Gamma(\textstyle\bigwedge^2 A^* \otimes B)$$

given by

$$R^\nabla_{X,Y} = \nabla \circ [X, Y]_A - [\nabla \circ X, \nabla \circ Y]_B$$

for $X, Y \in \Gamma(A)$.

Remark 2.1.3. Observe that if $(A, \varrho_A, [\cdot,\cdot]_A)$ is a Lie algebroid over M, $(B, \varrho_B, [\cdot,\cdot]_B)$ is a regular Lie algebroid over M, $\nabla : \Gamma(A) \longrightarrow \Gamma(B)$ is an A-connection in B, then from Lemma 2.1.1 it follows that

$$R^{\nabla} \in \Gamma(\textstyle\bigwedge^2 A^* \otimes \ker \varrho_B).$$

Definition 2.1.5. Let $(A, \varrho_A, [\cdot,\cdot]_A)$ and let $(B, \varrho_B, [\cdot,\cdot]_B)$ be Lie algebroids over the same manifold M. We say that an A-connection $\nabla : \Gamma(A) \longrightarrow \Gamma(B)$ in B is *flat* if $R^{\nabla} = 0$.

Hence, the following obvious characterization of flat connections in the language of commutative diagrams:

Theorem 2.1.1. *Let $(A, \varrho_A, [\cdot,\cdot]_A)$ and $(B, \varrho_B, [\cdot,\cdot]_B)$ be Lie algebroids over a manifold M. Then, a homomorphism of $C^\infty(M)$-modules $\nabla : \Gamma(A) \longrightarrow \Gamma(B)$ is a flat A-connection in B if and only if the diagrams*

$$\begin{tikzcd}
\Gamma(A) \arrow[rr, "\nabla"] \arrow[dr, "\mathrm{Sec}\,\varrho_A"'] & & \Gamma(A) \arrow[dl, "\mathrm{Sec}\,\varrho_B"] \\
& \Gamma(TM) &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(A) \arrow[r, "\nabla"] & \Gamma(B) \\
\Gamma(A)\times\Gamma(A) \arrow[u, "{[\cdot,\cdot]_A}"] \arrow[r, "\nabla\times\nabla"'] & \Gamma(B)\times\Gamma(B) \arrow[u, "{[\cdot,\cdot]_B}"']
\end{tikzcd}$$

are commutative.

Example 2.1.1. Every homomorphism $H : A \longrightarrow B$ of Lie algebroids (both over the same manifold M) is a flat A-connection in B.

Remark 2.1.4. Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a Lie algebroid over M and let E be a vector bundle over M. Since the Lie algebroid $A(E)$ of the vector bundle E is transitive (in particular $A(E)$ is regular) with the Atiyah sequence

$$0 \longrightarrow \mathrm{End}(E) \longrightarrow A(E) \xrightarrow{\varrho_{A(E)}} TM \longrightarrow 0,$$

we can consider the curvature of an A-connection ∇ in $A(E)$ as the 2-form

$$R^{\nabla} \in \Gamma(\textstyle\bigwedge^2 A^* \otimes \mathrm{End}(E))$$

given by

$$R^{\nabla}_{X,Y}(u) = \nabla_X(\nabla_Y u) - \nabla_X(\nabla_Y u) - \nabla_{[X,Y]_A} u$$

for $X, Y \in \Gamma(A)$, $u \in \Gamma(E)$.

The so-called representations in a given vector bundle, which are in fact some flat connections (cf. [61], [50]) are important for determining the characteristic classes.

Definition 2.1.6. Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a Lie algebroid over M and let E be a vector bundle over M. By a *representation of* A in E we mean any homomorphism of Lie algebroids

$$H : A \longrightarrow A(E).$$

Example 2.1.2. (Adjoint representation) Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a regular Lie algebroid over M and $\mathbf{g} = \ker \varrho_A$. The homomorphism of vector bundles

$$\mathrm{ad}_A : A \longrightarrow A(\mathbf{g})$$

defined by

$$\mathrm{ad}_A(X)\xi = [X, \xi]_A\,, \quad X \in \Gamma(A),\ \xi \in \Gamma(\mathbf{g})$$

is an A-connection in $A(\mathbf{g})$ called the *adjoint representation* of A.

Example 2.1.3. (Trivial representation) Given an arbitrary Lie algebroid $(A, \varrho_A, [\cdot,\cdot]_A)$ one can observe that $\varrho_A : A \longrightarrow TM$ is a flat A-connection in TM because at the level of sections, the anchor can be treated as a mapping $\mathrm{Sec}\,\varrho_A : \Gamma(A) \longrightarrow \mathcal{CDO}(M \times \mathbb{R})$.

Example 2.1.4. (Contragredient representation) Let

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

be an A-representation in E where $(A, \varrho_A, [\cdot,\cdot]_A)$ is a Lie algebroid over M and E is a vector bundle over M. Then,

$$\nabla^\natural : \Gamma(A) \longrightarrow \mathcal{CDO}(E^*)$$

given by

$$\left\langle \nabla^\natural_X (u^*)\,, u \right\rangle = (\varrho_A \circ X)\,(\langle u^*, u\rangle) - \langle u^*, \nabla_X u\rangle$$

for $X \in \Gamma(A)$, $u \in \Gamma(E)$, $u^* \in \Gamma(E^*)$, is an A-representation in the dual bundle E^*. The representation $\nabla^\natural$ is called the *contragredient* to ∇.

Example 2.1.5. (Induced representation in the bundle of homomorphism) Let $\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$ be an A-connection in $A(E)$ where $(A, \varrho_A, [\cdot,\cdot]_A)$ is a Lie algebroid over M and E is a vector bundle over M. Set in the A-module $\mathrm{Hom}^n(E;\mathbb{R})$,

$$\mathrm{Hom}^n(\nabla) : \Gamma(A) \longrightarrow \mathcal{CDO}(\mathrm{Hom}^n(E;\mathbb{R})),$$

by

$$\begin{aligned} &\left(\mathrm{Hom}^n(\nabla)_X(\Phi)\right)(u_1,\ldots,u_n) \\ &= (\varrho_A \circ X)(\Phi(u_1,\ldots,u_n)) - \sum_{i=1}^{n} \Phi(u_1,\ldots,u_{i-1},\nabla_X u_i,\ldots,u_n) \end{aligned}$$

for $n \in \mathbb{N}$, $\Phi \in \mathrm{Hom}^n(A;\mathbb{R})$, $X \in \Gamma(A)$, $u_1,\ldots,u_n \in \Gamma(E)$. For $n = 0$ we set

$$\mathrm{Hom}^0(\nabla) = \mathrm{Sec}\,\varrho_A : \Gamma(A) \to \mathcal{CDO}(M \times \mathbb{R}).$$

The map $\mathrm{Hom}(\nabla) : \Gamma(A) \to \mathcal{CDO}(\mathrm{Hom}(E;\mathbb{R}))$ defined by

$$\mathrm{Hom}(\nabla)_X = \bigoplus_{n\geq 0} \mathrm{Hom}^n(\nabla)_X$$

for $X \in \Gamma(A)$ is an A-connection in $\mathrm{Hom}(E;\mathbb{R}) = \bigoplus_{n\geq 0} \mathrm{Hom}^n(E;\mathbb{R})$.

Lemma 2.1.2. *Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a Lie algebroid over M and E is a vector bundle over M. If $\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$ is a flat A-connection in $A(E)$, then the connection $\mathrm{Hom}(\nabla)$ is flat.*

Proof. Let ∇ be a flat A-connection in $A(E)$, $n \in \mathbb{N}$, $\Phi \in \mathrm{Hom}^n(A;\mathbb{R})$, $X \in \Gamma(A)$, $u_1,\ldots,u_n \in \Gamma(E)$. Let us denote by $\overline{\nabla}$ the connection $\mathrm{Hom}(\nabla)$. Then,

$$\begin{aligned} &\left(\left[\overline{\nabla}_X, \overline{\nabla}_Y\right](\Phi)\right)(u_1,\ldots,u_n) \\ &= \left(\left(\overline{\nabla}_X \circ \overline{\nabla}_Y - \overline{\nabla}_Y \circ \overline{\nabla}_X\right)(\Phi)\right)(u_1,\ldots,u_n) \\ &= (\varrho_A \circ X)\left((\overline{\nabla}_Y \Phi)(u_1,\ldots,u_n)\right) - \sum_{i=1}^{n} (\overline{\nabla}_Y \Phi)(u_1,\ldots,\nabla_X u_i,\ldots,u_n) \\ &\quad -(\varrho_A \circ Y)\left((\overline{\nabla}_X \Phi)(u_1,\ldots,u_n)\right) + \sum_{i=1}^{n} (\overline{\nabla}_X \Phi)(u_1,\ldots,\nabla_Y u_i,\ldots,u_n) \end{aligned}$$

$$
\begin{aligned}
&= (\varrho_A \circ X)((\varrho_A \circ Y)((\overline{\nabla}_X \Phi)(u_1, \ldots, u_n))) \\
&\quad -(\varrho_A \circ X)\left(\sum_{i=1}^{n} \Phi(u_1, \ldots, u_{i-1}, \nabla_Y u_i, \ldots, u_n)\right) \\
&\quad -\sum_{i=1}^{n} (\varrho_A \circ Y)(\Phi(u_1, \ldots, u_{i-1}, \nabla_X u_i, \ldots, u_n)) \\
&\quad +\sum_{j<i} \Phi\left(u_1, \ldots, u_{j-1}, \nabla_Y u_j, \ldots, u_{i-1}, \nabla_X u_i, \ldots, u_n\right) \\
&\quad +\sum_{i=1}^{n} \Phi(u_1, \ldots, u_{i-1}, \nabla_Y(\nabla_X u_i), \ldots, u_n) \\
&\quad +\sum_{i<j} \Phi(u_1, \ldots, u_{i-1}, \nabla_X u_i, \ldots, u_{j-1}, \nabla_X u_i, \ldots, u_n) \\
&\quad -(\varrho_A \circ Y)((\varrho_A \circ X)(\Phi(u_1, \ldots, u_n))) \\
&\quad +(\varrho_A \circ Y)\left(\sum_{i=1}^{n} \Phi(u_1, \ldots, u_{i-1}, \nabla_X u_i, \ldots, u_n)\right) \\
&\quad +\sum_{i=1}^{n} (\varrho_A \circ X)(\Phi(u_1, \ldots, u_{i-1}, \nabla_Y u_i, \ldots, u_n)) \\
&\quad -\sum_{j<i} \Phi\left(u_1, \ldots, u_{j-1}, \nabla_X u_j, \ldots, u_{i-1}, \nabla_Y u_i, \ldots, u_n\right) \\
&\quad -\sum_{i=1}^{n} \Phi(u_1, \ldots, u_{i-1}, \nabla_X(\nabla_Y u_i), \ldots, u_n) \\
&\quad -\sum_{i<j} \Phi\left(u_1, \ldots, u_{i-1}, \nabla_Y u_i, \ldots, u_{j-1}, \nabla_Y u_j, \ldots, u_n\right) \\
&= [(\varrho_A \circ X, \varrho_A \circ Y]_{TM}\left(\Phi(u_1, \ldots, u_n)\right) \\
&\quad -\sum_{i=1}^{n} \Phi(u_1, \ldots, u_{i-1}, (\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X) u_i, \ldots, u_n) \\
&= (\varrho_A \circ [X, Y])(\Phi(u_1, \ldots, u_n)) - \sum_{i=1}^{n} \Phi(u_1, \ldots, [\nabla_X, \nabla_Y] u_i, \ldots, u_n) \\
&= (\varrho_A \circ [X, Y])(\Phi(u_1, \ldots, u_n)) - \sum_{i=1}^{n} \Phi(u_1, \ldots, \nabla_{[X,Y]} u_i, \ldots, u_n) \\
&= (\overline{\nabla}_{[X,Y]} \Phi)(u_1, \ldots, u_n).
\end{aligned}
$$

Therefore, $\overline{\nabla} = \mathrm{Hom}(\nabla)$ is a flat A-connection in $\mathrm{Hom}\,(E; \mathbb{R})$ which is a consequence of the flatness of ∇. ■

Example 2.1.6. Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a regular Lie algebroid over M and $\mathbf{g} = \ker \varrho_A$. The adjoint representation $\mathrm{ad}_A : A \longrightarrow A(\mathbf{g})$ is a flat A-connection. Now Lemma 2.1.2 shows that $\mathrm{Hom}(\mathrm{ad}_A)$ is flat. Consequently, $\mathrm{Hom}(\mathrm{ad}_A)$ is a representation of A in $\mathrm{Hom}(\mathbf{g};\mathbb{R})$.

Theorem 2.1.2. *Let $(A, \varrho_A, [\cdot,\cdot]_A)$, $(B, \varrho_B, [\cdot,\cdot]_B)$, and $(C, \varrho_C, [\cdot,\cdot]_C)$ be Lie algebroids, all over the same manifold M. If*

$$0 \longrightarrow A \xrightarrow{F} B \xrightarrow{G} C \longrightarrow 0$$

is an exact sequences of Lie algebroids, then any of its splitting $H : C \longrightarrow B$ is a C-connection in B.

Proof. Let $H : C \longrightarrow B$ be a homomorphism of vector bundle such that $G \circ H = \mathrm{id}_C$. Since G commutes with anchors, i.e., $\varrho_C \circ G = \varrho_B$, we obtain

$$\varrho_B \circ H = (\varrho_C \circ G) \circ H = \varrho_C \circ (G \circ H) = \varrho_C \circ \mathrm{id}_C = \varrho_C.$$

■

Corollary 2.1.1. *Let $(A, \varrho_A, [\cdot,\cdot]_A)$ be a regular Lie algebroid over M. Then, any splitting $H : \mathrm{Im}\,\varrho_A \longrightarrow A$ of the Atiyah sequence*

$$0 \longrightarrow \ker \varrho_A \longrightarrow A \xrightarrow{\varrho_A} \mathrm{Im}\,\varrho_A \longrightarrow 0$$

is a connection in A.

Definition 2.1.7. The *torsion* of an A-connection ∇ in A is the tensor $T^\nabla \in \Gamma(\bigwedge^2 A^* \otimes A)$ defined by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for $X, Y \in \Gamma(A)$.

Definition 2.1.8. We say that an A-connection in A is *torsion-free* if its torsion equals zero.

In the following sections we consider the torsion-free connections related to Lie algebroids equipped with metric structures.

2.2 Exterior derivative operator

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M and let E be a vector bundle over M.

Definition 2.2.1. Let $X \in \Gamma(A)$. The *substitution operator*

$$i_X : \Gamma(\bigotimes^k A^* \otimes E) \longrightarrow \Gamma(\bigotimes^{k-1} A^* \otimes E)$$

is defined by

$$(i_X\zeta)(X_1, \ldots, X_{k-1}) = \zeta(X, X_1, \ldots, X_{k-1})$$

for $\zeta \in \Gamma(\bigotimes^k A^* \otimes E)$, $X_1, \ldots, X_{k-1} \in \Gamma(A)$.

Let

$$\nabla^E : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

be an A-connection in $A(E)$.

Definition 2.2.2. The *Lie derivative*

$$\mathcal{L}_X^{\nabla^E} : \Gamma(\bigotimes^k A^* \otimes E) \longrightarrow \Gamma(\bigotimes^k A^* \otimes E)$$

for $X \in \Gamma(A)$ is defined by

$$(\mathcal{L}_X^{\nabla^E}\Omega)(X_1, \ldots, X_k) = \nabla^E_X(\Omega(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \Omega(X_1, \ldots, [X, X_i], \ldots, X_k)$$

for $\Omega \in \Gamma(\bigotimes^k A^* \otimes E)$, $X_1, \ldots, X_k \in \Gamma(A)$.

Remark 2.2.1. Observe that $\mathcal{L}_X^{\nabla^E}(\eta) \in \Gamma(\bigwedge A^* \otimes E)$ if $\eta \in \Gamma(\bigwedge A^* \otimes E)$.

Moreover, let

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(A)$$

be an A-connection in $A(A)$. We define the A-connection $\bar{\nabla}$ in the dual bundle in a classical way by the following formula

$$(\bar{\nabla}_X\omega)Y = (\varrho_A \circ X)(\omega(Y)) - \omega(\nabla_X Y)$$

for $\omega \in \Gamma(A^*)$, $X, Y \in \Gamma(A)$ (cf. contragredient representation in Example 2.1.4). Next, we take the tensor product of connections ∇^E and

∇; by the Leibniz rule, we extend this connection to the A-connection in the whole tensor bundle $\bigotimes A^* \otimes E$, which will also be denoted by ∇. Then, for $\zeta \in \Gamma(\bigotimes^k A^* \otimes E)$, $X, X_1, \ldots, X_k \in \Gamma(A)$, we have

$$(\nabla_X \zeta)(X_1, \ldots, X_k) = \nabla^E_X(\zeta(X_1, \ldots, X_k)) - \sum_{j=1}^{k} \zeta(X_1, \ldots, \nabla_X X_j, \ldots, X_k).$$

Definition 2.2.3. We define the connection operator

$$\nabla : \Gamma(\bigotimes^k A^* \otimes E) \longrightarrow \Gamma(\bigotimes^{k+1} A^* \otimes E)$$

by

$$(\nabla\zeta)(X_1, X_2 \ldots, X_{k+1}) = (\nabla_{X_1}\zeta)(X_2, \ldots, X_{k+1})$$

for $\zeta \in \Gamma(\bigotimes^k A^* \otimes E)$, $X_1, \ldots, X_{k+1} \in \Gamma(A)$.

We recall that the exterior derivative operator

$$d^{\nabla^E} : \Gamma(\bigwedge^k A^* \otimes E) \longrightarrow \Gamma(\bigwedge^{k+1} A^* \otimes E)$$

on the Lie algebroid $(A, \varrho_A, [\cdot,\cdot])$ is defined by

$$\begin{aligned} &\left(d^{\nabla^E}\eta\right)(X_1, \ldots, X_{k+1}) \\ &= \sum_{j=1}^{k+1} (-1)^{j+1}\, \nabla^E_{X_j}\Big(\eta(X_1, \ldots \widehat{X}_j \ldots, X_{k+1})\Big) \\ &\quad + \sum_{i<j} (-1)^{i+j}\, \eta\Big([X_i, X_j], X_1, \ldots \widehat{X}_i \ldots \widehat{X}_j \ldots, X_{k+1}\Big) \end{aligned} \tag{2.2}$$

for $\eta \in \Gamma(\bigwedge^k A^* \otimes E)$, $X_1, \ldots, X_{k+1} \in \Gamma(A)$.

If ∇ is torsion-free A-connection in A, then d^{∇^E} can be written as the alternation of the operator ∇ (cf. [8], [12], [40]), i.e.,

$$d^{\nabla^E} = (k+1) \cdot (\operatorname{Alt} \circ \nabla) \quad \text{on} \quad \Gamma(\bigwedge^k A^*),$$

where Alt is the *alternator* given by

$$(\operatorname{Alt} \zeta)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn} \sigma \; \zeta(X_{\sigma(1)}, \ldots, X_{\sigma(k)})$$

for $\zeta \in \Gamma(\bigotimes^k A^* \otimes E)$. Equivalently,

$$\left(d^{\nabla^E}\eta\right)(X_1,\ldots,X_{k+1}) = \sum_{j=1}^{k+1}(-1)^{j+1}\left(\nabla^E_{X_j}\eta\right)\left(X_1,\ldots\widehat{X}_j\ldots,X_{k+1}\right)$$

for $\eta \in \Gamma(\bigwedge^k A^* \otimes E)$, $X_1,\ldots,X_{k+1} \in \Gamma(A)$.

Here, we recall the classical Cartan's formulas:

Lemma 2.2.1. *For any* $X, Y \in \Gamma(A)$ *we have*

(*a*) $\mathcal{L}^{\nabla^E}_X = i_X d^{\nabla^E} + d^{\nabla^E} i_Y$,

(*b*) $\mathcal{L}^{\nabla^E}_X i_Y - i_Y \mathcal{L}^{\nabla^E}_X = i_{[X,Y]}$.

2.3 The Bianchi identity

Let $(A, \varrho_A, [\cdot,\cdot])$ be a Lie algebroid over a manifold M and let E be a vector bundle over M. Moreover, let

$$\nabla^E : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

be an A-connection.

Definition 2.3.1. We define $\mathscr{A}^k(A,E)$ to be $\Gamma(\bigwedge^k A^* \otimes E)$.

Recall that

$$d^{\nabla^E} \circ d^{\nabla^E} = \mathcal{R}^{\nabla^E} \,\overline{\wedge}\, (\cdot), \tag{2.3}$$

where

$$(\mathcal{R}^{\nabla^E} \overline{\wedge} \eta)(X_1,\ldots,X_{k+2}) = \sum_{i<j}(-1)^{i+j}\,\mathcal{R}^{\nabla^E}_{X_i,X_j}(\eta(X_1,\ldots\widehat{i}\ldots\widehat{j}\ldots,X_{k+2}))$$

for $\eta \in \mathscr{A}^k(A,E)$, $X_1,\ldots,X_{k+2} \in \Gamma(A)$ (e.g. cf. [42], [83]).

Now, let

$$\nabla = \nabla^A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$$

be a connection in A. Then we can consider the differential operator $d^\nabla : \mathscr{A}^k(A,A) \longrightarrow \mathscr{A}^{k+1}(A,A)$ given by

$$\begin{aligned}(d^\nabla\eta)(X_1,\ldots,X_{k+1}) &= \sum_{j=1}^{k+1}(-1)^{j+1}\nabla_{X_j}\left(\eta(X_1,\ldots\widehat{X}_j\ldots,X_{k+1})\right)\\ &\quad+\sum_{i<j}(-1)^{i+j}\,\eta\left([X_i,X_j],X_1,\ldots\widehat{X}_i\ldots\widehat{X}_j\ldots,X_{k+1}\right)\end{aligned}$$

for $\eta \in \mathscr{A}^k(A,A)$, $X_1,\dots,X_{k+1} \in \Gamma(A)$.

Observe that

$$\begin{aligned}(d^\nabla(\mathrm{id}_A))(X,Y) &= \nabla_X(\mathrm{id}_A(Y)) - \nabla_Y(\mathrm{id}_A(X)) - \mathrm{id}_A([X,Y]) \\ &= \nabla_X Y - \nabla_Y X - [X,Y] \\ &= T^\nabla(X,Y)\end{aligned}$$

for any $X,Y \in \Gamma(A)$. Consequently, the torsion of ∇ is the differential of the identity:

Lemma 2.3.1. $T^\nabla = d^\nabla(\mathrm{id}_A)$.

Combining (2.3) with the equality in Lemma 2.3.1 we get the first Bianchi identity.

Theorem 2.3.1. (*The first Bianchi identity.*)

$$d^\nabla(T^\nabla) = \mathcal{R}^\nabla \,\overline{\wedge}\, \mathrm{id}_A\,.$$

The first Bianchi identity can be written directly as in the following corollary.

Corollary 2.3.1. (*The first Bianchi identity.*) *For any* $X,Y,Z \in \Gamma(A)$, *we have*

$$\begin{aligned}&\mathcal{R}^\nabla_{X,Y}Z - \mathcal{R}^\nabla_{X,Z}Y + \mathcal{R}^\nabla_{Y,Z}X \\ &= \nabla_X(T^\nabla(Y,Z)) - \nabla_Y(T^\nabla(X,Z)) + \nabla_Z(T^\nabla(X,Y)) \\ &\quad -T^\nabla([X,Y],Z) + T^\nabla([X,Z],Y) - T^\nabla([Y,Z],X).\end{aligned}$$

Corollary 2.3.2. *If* ∇ *is torsion-free, for any* $X,Y,Z \in \Gamma(A)$ *we have*

$$\mathcal{R}^\nabla_{X,Y}Z + \mathcal{R}^\nabla_{Z,X}Y + \mathcal{R}^\nabla_{Y,Z}X = 0.$$

2.4 The Jacobi identity as the Bianchi identity

Let A be a vector bundle equipped with an anchor $\varrho_A : A \longrightarrow TM$. Moreover, let

$$\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$$

be a connection in a vector bundle A, i.e., ∇ is $\mathbb{R}$-bilinear and satisfies the following properties:

$$\nabla_{f\cdot X}(Y) = f \cdot \nabla_X(Y),$$

$$\nabla_X(f \cdot Y) = f \cdot \nabla_X(Y) + (\varrho_A \circ X)(f) \cdot Y$$

for any $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$.

Define

$$[\![X, Y]\!] = \nabla_X Y - \nabla_Y X$$

for any $X, Y \in \Gamma(A)$.

Observe that $[\![\cdot,\cdot]\!] : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ is bilinear over $\mathbb{R}$ and skew-symmetric. Moreover,

$$\begin{aligned}[\![X, fY]\!] &= \nabla_X(fY) - \nabla_{fY} X \\ &= f\nabla_X(Y) + (\varrho_A \circ X)(f) \cdot Y - f\nabla_Y X \\ &= f[\![X, Y]\!] + (\varrho_A \circ X)(f) \cdot Y\end{aligned}$$

for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$. It follows that $[\![\cdot,\cdot]\!]$ introduces the structure skew-symmetric algebroid into A. Theorem 2.4.1 presents the relationship between the Jacobi identity of $[\![\cdot,\cdot]\!]$ and the Bianchi identity for ∇ in the skew-symmetric algebroid $(A, \varrho_A, [\![\cdot,\cdot]\!])$.

Theorem 2.4.1. *For any $X, Y, Z \in \Gamma(A)$ we have*

$$\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(X, Y, Z) = -\sum_{\substack{cycl \\ X,Y,Z}} R^\nabla_{X,Y} Z.$$

Proof. Let $X, Y, Z \in \Gamma(A)$. Since

$$\begin{aligned}[\![[\![X, Y]\!], Z]\!] &= [\![\nabla_X Y - \nabla_Y X, Z]\!] \\ &= \nabla_{\nabla_X Y - \nabla_Y X}(Z) - \nabla_Z(\nabla_X Y - \nabla_Y X) \\ &= \nabla_Z(\nabla_Y X) - \nabla_Z(\nabla_X Y) + \nabla_{[\![X,Y]\!]}(Z),\end{aligned}$$

we have

$$\begin{aligned}\mathrm{Jac}_{[\![\cdot,\cdot]\!]}(X, Y, Z) &= [\![[\![X, Y]\!], Z]\!] + [\![[\![Z, X]\!], Y]\!] + [\![[\![Y, Z]\!], X]\!] \\ &= \nabla_Z(\nabla_Y X) - \nabla_Z(\nabla_X Y) + \nabla_{[\![X,Y]\!]}(Z) \\ &\quad + \nabla_Y(\nabla_X Z) - \nabla_Y(\nabla_Z X) + \nabla_{[\![Z,X]\!]}(Y) \\ &\quad + \nabla_X(\nabla_Z Y) - \nabla_X(\nabla_Y Z) + \nabla_{[\![Y,Z]\!]}(X) \\ &= -\nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) + \nabla_{[\![X,Y]\!]}(Z) \\ &\quad - \nabla_Z(\nabla_X Y) + \nabla_X(\nabla_Z Y) + \nabla_{[\![Z,X]\!]}(Y) \\ &\quad - \nabla_Y(\nabla_Z X) + \nabla_Z(\nabla_Y X) + \nabla_{[\![Y,Z]\!]}(X) \\ &= -R^\nabla_{X,Y} Z - R^\nabla_{Z,X} Y - R^\nabla_{Y,Z} X.\end{aligned}$$

■

We show that the family of examples of Lie algebroids on the cotangent bundle considered in Section 1.10 and introduced in [22] come from some flat linear connections in skew-symmetric algebroids defined by these connections. We notice that the defined Lie brackets on 1-differential forms are in fact skew-symmetric parts of some torsion-free and flat connections. Moreover, we obtain a slight generalization of considered relations between vector fields.

Let M be a differential manifold and let $X, Y \in \varGamma(TM)$ be vector fields on M such that

$$[X, Y] = fY \text{ for some } f \in C^\infty(M).$$

Define

$$\varrho_{X,Y} : T^*M \longrightarrow TM$$

such that

$$\varrho_{X,Y}(\alpha) = -\alpha(Y)X \text{ for } \alpha \in \varGamma(T^*M).$$

Next, define ∇ by

$$\nabla : \varGamma(T^*M) \times \varGamma(T^*M) \longrightarrow \varGamma(T^*M),$$

$$\nabla_\alpha \beta = -\alpha(Y) \cdot \mathcal{L}_X \beta$$

for $\alpha, \beta \in \varGamma(T^*M)$.

Lemma 2.4.1. *For $g \in C^\infty(M)$, $X \in \varGamma(TM)$, $\omega \in \varGamma(T^*M)$, we have*

(a) $\mathcal{L}_{g\cdot X}\omega = g \cdot \mathcal{L}_X \omega + \omega(X) \cdot dg$,
(b) $\mathcal{L}_X(g \cdot \omega) = g \cdot \mathcal{L}_X \omega + X(g) \cdot \omega$.

Lemma 2.4.2. *∇ is a linear connection in T^*M.*

Proof. It is evident that ∇ is bilinear over $\mathbb{R}$ and that $\nabla_{g\alpha}\beta = g\nabla_\alpha\beta$ for $\alpha, \beta \in \varGamma(T^*M)$, $g \in C^\infty(M)$. The property (b) in Lemma 2.4.1 implies that

$$\begin{aligned}
\nabla_\alpha(g\beta) &= -\alpha(Y) \cdot \mathcal{L}_X(g\beta) \\
&= -\alpha(Y) \cdot (g\mathcal{L}_X\beta + X(g) \cdot \beta) \\
&= g\left(-\alpha(Y) \cdot \mathcal{L}_X\beta\right) + \left(-\alpha(Y) \cdot X\right)(g) \cdot \beta \\
&= g\nabla_\alpha\beta + \varrho_{X,Y}(\alpha)(g) \cdot \beta
\end{aligned}$$

for any $\alpha, \beta \in \varGamma(T^*M)$, $g \in C^\infty(M)$. ∎

Recall the bracket $[\cdot,\cdot]_{X,Y} : \Gamma(T^*M) \times \Gamma(T^*M) \longrightarrow \Gamma(T^*M)$ from Section 1.10:

$$[\alpha,\beta]_{X,Y} = \beta(Y)\mathcal{L}_X\alpha - \alpha(Y)\mathcal{L}_X\beta$$

for $\alpha,\beta \in \Gamma(T^*M)$. Observe that $[\cdot,\cdot]_{X,Y}$ can be written as follows

$$[\alpha,\beta]_{X,Y} = \nabla_\alpha\beta - \nabla_\beta\alpha$$

for $\alpha,\beta \in \Gamma(T^*M)$. It follows immediately that $[\cdot,\cdot]_{X,Y}$ specifies a skew-symmetric $\mathbb{R}$-bilinear bracket. We show that the Jacobi identity is a consequence of the flatness the considered connection.

Lemma 2.4.3. ∇ *is flat with respect to the bracket* $[\cdot,\cdot]_{X,Y}$.

Proof. Let $\alpha,\beta,\gamma \in \Gamma(T^*M)$. Then

$$\begin{aligned}(I) &\triangleq \nabla_\alpha\left(\nabla_\beta\gamma\right)\\ &= \nabla_\alpha\left(-\beta(Y)\cdot\mathcal{L}_X\gamma\right)\\ &= -\beta(Y)\cdot\nabla_\alpha\left(\mathcal{L}_X\gamma\right) + \nabla_\alpha\left(-\beta(Y)\cdot\mathcal{L}_X\gamma\right)\\ &= -\beta(Y)\cdot\nabla_\alpha\left(\mathcal{L}_X\gamma\right) + \left(\varrho_{X,Y}\circ\alpha\right)\left(-\beta(Y)\right)\cdot\mathcal{L}_X\gamma\\ &= \alpha(Y)\cdot\beta(Y)\cdot\mathcal{L}_X\left(\mathcal{L}_X\gamma\right) + \alpha(Y)\cdot X\left(\beta(Y)\right)\cdot\mathcal{L}_X\gamma.\end{aligned}$$

Likewise,

$$(II) \triangleq \nabla_\beta\left(\nabla_\alpha\gamma\right) = \alpha(Y)\cdot\beta(Y)\cdot\mathcal{L}_X\left(\mathcal{L}_X\gamma\right) + \beta(Y)\cdot X\left(\alpha(Y)\right)\cdot\mathcal{L}_X\gamma.$$

Subtracting (II) from (I), we conclude that

$$\nabla_\alpha\left(\nabla_\beta\gamma\right) - \nabla_\beta\left(\nabla_\alpha\gamma\right) = \left(\alpha(Y)\cdot X\left(\beta(Y)\right) - \beta(Y)\cdot X\left(\alpha(Y)\right)\right)\cdot\mathcal{L}_X\gamma.$$

Furthermore, observe that

$$\begin{aligned}(III) &\triangleq \nabla_{[\alpha,\beta]_{X,Y}}\gamma = \nabla_{\beta(Y)\cdot\mathcal{L}_X\alpha-\alpha(Y)\cdot\mathcal{L}_X\beta}\gamma\\ &= \beta(Y)\cdot\nabla_{\mathcal{L}_X\alpha}\gamma - \alpha(Y)\cdot\nabla_{\mathcal{L}_X\beta}\gamma\\ &= -\beta(Y)\cdot(\mathcal{L}_X\alpha)(Y)\cdot\mathcal{L}_X\gamma + \alpha(Y)\cdot(\mathcal{L}_X\beta)(Y)\cdot\mathcal{L}_X\gamma\\ &= \left(-\beta(Y)\cdot(\mathcal{L}_X\alpha)(Y) + \alpha(Y)\cdot(\mathcal{L}_X\beta)(Y)\right)\cdot\mathcal{L}_X\gamma\\ &= \beta(Y)\cdot\left(-X(\alpha(Y)) + \alpha([X,Y])\right)\cdot\mathcal{L}_X\gamma\\ &\quad +\alpha(Y)\cdot\left(X(\beta(Y)) - \beta([X,Y])\right)\cdot\mathcal{L}_X\gamma.\end{aligned}$$

The result is

$$\begin{aligned}
R^{\nabla}_{\alpha,\beta}\gamma &= (I) - (II) - (III) \\
&= (\alpha(Y) \cdot X(\beta(Y)) - \beta(Y) \cdot X(\alpha(Y))) \cdot \mathcal{L}_X\gamma \\
&\quad + \beta(Y) \cdot (X(\alpha(Y)) - \alpha([X,Y])) \cdot \mathcal{L}_X\gamma \\
&\quad - \alpha(Y) \cdot (X(\beta(Y)) + \beta([X,Y])) \cdot \mathcal{L}_X\gamma \\
&= (\alpha(Y) \cdot \beta([X,Y]) - \beta(Y) \cdot \alpha([X,Y])) \cdot \mathcal{L}_X\gamma.
\end{aligned}$$

Since $[X,Y] = fY$, we conclude that

$$\begin{aligned}
R^{\nabla}_{\alpha,\beta}\gamma &= (\alpha(Y) \cdot \beta(fY) - \beta(Y) \cdot \alpha(fY)) \cdot \mathcal{L}_X\gamma \\
&= f\,(\alpha(Y) \cdot \beta(Y) - \beta(Y) \cdot \alpha(Y)) \cdot \mathcal{L}_X\gamma \\
&= 0.
\end{aligned}$$

■

Since ∇ is a torsion-free and flat connection, by Theorem 2.4.1 the Jacobi identity follows from the Bianchi identity:

$$\mathrm{Jac}_{[\cdot,\cdot]_{X,Y}}(\alpha,\beta,\gamma) = -\left(R^{\nabla}_{\alpha,\beta}\gamma + R^{\nabla}_{\gamma,\alpha}\beta + R^{\nabla}_{\beta,\gamma}\alpha\right) = 0.$$

As mentioned in [22], the starting point can be not only a special Lie algebroid, which is the tangent bundle, but any Lie algebroid. Now take a finite-dimensional real Lie algebra $\mathfrak{g}$. Then in $\mathfrak{g}^*$ for $x, y \in \mathfrak{g}$ satisfying $[x,y] = cy$ for some $c \in \mathbb{R}$, we can consider the connection ∇ in $\mathfrak{g}$ such that

$$\nabla_\alpha\beta = -\alpha(y) \cdot \mathcal{L}_x\beta = \alpha(y) \cdot (\beta \circ \mathrm{ad}(x))$$

for $\alpha, \beta \in \mathfrak{g}^*$, where ad denotes the adjoint representation of the Lie algebra $\mathfrak{g}$. The related Lie bracket for $\alpha, \beta \in \mathfrak{g}^*$ is then given by the formula

$$\begin{aligned}
[\alpha,\beta]_{x,y} &= \nabla_\alpha\beta - \nabla_\beta\alpha \\
&= (\alpha(y) \cdot \beta - \beta(y) \cdot \alpha) \circ \mathrm{ad}(x).
\end{aligned}$$

2.5 Exterior derivative on a Lie algebroid

Let $(A, \varrho_A, [\cdot,\cdot])$ be a Lie algebroid over a manifold M. Consider the anchor of the algebroid A as an A-connection in the trivial vector bundle $M \times \mathbb{R}$. Then

$$d = d^{\mathrm{Sec}\,\varrho_A} : \Gamma(\textstyle\bigwedge^k A^*) \longrightarrow \Gamma(\textstyle\bigwedge^{k+1} A^*)$$

is the *exterior derivative operator on the Lie algebroid* $(A, \varrho_A, [\cdot,\cdot])$, and is given by

$$\begin{aligned}&(d\eta)(X_1,\dots,X_{k+1})\\&=\sum_{j=1}^{k+1}(-1)^{j+1}(\varrho_A\circ X_j)\Big(\eta(X_1,\dots\widehat{X}_j\dots,X_{k+1})\Big)\\&\quad+\sum_{i<j}(-1)^{i+j}\eta([X_i,X_j],X_1,\dots\widehat{X}_i\dots\widehat{X}_j\dots,X_{k+1})\end{aligned}$$

for $\eta\in\Gamma(\bigwedge^k A^*)$, $X_1,\dots,X_{k+1}\in\Gamma(A)$.

Since the bracket $[\cdot,\cdot]$ satisfies the Jacobi identity, i.e., $\mathrm{Jac}_{[\cdot,\cdot]}=0$,

$$d\circ d=0$$

(cf. [42], [66]). For the first time in the context of Lie algebroids, this operator was discussed in [66].

Definition 2.5.1. The cohomology of the chain complex

$$(\bigoplus_{k\geq 0}\Gamma(\bigwedge^k A^*), d)$$

is called the *Lie algebroid cohomology* of A and denoted by $\mathsf{H}^\bullet(A)$.

Lemma 2.5.1. *For any $X\in\Gamma(A)$, we have*

$$\mathrm{ad}^\natural_A(X)=d\circ\iota_X+\iota_X\circ d$$

where $d=d^{\varrho_A}$ is the exterior differential operator in A.

Proof. Let $X,Y\in\Gamma(A)$, $\eta\in\Gamma(A^*)$. Then,

$$\begin{aligned}&\langle(d\circ\iota_X+\iota_X\circ d)\eta,Y\rangle\\&=\langle d(\langle\eta,X\rangle,Y\rangle+\langle(\iota_X(d\eta),Y\rangle\\&=(d(\langle\eta,X\rangle)(Y)+(d\eta)(X,Y)\\&=(\varrho_A\circ Y)(\langle\eta,X\rangle)+(\varrho_A\circ X)(\eta(Y))-(\varrho_A\circ Y)(\eta(X))-\eta([X,Y])\\&=(\varrho_A\circ X)(\langle\eta,Y\rangle)-\langle\eta,\mathrm{ad}_A(X)(Y)\rangle\\&=\Big\langle\mathrm{ad}^\natural_A(X)(\eta),Y\Big\rangle.\end{aligned}$$

■

Similarly, the connection $\mathrm{Hom}^n(\mathrm{ad}_A)$ can be written in the language of the differential operator d and the substitution operator. By Lemma 2.2.1, we have immediately the following corollary.

Corollary 2.5.1. *For any $X\in\Gamma(A)$,*

$$\mathrm{Hom}^n(\mathrm{ad}_A)(X)=d\circ\iota_X+\iota_X\circ d$$

where $d=d^{\varrho_A}$ is the exterior differential operator on A.

2.6 Connections associated with metric structures

2.6.1 Symmetric products and the symmetrized covariant derivative

In this section we discuss symmetric products on skew-symmetric algebroids. We note that linear connections on skew-symmetric algebroids are the source of such objects. Such symmetric products appeared in the expansion of the symmetrized connection, and this is the subject of our considerations. In particular, we remark that symmetric products have similar properties to the exterior derivative operator. We depend on our paper [7]. Some proofs has been supplemented here and presented for the sake of clarity and completeness.

Let $(A, \varrho_A, [\cdot, \cdot])$ be a skew-symmetric algebroid over a manifold M.

Definition 2.6.1. A *symmetric bracket* on the anchored vector bundle (A, ϱ_A) is an $\mathbb{R}$-bilinear symmetric mapping

$$\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$$

satisfying the following Leibniz-kind rule:

$$\langle X : f \cdot Y \rangle = f \cdot \langle X : Y \rangle + (\varrho_A \circ X)(f) \cdot Y$$

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$.

Example 2.6.1. Let $\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ be a connection in A. Then, $\langle \cdot : \cdot \rangle^\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ given by

$$\langle X : Y \rangle^\nabla = \nabla_X Y + \nabla_Y X \tag{2.4}$$

for $X, Y \in \Gamma(A)$, is a symmetric bracket on A.

Definition 2.6.2. Let $\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ be a connection in A. The symmetric bracket on A defined by (2.4) is called *the symmetric product induced by* ∇.

Remark 2.6.1. The symmetric product in the case $A = TM$ was first introduced by Crouch in [20]. However, the symmetric product for Lie algebroids was first considered in the context of control systems by Cortés and Martínez in [15]. Lewis in [58] gives some geometrical interpretation of the symmetric product associated with the geodesically invariant property of a distribution. Namely, we say that a smooth distribution D on a manifold M with an affine connection ∇^{TM} is *geodesically invariant* if for every geodesic $c: I \to M$ satisfying the property $c'(s) \in D_{c(s)}$ for some $s \in I$, we have $c'(s) \in D_{c(s)}$ for every $s \in I$. Lewis proved in [58] that a distribution D on a manifold M equipped with an affine connection ∇^{TM} is geodesically invariant if and only if the symmetric product $\langle \cdot : \cdot \rangle^{\nabla_{TM}}$ induced by ∇^{TM} is closed under D, i.e.,

$$\langle D : D \rangle^{\nabla^{TM}} \subset D.$$

Let us assume that the skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$ is equipped with a symmetric bracket $\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$.

We define $d^s : \Gamma(\bigotimes^k A^*) \longrightarrow \Gamma(\bigotimes^{k+1} A^*)$ on the whole tensor bundle by

$$\begin{aligned}(d^s\Omega)(X_1,\ldots,X_{k+1}) = &\sum_{j=1}^{k+1}(\varrho_A \circ X_j)\Big(\Omega(X_1,\ldots\widehat{X}_j\ldots,X_{k+1})\Big)\\ &-\sum_{i<j}\Omega\Big(X_1,\ldots\widehat{X}_i\ldots,\langle X_i : X_j\rangle,\ldots,X_{k+1}\Big)\end{aligned}$$

for $\Omega \in \Gamma(\bigotimes^k A^*)$, $X_1,\ldots,X_{k+1} \in \Gamma(A)$. We denote the restriction of d^s to the symmetric power bundle $\mathsf{S}(A)$ by the same symbol.

The *symmetric Lie derivative* $\mathcal{L}^s_X : \Gamma(\bigotimes^k A^*) \longrightarrow \Gamma(\bigotimes^k A^*)$ for $X \in \Gamma(A)$ is defined by

$$\begin{aligned}(\mathcal{L}^s_X\Omega)(X_1,\ldots,X_k) = &(\varrho_A \circ X)(\Omega(X_1,\ldots,X_k))\\ &-\sum_{i=1}^{k}\Omega(X_1,\ldots,\langle X : X_i\rangle,\ldots,X_k)\end{aligned}$$

for $\Omega \in \Gamma(\bigotimes^k A^*)$, $X_1,\ldots,X_k \in \Gamma(A)$. Remark that the image $\mathcal{L}^s_X(\zeta)$ of a symmetric tensor ζ is also a symmetric tensor.

By using definitions, one can prove that the symmetric Lie derivative satisfies the following Cartan's identities analogous to these Cartan identities on exterior forms:

Theorem 2.6.1. *For any $X, Y \in \Gamma(A)$ we have:*

(a) $\mathcal{L}^s_X = i_X d^s - d^s i_X$,

(b) $\mathcal{L}^s_X i_Y - i_Y \mathcal{L}^s_X = i_{\langle X:Y\rangle}$.

Proof. Let $X, Y, X_1, \ldots, X_k \in \Gamma(A)$ and $\Omega \in \Gamma(\bigotimes^k A^*)$. Observe that

$$\begin{aligned}
&(i_X d^s \Omega)(X_1, \ldots, X_k) \\
&= (\varrho_A \circ X)(\Omega(X_1, \ldots, X_k)) + \sum_{i=1}^{k} (\varrho_A \circ X_i)(\Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, X_k)) \\
&\quad - \sum_{i=1}^{k} \Omega(X_1, \ldots, \langle X : X_i \rangle, \ldots, X_k) \\
&\quad - \sum_{i<j}^{k} \Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, \langle X_i : X_j \rangle, \ldots, X_k)
\end{aligned}$$

and

$$\begin{aligned}
(d^s i_X \Omega)(X_1, \ldots, X_k) &= \sum_{i=1}^{k} (\varrho_A \circ X_i)(\Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, X_k)) \\
&\quad - \sum_{i<j}^{k} \Omega(X, X_1, \ldots, \widehat{X}_i, \ldots, \langle X_i : X_j \rangle, \ldots, X_k).
\end{aligned}$$

Hence, we obtain (a) in Theorem 2.6.1.

Moreover,

$$\begin{aligned}
&(\mathcal{L}^s_X i_Y \Omega)(X_1, \ldots, X_{k-1}) \\
&= (\varrho_A \circ X)(\Omega(Y, X_1, \ldots, X_{k-1})) - \sum_{i=1}^{k-1} \Omega(Y, X_1, \ldots, \langle X : X_i \rangle, \ldots, X_{k-1})
\end{aligned}$$

and

$$\begin{aligned}
(i_Y \mathcal{L}^s_X \Omega)(X_1, \ldots, X_{k-1}) &= (\varrho_A \circ X)(\Omega(Y, X_1, \ldots, X_{k-1})) \\
&\quad - \Omega(\langle X : Y \rangle, X_1, \ldots, X_{k-1}) - \sum_{i=1}^{k-1} \Omega(Y, X_1, \ldots, \langle X : X_i \rangle, \ldots, X_{k-1}),
\end{aligned}$$

which give immediately (b) in Theorem 2.6.1. ■

Lemma 2.6.1. *For $f \in C^\infty(M)$, $X \in \Gamma(A)$, $\eta \in \Gamma(A^*)$, we have:*

(a) $\mathcal{L}^s_{f \cdot X} \eta = f \cdot \mathcal{L}^s_X \eta - (i_X \eta) \cdot d^s f$,

(b) $\mathcal{L}^s_X (f \cdot \eta) = f \cdot \mathcal{L}^s_X \eta + (\varrho_A \circ X)(f) \cdot \eta$.

When the symmetric bracket comes from a connection, i.e., is the symmetric part of a linear connection $\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$, d^s on symmetric tensors is simply a symmetrization of the connection operator. We have the following theorem.

Theorem 2.6.2. *Let* $\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ *be an* A*-connection and* d^s *is induced by the symmetric product* $\langle X : Y \rangle^{\nabla} = \nabla_X Y + \nabla_Y X$*, i.e.,*

$$(d^s\eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (\varrho_A \circ X_j)\Big(\eta(X_1, \ldots \widehat{X}_j \ldots, X_{k+1})\Big) - \sum_{i<j} \eta\Big(\langle X_i : X_j \rangle^{\nabla}, X_1, \ldots \widehat{X}_i \ldots \widehat{X}_j \ldots, X_{k+1}\Big)$$

for $\eta \in \Gamma(\mathsf{S}^k A^*)$*,* $X_1, \ldots, X_{k+1} \in \Gamma(A)$*. Then*

$$d^s = (k+1) \cdot (\mathrm{Sym} \circ \nabla) : \Gamma(\mathsf{S}^k A^*) \longrightarrow \Gamma(\mathsf{S}^{k+1} A^*),$$

where Sym *is the* symmetrizer *defined by*

$$(\mathrm{Sym}\,\zeta)(X_1, \ldots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \zeta(X_{\sigma(1)}, \ldots, X_{\sigma(k)})$$

for $\zeta \in \Gamma(\bigotimes^k A^*)$*. Equivalently,*

$$(d^s\eta)(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} \big(\nabla_{X_j}\eta\big)(X_1, \ldots \widehat{X}_j \ldots, X_{k+1}) \tag{2.5}$$

for $\eta \in \Gamma(\mathsf{S}^k A^*)$*,* $X_1, \ldots, X_{k+1} \in \Gamma(A)$*.*

Definition 2.6.3. We call d^s the *symmetrized covariant derivative.*

Remark 2.6.2. The mapping d^s in the case of tangent bundles was introduced by Sampson in [76], where a symmetric version of Chern's theorem is proved. The Koszul-type shape of d^s for tangent bundles was first obtained by Heydari, Boroojerdian, and Peyghan in [40], and next under the study of generalized gradients on Lie algebroids in the sense of Stein-Weiss in [12]. Thus, the symmetrized covariant derivative is a symmetric counterpart of the exterior derivative operator, except that the role of the skew-symmetric bracket is taken over by the symmetric product.

The symmetrized covariant derivative was also considered by Mikeš, Rovenski, Stepanov and Tsyganok in the study of the Lichnerowicz-type Laplacian on symmetric tensors [77], [67].

2.6.2 Connections compatible with Riemannian pseudometrics

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M equipped with a pseudo-Riemannian metric $g \in \Gamma(S^2 A^*)$ in the vector bundle A and an A-connection ∇ in A. Let $\langle \cdot : \cdot \rangle^{\nabla}$ be the symmetric product induced by ∇, i.e.,

$$\langle X : Y \rangle^{\nabla} = \nabla_X Y + \nabla_Y X$$

for $X, Y \in \Gamma(A)$, and let d^s be the symmetrized covariant derivative.

Definition 2.6.4. A connection ∇ in is said to be *compatible with the metric* g if $\nabla g = 0$.

The pseudo-Riemannian metric defines two homomorphisms of vector bundles

$$\flat : A \longrightarrow A^*, \qquad \sharp : A^* \longrightarrow A$$

by

$$\flat(X) = i_X g, \qquad g(\sharp(\omega), X) = \omega(X)$$

for $X \in \Gamma(A)$, $\omega \in \Gamma(A^*)$, respectively.

Definition 2.6.5. Let $X \in \Gamma(A)$. We will use the symbol $X^\flat$ to denote the 1-form $i_X g = g(X, \cdot)$, i.e.,

$$X^\flat = i_X g.$$

Definition 2.6.6. We say that ∇ is a *connection with totally skew-symmetric torsion* with respect to a pseudo-Riemannian metric g if the tensor $T^g \in \Gamma(\bigotimes^3 A^*)$ given by

$$T^g(X, Y, Z) = g(T^{\nabla}(X, Y), Z)$$

for $X, Y, Z \in \Gamma(A)$, is a 3-form on A, i.e., $T^g \in \Gamma(\bigwedge^3 A^*)$.

In the following theorem we show the relationship determined by the metric tensor between the connection, its torsion, and the symmetrized covariant derivative of the metric.

Theorem 2.6.3. [7] *Let $X, Z \in \Gamma(A)$. Then,*

$$\begin{aligned} g(\nabla_X X, Z) = g(\sharp(\mathcal{L}_X^{\varrho_A} X^\flat - \tfrac{1}{2} d(g(X,X))), Z) \\ -g(T^\nabla(X,Z), X) \\ +(\nabla g)(Z,X,X) - \tfrac{1}{2}(d^s g)(X,X,Z). \end{aligned} \tag{2.6}$$

Proof. [7] Let $X, Z \in \Gamma(A)$. Observe that

$$\begin{aligned} (d^s g)(X,X,Z) &= (\nabla_X g)(X,Z) + (\nabla_X g)(X,Z) + (\nabla_Z g)(X,X) \\ &= 2(\nabla g)(X,X,Z) + (\nabla g)(Z,X,X). \end{aligned}$$

Therefore,

$$\begin{aligned} &(\nabla g)(Z,X,X) - \tfrac{1}{2}(d^s g)(X,X,Z) \\ &= (\nabla g)(Z,X,X) - \tfrac{1}{2}\left(2(\nabla g)(X,X,Z) + (\nabla g)(Z,X,X)\right) \\ &= \tfrac{1}{2}(\nabla g)(Z,X,X) - (\nabla g)(X,X,Z). \end{aligned}$$

Now observe that

$$\begin{aligned} &\tfrac{1}{2}(\nabla g)(Z,X,X) - (\nabla g)(X,X,Z) \\ &= \tfrac{1}{2}(\nabla_Z g)(X,X) - (\nabla_X g)(X,Z) \\ &= \tfrac{1}{2}\varrho_A(Z)(g(X,X)) - g(\nabla_Z X, X) - \varrho_A(X)(g(X,Z)) \\ &\quad + g(\nabla_X X, Z) + g(X, \nabla_X Z) \\ &= \tfrac{1}{2}\varrho_A(Z)(g(X,X)) + g(\nabla_X Z - \nabla_Z X - [X,Z], X) \\ &\quad - \varrho_A(X)(g(X,Z)) + g([X,Z], X) + g(\nabla_X X, Z). \end{aligned}$$

Since

$$\varrho_A(Z)(g(X,X)) = d(g(X,X))(Z) = g(\sharp(d(g(X,X))), Z)$$

and

$$\left(\mathcal{L}_X^{\varrho_A} X^\flat\right)(Z) = \varrho_A(X)(g(X,Z)) - g(X, [X,Z]),$$

we have

$$\begin{aligned} &\tfrac{1}{2}(\nabla g)(Z,X,X) - (\nabla g)(X,X,Z) \\ &= \tfrac{1}{2} d(g(X,X))(Z) + g(T^\nabla(X,Z), X) - \left(\mathcal{L}_X^{\varrho_A} X^\flat\right)(Z) + g(\nabla_X X, Z). \end{aligned}$$

In consequence, we obtain the formula given in (2.6). This completes the proof. ∎

Moreover, if ∇ is a metric connection with totally skew-symmetric torsion, then $\nabla g = 0$, $d^s g = 0$, and

$$g(T^\nabla(X,Z), X) = -g(T^\nabla(X,X), Z) = 0$$

for $X, Y, Z \in \Gamma(A)$. In consequence, we obtain the following conclusion for connections compatible with the metric.

Corollary 2.6.1. *If ∇ is a connection with totally skew-symmetric torsion compatible with g, then*

$$\nabla_X X = \sharp(\mathcal{L}_X^{\varrho_A} X^\flat - \tfrac{1}{2} d(g(X,X))) \tag{2.7}$$

for $X \in \Gamma(A)$.

Applying Theorem 2.6.3, we obtain the aforementioned relationship between the connection, its torsion, and the symmetrized covariant derivative of the metric:

Theorem 2.6.4. [7] *Let $\langle \cdot : \cdot \rangle^\nabla$ be the symmetric bracket of sections induced by ∇. Then,*

$$\begin{aligned} g(\langle X : Y\rangle^\nabla, Z) = {} & g(\sharp(\mathcal{L}_X^{\varrho_A} Y^\flat + \mathcal{L}_Y^{\varrho_A} X^\flat - d(g(X,Y))), Z) \\ & - g(T^\nabla(X,Z),Y) - g(T^\nabla(Y,Z),X) \\ & + 2(\nabla g)(Z,X,Y) - (d^s g)(X,Y,Z) \end{aligned} \tag{2.8}$$

for $X, Y, Z \in \Gamma(A)$.

Proof. [7] Using the following polarization formula

$$\langle X : Y\rangle^\nabla = \nabla_{X+Y}(X+Y) - \nabla_X X - \nabla_Y Y$$

and Theorem 2.6.3, we obtain

$$\begin{aligned} g(\langle X : Y\rangle^\nabla, Z) = {} & g(\sharp(\mathcal{L}_{X+Y}^{\varrho_A}(X+Y)^\flat - \tfrac{1}{2} d(g(X+Y,X+Y))), Z) \\ & - g(T^\nabla(X+Y,Z), X+Y) \\ & + (\nabla g)(Z, X+Y, X+Y) \\ & - \tfrac{1}{2}(d^s g)(X+Y, X+Y, Z) \\ & - g(\sharp(\mathcal{L}_X^{\varrho_A} X^\flat - \tfrac{1}{2} d(g(X,X))), Z) \\ & + g(T^\nabla(X,Z),X) - (\nabla g)(Z,X,X) \\ & + \tfrac{1}{2}(d^s g)(X,X,Z) \\ & - g(\sharp(\mathcal{L}_Y^{\varrho_A} Y^\flat - \tfrac{1}{2} d(g(Y,Y))), Z) \\ & + g(T^\nabla(Y,Z),Y) - (\nabla g)(Z,Y,Y) + \tfrac{1}{2}(d^s g)(Y,Y,Z). \end{aligned}$$

First, observe that

$$\mathcal{L}_{X+Y}^{\varrho_A}(X+Y)^\flat - \mathcal{L}_X^{\varrho_A} X^\flat - \mathcal{L}_Y^{\varrho_A} Y^\flat = \mathcal{L}_X^{\varrho_A} Y^\flat + \mathcal{L}_Y^{\varrho_A} X^\flat$$

and

$$-\tfrac{1}{2} d(g(X+Y,X+Y)) + \tfrac{1}{2} d(g(X,X)) + \tfrac{1}{2} d(g(Y,Y)) = -d(g(X,Y)).$$

Since g is a symmetric tensor and T^∇ is skew-symmetric, we conclude that

$$-g(T^\nabla(X+Y,Z),X+Y)+g(T^\nabla(X,Z),X)+g(T^\nabla(Y,Z),Y)$$

is equal to

$$-g(T^\nabla(X,Z),Y)-g(T^\nabla(Y,Z),X).$$

Moreover,

$$\begin{aligned}2(\nabla g)(Z,X,Y)&=(\nabla g)(Z,X+Y,X+Y)\\&\quad-(\nabla g)(Z,X,X)-(\nabla g)(Z,Y,Y)\end{aligned}$$

and

$$\begin{aligned}(d^s g)(X,Y,Z)&=\tfrac{1}{2}(d^s g)(X,Y,Z)+\tfrac{1}{2}(d^s g)(Y,X,Z)\\&=\tfrac{1}{2}(d^s g)(X+Y,X+Y,Z)-\tfrac{1}{2}(d^s g)(X,X,Z)\\&-\tfrac{1}{2}(d^s g)(Y,Y,Z).\end{aligned}$$

Hence, it is clear that some terms of $\langle X:Y\rangle$ cancel themselves. In fact, we obtain

$$\begin{aligned}g(\langle X:Y\rangle^\nabla,Z)=g(&\sharp(\mathcal{L}_X^{\varrho_A}Y^\flat+\mathcal{L}_Y^{\varrho_A}X),Z)\\&-g(\sharp(-d(g(X,Y))),Z)\\&-g(T^\nabla(X,Z),Y)-g(T^\nabla(Y,Z),X)\\&-(d^s g)(X,Y,Z).\end{aligned}$$

This proves (2.8). ■

The formula in Theorem 2.6.4 gives an explicit one of symmetric bracket defined by any metric connection with totally skew-symmetric torsion.

Corollary 2.6.2. *Let ∇ be any metric A-connection in A with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric g. Then,*

$$\langle X:Y\rangle^\nabla=\sharp(\mathcal{L}_X^{\varrho_A}Y^\flat+\mathcal{L}_Y^{\varrho_A}X^\flat-d(g(X,Y)))$$

for $X,Y\in\Gamma(A)$.

2.6.3 Fundamental theorem of pseudo-Riemannian geometry and the Levi-Civita connection

In this section we want to present the fundamental theorem of pseudo-Riemann geometry. In particular, we want to demonstrate the uniqueness of the torsion-free and compatible with the given metric structure connection. The starting point is a skew-symmetric algebroid with a given metric and additionally equipped with a symmetric bracket.

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid equipped with a metric $g \in \Gamma(S^2 A^*)$ and a symmetric bracket $\langle \cdot : \cdot \rangle$ in $\Gamma(A)$, i.e., an $\mathbb{R}$-bilinear symmetric mapping $\langle \cdot : \cdot \rangle : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ satisfying

$$\langle X : fY \rangle = f \langle X : Y \rangle + (\varrho_A \circ X)(f) \cdot Y$$

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$.

With the given symmetric bracket $\langle \cdot : \cdot \rangle$ we associate the symmetric Lie derivative $\mathcal{L}^s$ and the symmetrized covariant derivative d^s.

Theorem 2.6.5. [7] *Let ∇ be an A-connection in A with totally skew-symmetric torsion with respect to a pseudo-Riemannian metric g on A given by*

$$\nabla_X Y = \tfrac{1}{2}\left([X,Y] + \langle X : Y \rangle\right) + \tfrac{1}{2} T(X,Y) \tag{2.9}$$

for $X, Y \in \Gamma(A)$, and some $T \in \Gamma(\bigwedge^2 A^* \otimes A)$. Then,

$$(i_X \circ \nabla) g = \tfrac{1}{2}\left(\mathcal{L}_X^{\varrho_A} + \mathcal{L}_X^s\right) g \quad \text{for} \quad X \in \Gamma(A).$$

Proof. [7] Let $X, Y, Z \in \Gamma(A)$. Since $T \in \Gamma\left(\bigwedge^2 A^* \otimes A\right)$ is a 2-skew-symmetric tensor with the property that

$$g(Y, T(X,Z)) = g(T(X,Z), Y) = -g(T(X,Y), Z),$$

we have

$$\begin{aligned}
(\nabla_X g)(Y,Z) &= \rho_A(X)(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\
&= \tfrac{1}{2}\left(\rho_A(X)(g(Y,Z)) - g([X,Y],Z) - g(Y,[X,Z])\right) \\
&\quad + \tfrac{1}{2}\left(\rho_A(X)(g(Y,Z)) - g(\langle X : Y \rangle, Z) - g(Y, \langle X : Z \rangle)\right) \\
&\quad - \tfrac{1}{2} g(T(X,Y),Z) - \tfrac{1}{2} g(Y, T(X,Z)) \\
&= \tfrac{1}{2}\left(\mathcal{L}_X^a g + \mathcal{L}_X^s g\right)(Y,Z).
\end{aligned}$$

■

We conclude with the following condition ensuring that a connection with totally skew-symmetric torsion is a metric connection.

Corollary 2.6.3. *If ∇ is an A-connection in A with totally skew-symmetric torsion with respect to g given by* (2.9)*, then ∇ is metric with respect to g if and only if*

$$\mathcal{L}^{\varrho_A}_X g = -\mathcal{L}^s_X g \quad \text{for any} \quad X \in \Gamma(A).$$

Now, we recall some properties of the classical Lie derivative.

Lemma 2.6.2. *For $f \in C^\infty(M)$, $X \in \Gamma(A)$, $\omega \in \Gamma(A^*)$, we have*

$$(a) \quad \mathcal{L}^{\varrho_A}_{f\cdot X}\omega = f \cdot \mathcal{L}^{\varrho_A}_X \omega + (i_X\omega)\cdot df,$$

$$(b) \quad \mathcal{L}^{\varrho_A}_X (f\cdot\omega) = f \cdot \mathcal{L}^{\varrho_A}_X \omega + (\varrho_A \circ X)(f)\cdot\omega.$$

Theorem 2.6.6. [7] *Given a skew-symmetric algebroid $(A, \varrho_A, [\cdot,\cdot])$, we define*

$$\langle X : Y\rangle^s : \Gamma(A)\times\Gamma(A) \longrightarrow \Gamma(A)$$

by

$$\langle X : Y\rangle^s = \sharp(\mathcal{L}^{\varrho_A}_X Y^\flat + \mathcal{L}^{\varrho_A}_Y X^\flat - d(g(X,Y))) \tag{2.10}$$

for $X, Y \in \Gamma(A)$. Then, $\langle\cdot : \cdot\rangle^s$ is a symmetric bracket that defines the symmetric Lie derivative $\mathcal{L}^s$ satisfying $\mathcal{L}^s_X g = -\mathcal{L}^{\varrho_A}_X g$.

The following conclusion immediately follows from Theorem 2.6.5.

Corollary 2.6.4. *The torsion-free connection ∇ given by*

$$\nabla_X Y = \tfrac{1}{2}\left([X,Y] + \langle X : Y\rangle^s\right),$$

where

$$\langle X : Y\rangle^s = \sharp(\mathcal{L}^{\varrho_A}_X Y^\flat + \mathcal{L}^{\varrho_A}_Y X^\flat - d(g(X,Y))) \tag{2.11}$$

for $X, Y \in \Gamma(A)$, is compatible with g.

Now, we show that for the skew-symmetric algebroid structure equipped with additional pseudometric g, the following generalization of the fundamental theorem of Riemannian geometry holds:

Theorem 2.6.7. [7] *Let g be a pseudo-Riemannian metric in the vector bundle A and $\Omega \in \Gamma(\bigwedge^2 A^* \otimes A)$ be a 2-form on A with values in A. Then, there exists a unique connection ∇ on A compatible with g such that its torsion tensor equals Ω, i.e.,*

$$\nabla g = 0 \quad \text{and} \quad T^{\nabla} = \Omega.$$

The connection ∇ is given by the formula

$$\nabla_X Y = \tfrac{1}{2}\left([X,Y] + \langle X : Y\rangle^s\right) + \tfrac{1}{2}\Omega(X,Y) + S(X,Y),$$

where

$$\langle X : Y\rangle^s = \sharp(\mathcal{L}_X^{\varrho_A} Y^\flat + \mathcal{L}_Y^{\varrho_A} X^\flat - d(g(X,Y))), \tag{2.12}$$

and $S \in \Gamma(\mathsf{S}^2 A^* \otimes A)$ is the symmetric 2-tensor on A with values in A determined uniquely by

$$g(S(X,Y),Z) = g(\Omega(Z,X),Y) + g(\Omega(Z,Y),X)$$

for $X, Y, Z \in \Gamma(A)$.

Proof. (cf. [7]) Let $X, Y \in \Gamma(A)$. To prove the existence of the suitable connection, take the linear connection ∇^g defined by

$$\nabla^g_X Y = \tfrac{1}{2}\left([X,Y] + \langle X : Y\rangle^s\right),$$

where

$$\langle X : Y\rangle^s = \sharp(\mathcal{L}_X^{\varrho_A} Y^\flat + \mathcal{L}_Y^{\varrho_A} X^\flat - d^a(g(X,Y))).$$

Let ∇ be a linear connection compatible with g and with torsion T^{∇} equals Ω. There exists some 2-tensor $\Phi \in \Gamma(\otimes^2 A^* \otimes A)$ such that

$$\nabla_X Y = \nabla^g_X Y + \Phi(X,Y).$$

Hence

$$\begin{aligned}\Omega(X,Y) &= T^{\nabla}(X,Y)\\ &= \nabla^g_X Y + \Phi(X,Y) - \nabla^g_Y X + \Phi(Y,X) - [X,Y]\\ &= \Phi(X,Y) - \Phi(Y,X)\end{aligned}$$

because $\nabla^g_X Y - \nabla^g_Y X = [X,Y]$ (∇^g is torsion-free). Therefore,

$$\begin{aligned}\nabla_X Y &= \nabla^g_X Y + \tfrac{1}{2}\left(\Phi(X,Y) - \Phi(Y,X)\right) + \tfrac{1}{2}\left(\Phi(X,Y) + \Phi(Y,X)\right)\\ &= \nabla^g_X Y + \tfrac{1}{2}\Omega(X,Y) + \tfrac{1}{2}\left(\Phi(X,Y) + \Phi(Y,X)\right).\end{aligned}$$

It follows that there exists some symmetric tensor $S \in \Gamma(\mathsf{S}^2 A^* \otimes A)$ such that

$$\nabla_X Y = \nabla^g_X Y + \tfrac{1}{2}\Omega(X,Y) + \tfrac{1}{2}S(X,Y).$$

Thus we get

$$\begin{aligned}
\langle X : Y \rangle^{\nabla} &= \nabla_X Y + \nabla_Y X \qquad (2.13)\\
&= \nabla^g_X Y + \tfrac{1}{2}\Omega(X,Y) + \tfrac{1}{2}S(X,Y) + \nabla^g_Y X + \tfrac{1}{2}\Omega(Y,X) + \tfrac{1}{2}S(Y,X)\\
&= (\nabla^g_X Y + \nabla^g_Y X) + \tfrac{1}{2}\cdot 0 + S(X,Y)\\
&= \langle X : Y \rangle^{s} + S(X,Y).
\end{aligned}$$

This shows immediately that S is determined uniquely. Since $\nabla g = 0$, Theorem 2.6.4 and (2.13) now lead to

$$\begin{aligned}
&g(\langle X : Y \rangle^{s} + S(X,Y), Z)\\
&= g(\langle X : Y \rangle^{\nabla}, Z)\\
&= g(\langle X : Y \rangle^{s}, Z) - g(T^{\nabla}(X,Z),Y) - g(T^{\nabla}(Y,Z),X)\\
&= g(\langle X : Y \rangle^{s}, Z) + g(T^{\nabla}(Z,X),Y) + g(T^{\nabla}(Z,Y),X)\\
&= g(\langle X : Y \rangle^{s}, Z) + g(\Omega(Z,X),Y) + g(\Omega(Z,Y),X)
\end{aligned}$$

where the last equalities are the consequence of skew-symmetricity of the torsion and the equality $T^{\nabla} = \Omega$. From what has already been proved, we see that

$$g(S(X,Y),Z) = g(\Omega(Z,X),Y) + g(\Omega(Z,Y),X).$$

■

One can immediately see that Theorem 2.6.7 allows us to write formulas of some connections related to the given 2-skew-symmetric form on A with values in A. In the case, if ∇ is a metric connection in the bundle A with torsion $T \in \Gamma(\bigwedge^2 A^* \otimes A)$ which is totally skew-symmetric with respect to g. The connection is given by the formula

$$\nabla_X Y = \tfrac{1}{2}\left([X,Y] + \langle X : Y \rangle^{s}\right) + \tfrac{1}{2}T(X,Y),$$

where $\langle X : Y \rangle^{s}$ is given in (2.10).

As a special case of Theorem 2.6.7, we obtain Theorem 2.6.8 below when the considered connection is torsion-free and compatible with g (i.e., $T^{\nabla} = 0$ and $\nabla g = 0$).

Theorem 2.6.8. *Given a bundle metric g on A, there is a unique connection in A which is torsion-free and metric-compatible.*

Definition 2.6.7. We call such a unique torsion-free connection in A compatible with g the *Levi-Civita connection* with respect to g.

The explicit formula of the Levi-Civita connection compatible with g is provided by Corollary 2.6.4.

2.6.4 Dual connection with respect to a metric tensor

Let $(A, \varrho_A, [\cdot,\cdot])$ be a skew-symmetric algebroid over a manifold M equipped with a pseudo-Riemannian metric $g \in \Gamma(S^2A^*)$ in the vector bundle A. Let $\nabla : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ be a connection in the bundle A.

Definition 2.6.8. The *dual connection* ∇^{*g} to ∇ with respect to g is given by

$$\nabla^{*g} : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A),$$

$$g(\nabla^{*g}_X Y, Z) = (\varrho_A \circ X)(g(Y,Z)) - g(Y, \nabla_X Z)$$

for $X, Y, Z \in \Gamma(A)$.

Definition 2.6.9. The *affine combination of two connections* $\nabla^0, \nabla^1 : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ in A is the connection $\nabla^{\mathrm{aff},t}$ given by the formula

$$\nabla^{\mathrm{aff},t} = (1-t)\,\nabla^0 + t\nabla^1$$

where $t \in C^\infty(M)$.

Theorem 2.6.9. *If ∇^0, ∇^1 are two connections in A, $t \in C^\infty(M)$, $X \in \Gamma(A)$, then*

$$(a)\quad T^{\nabla^{\mathrm{aff},t}} = (1-t)\,T^{\nabla^0} + tT^{\nabla^1},$$
$$(b)\quad \nabla^{\mathrm{aff},t} g = (1-t)\cdot\left(\nabla^0 g\right) + t\cdot\left(\nabla^1 g\right),$$
$$(c)\quad \mathcal{L}_X^{\nabla^{\mathrm{aff},t}} = (1-t)\cdot\mathcal{L}_X^{\nabla^0} + t\cdot\mathcal{L}_X^{\nabla^1}.$$

Proof. (c) Let $X, X_1, \ldots, X_k \in \Gamma(A)$, $\eta \in \mathscr{A}^k(A, A)$. Then,

$$
\begin{aligned}
&(\mathcal{L}_X^{\nabla^{\mathrm{aff},t}}\eta)(X_1,\ldots,X_k)\\
&= \nabla_X^{\mathrm{aff},t}(\eta(X_1,\ldots,X_k)) - \sum_{i=1}^{k}\eta(X_1,\ldots,[X,X_i],X_{i+1},\ldots,X_k)\\
&= (1-t)\,\nabla_X^0(\eta(X_1,...,X_k)) - (1-t)\sum_{i=1}^{k}\eta(X_1,...,[X,X_i],...,X_k)\\
&\quad + t\cdot\nabla_X^1(\eta(X_1,\ldots,X_k)) - t\sum_{i=1}^{k}\eta(X_1,\ldots,[X,X_i],\ldots,X_k)\\
&= ((1-t)\cdot\mathcal{L}_X^{\nabla^0}\eta + t\cdot\mathcal{L}_X^{\nabla^1}\eta)(X_1,\ldots,X_k).
\end{aligned}
$$

■

Important for secondary characteristic classes are the affine combinations of a given connection and a connection dual to it with respect to the metric g. It turns out that the only affine combination of the ∇ and ∇^{*g} compatible with the pseudo-metric is the connection $\nabla^{\mathrm{aff},t}$ for $t=\frac{1}{2}$. We have the following theorem.

Theorem 2.6.10. *Let $t \in C^\infty(M)$. If $\nabla^{\mathrm{aff},t} = (1-t)\,\nabla + t\nabla^{*g}$, then we have*

$$\nabla^{\mathrm{aff},t} g = (1-2t)\,\nabla g.$$

Proof. Let $X, Y, Z \in \Gamma(A)$. Then,

$$
\begin{aligned}
&(\nabla^{\mathrm{aff},t}g)(X,Y,Z)\\
&= (\varrho_A \circ X)\,(g(Y,Z)) - g(\nabla_X^{\mathrm{aff},t}Y,Z) - g(Y,\nabla_X^{\mathrm{aff},t}Z)\\
&= (\varrho_A \circ X)\,(g(Y,Z)) - g((1-t)\,\nabla_X Y,Z) - g(t\nabla_X^{*g}Y,Z)\\
&\quad - g(Y,(1-t)\,\nabla_X Z) - g(Y,t\nabla_X^{*g}Z)\\
&= (\varrho_A \circ X)\,(g(Y,Z))\\
&\quad - (1-t)\,g(\nabla_X Y,Z) - t\,(\varrho_A \circ X)\,(g(Y,Z)) + tg(Y,\nabla_X Z)\\
&\quad - (1-t)\,g(Y,\nabla_X Z) - t\,(\varrho_A \circ X)\,(g(Z,Y)) + tg(\nabla_X Y,Z)\\
&= (1-2t)\,((\varrho_A \circ X)\,(g(Y,Z)) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z))\\
&= ((1-2t)\,\nabla g)(X,Y,Z).
\end{aligned}
$$

■

Corollary 2.6.5. *If ∇^{*g} is the dual connection to ∇ with respect to g, then we have*

$$\left(\frac{1}{2}\,(\nabla + \nabla^{*g})\right) g = 0.$$

2.7 Characterization of regular Lie algebroids by a splitting of the Atiyah sequence

Let $(A, \varrho_A, [\cdot,\cdot])$ be a regular Lie algebroid over M with the Atiyah sequence

$$0 \longrightarrow \mathbf{g} \longrightarrow A \xrightarrow{\varrho_A} \mathbf{F} \longrightarrow 0. \tag{2.14}$$

We note that A is isomorphic with the direct sum $\mathbf{g} \oplus \mathbf{F}$, where the structure of a Lie algebroid is determined by some splitting of the Atiyah sequence (cf. [61], [53]).

Let

$$\Lambda : \mathbf{F} \to A$$

be a splitting of the Atiyah sequence (2.14), i.e., Λ is a homomorphism of vector bundles with

$$\varrho_A \circ \Lambda = \mathrm{id}_{\mathbf{F}}\,.$$

Recall form Theorem 2.1.2 and Corollary 2.1.1 that Λ commutes with the anchors of $\mathbf{F}$ and A. Define the following homomorphism of $C^\infty(M)$-modules

$$\nabla^\Lambda : \Gamma(\mathbf{F}) \longrightarrow \mathcal{CDO}(\mathbf{g}),\ \ \nabla^\Lambda_Y(u) = [\Lambda(Y), u]$$

for $Y \in \Gamma(\mathbf{F})$, $u \in \Gamma(\mathbf{g})$. ∇^Λ is an $\mathbf{F}$-connection in $\mathbf{g}$. Let

$$R^\Lambda \in \Gamma(\textstyle\bigwedge^2 F^* \otimes \mathbf{g})$$

given by

$$R^\Lambda(X,Y) = [\Lambda(X), \Lambda(Y)] - \Lambda[X,Y]_F\,,$$

be the curvature of Λ. Observe that the curvature

$$R^{\nabla^\Lambda} \in \Gamma(\textstyle\bigwedge^2 F^* \otimes \mathrm{End}(\mathbf{g}))$$

of ∇^Λ is given by

$$R^{\nabla^\Lambda}_{X,Y}u = \left[R^\Lambda(X,Y), u\right] \tag{2.15}$$

for $X, Y \in \Gamma(\mathbf{F})$, $u \in \Gamma(\mathbf{g})$. Moreover, ∇^Λ has the property:

$$\nabla^{\Lambda}_X([u,\nu]) = \left[\nabla^{\Lambda}_X(u),\nu\right] + \left[u,\nabla^{\Lambda}_X(\nu)\right]$$

for $X \in \Gamma(\mathbf{F})$, $u,\nu \in \Gamma(\mathbf{g})$. ∇^{Λ} induces the covariant exterior derivative $d^{\nabla^{\Lambda}}$ on $\mathscr{A}^k(F,\mathbf{g})$. There is (well-known) the following *second Bianchi identity*

$$d^{\nabla^{\Lambda}}(R^{\Lambda}) = 0. \tag{2.16}$$

Since

$$\left(d^{\nabla^{\Lambda}} d^{\nabla^{\Lambda}}(a)\right)(X,Y) = R^{\nabla^{\Lambda}}_{X,Y} a$$

for all $a \in \Gamma(\mathbf{g}) = \mathscr{A}^0(F,\mathbf{g})$, $X,Y \in \Gamma(\mathbf{F})$, (2.15) shows a *generalized Ricci identity*:

$$\left(d^{\nabla^{\Lambda}} d^{\nabla^{\Lambda}}(a)\right)(\cdot,\cdot) = \left[R^{\Lambda}(\cdot,\cdot),a\right] \tag{2.17}$$

for all $a \in \Gamma(\mathbf{g})$. In the direct sum $\mathbf{g} \oplus \mathbf{F}$ we have the structure of an $\mathbb{R}$-Lie algebra with the bracket $[\cdot,\cdot]^{\Lambda}$ defined by

$$[(a,X),(b,Y)]^{\Lambda} = \left([a,b] + R^{\Lambda}(X,Y) + \nabla^{\Lambda}_X b - \nabla^{\Lambda}_Y a, [X,Y]\right)$$

for $a,b \in \Gamma(\mathbf{g})$, $X,Y \in \Gamma(\mathbf{F})$. It is evident that $[\cdot,\cdot]^{\Lambda}$ is $\mathbb{R}$-multilinear and skew-symmetric. Moreover, $[\cdot,\cdot]^{\Lambda}$ satisfies the Jacobi identity. In fact, let $a,b,c \in \Gamma(\mathbf{g})$ and $X,Y,Z \in \Gamma(\mathbf{F})$. One can check using the Jacobi identity in $\mathbf{g}$ and reduction of similar terms that

$$\begin{aligned}
&[[(a,X),(b,Y)],(c,Z)] + \text{cyclic} \\
&= \left[R^{\Lambda}(X,Y),c\right] - (d^{\nabla^{\Lambda}} d^{\nabla^{\Lambda}} c)(X,Y) - (d^{\nabla^{\Lambda}} R^{\Lambda})(X,Y,Z) \\
&\quad - \left[R^{\Lambda}(X,Z),b\right] - (d^{\nabla^{\Lambda}} d^{\nabla^{\Lambda}} b)(X,Z) \\
&\quad + \left[R^{\Lambda}(Y,Z),a\right] - (d^{\nabla^{\Lambda}} d^{\nabla^{\Lambda}} a)(Y,Z) \\
&= 0
\end{aligned}$$

which equals zero because of the Bianchi and the Ricci identities (2.16), (2.17).

The vector bundle $\mathbf{g} \oplus \mathbf{F}$ is a Lie algebroid with the bracket $[\cdot,\cdot]^{\Lambda}$ and the projection to the second factor

$$\varrho = \mathrm{pr}_2 : \mathbf{g} \oplus \mathbf{F} \longrightarrow \mathbf{F}$$

as an anchor. This Lie algebroid is denoted by $\mathbf{g} \oplus_{\Lambda} \mathbf{F}$. We call $\mathbf{g} \oplus_{\Lambda} \mathbf{F}$ the *semidirect sum* of $\mathbf{g}$ and $\mathbf{F}$. Note that the mapping

$$\psi : \mathbf{g} \oplus \mathbf{F} \to A$$

given by

$$\psi(a, X) = a + \Lambda(X)$$

is an isomorphism of vector bundles. Moreover, $\operatorname{Sec} \psi$ is a homomorphism of $\mathbb{R}$-Lie algebras ($\operatorname{Sec} \psi$ preserves the Lie brackets $[\cdot, \cdot]^{\Lambda}$ and $[\cdot, \cdot]$) and ψ commutes with anchors ϱ and ϱ_A. It follows that Lie algebroids A and $\mathbf{g} \oplus_{\Lambda} \mathbf{F}$ are isomorphic,

$$A \cong \mathbf{g} \oplus_{\Lambda} \mathbf{F}.$$

Part III

Secondary characteristic classes of Lie algebroids

3. Relative cohomology

3.1 The differential $\bar{\delta}$

We consider a real Lie algebra **g** and its Lie subalgebra **h**. In the space $\bigwedge(\mathbf{g}/\mathbf{h})^*$ of forms on the quotient space $\mathbf{g}/\mathbf{h}$, we can define the relation $\overline{\delta}$, which after restriction to invariant forms is a function such that $\overline{\delta}\circ\overline{\delta}=0$. We make elementary proofs for the properties of this relation because **h** does not have to be an ideal of the Lie algebra **g**, and we generally do not use properties of the exterior derivative operator in **g**. In proofs showing that some relation $\overline{\delta}$ is anti-differentiation and satisfies the condition $\overline{\delta}\circ\overline{\delta}=0$, we use some properties of permutations. We show some useful properties at the beginning.

Some properties of permutations.

Definition 3.1.1. Let $m,n\in\mathbb{N}$. Write

$$N(m,n)=\mathbb{N}\cap[m,n]\,.$$

Definition 3.1.2. Let $m,n,p\in\mathbb{N}$. Denote by $S_{m,n}$ the set of (m,n)-shuffles, that is, the set of permutations $\sigma\in S_{m+n}$ satisfying

$$\sigma(1)<\cdots<\sigma(m)\quad\text{and}\quad\sigma(m+1)<\cdots<\sigma(m+n).$$

Definition 3.1.3. Let $m,n,p\in\mathbb{N}$. Let $S_{m,n,p}$ denote the set of all permutations $\sigma\in S_{m+n+p}$ such that functions

$$\sigma|N(1,m),\quad \sigma|N(m+1,m+n),\quad\text{and}\quad \sigma|N(m+n+1,m+n+p)$$

are increasing, i.e., $\sigma(1)<\cdots<\sigma(m)$, $\sigma(m+1)<\cdots<\sigma(m+n)$, and $\sigma(m+n+1)<\cdots<\sigma(m+n+p)$.

Theorem 3.1.1. *For any $r,s,t\in\mathbb{N}$ sets $S_{r,s+t}\times S_{t,s}$ and $S_{r,t,s}$ are bijective. The mapping*

$$\Xi^{r,s,t}:S_{r,s+t}\times S_{t,s}\xrightarrow{1-1}S_{r,t,s},$$

$$(\sigma,\rho)\longmapsto\sigma\circ(r+\rho)$$

is a bijection, where $r+\rho \in S_{r,t,s} \subset S_{r+s+t}$ *is defined by*

$$(r+\rho)(i) = \begin{cases} i & \text{if} \;\; i \in N(1,r), \\ r+\rho(i-r) & \text{if} \;\; i \in N(r+1, r+t+s). \end{cases}$$

Moreover,

$$\operatorname{sgn}(\Xi^{r,s,t}(\sigma,\rho)) = \operatorname{sgn}(\sigma \circ (r+\rho)) = \operatorname{sgn}\sigma \cdot \operatorname{sgn}\rho.$$

Proof. Let σ, $\sigma' \in S_{r,s+t}$, ρ, $\rho' \in S_{t,s}$, and let

$$\sigma \circ (r+\rho) = \sigma' \circ (r+\rho')\,.$$

Observe that $(r+\rho)|N(1,r) = \mathrm{id}_{N(1,r)} = (r+\rho')|N(1,r)$. Thus, $\sigma|N(1,r) = \sigma'|N(1,r)$. Due to the fact that $\sigma|N(r+1, r+t+s)$ and $\sigma'|N(r+1,r+t+s)$ are increasing, it follows that $\sigma = \sigma'$. The equality $\sigma\circ(r+\rho) = \sigma\circ(r+\rho')$ and injectivity of σ imply $r+\rho = r+\rho'$. In particular,

$$(r+\rho)|N(r+1,r+t+s) = (r+\rho')|N(r+1,r+t+s).$$

Therefore,

$$\begin{aligned} \rho &= (r+\rho)|N(r+1,r+t+s)\circ(j\mapsto j+r) - r \\ &= (r+\rho')|N(r+1,r+t+s)\circ(j\mapsto j+r) - r \\ &= \rho'. \end{aligned}$$

Thus $\Xi^{r,s,t}$ is injective.

Now, let $\tau \in S_{r,t,s}$, i.e., $\tau \in S_{r+t+s}$ satisfies the conditions $\tau(1) < \cdots < \tau(r)$, $\tau(r+1) < \cdots < \tau(r+t)$, and $\tau(r+t+1) < \cdots < \tau(r+t+s)$. We denote by K the set $\tau(N(r+1,r+t+s))$.

Define $\sigma \in S_{r,t+s}$ in such a way that

$$\sigma|N(1,r) = \tau|N(1,r), \quad \sigma(N(r+1,r+t+s)) = K,$$

and $\sigma|N(r+1,r+t+s)$ is increasing. Take an injective mapping

$$\rho : N(1,t+s) \longrightarrow \mathbb{N}$$

given by

$$\rho(j) = \sigma^{-1}(\tau(r+j)) - r \;\; \text{for} \;\; j \in N(1,t+s).$$

The injectivity of σ implies

$$\begin{aligned} N(r+1, r+t+s) &= \sigma^{-1}(\sigma(N(r+1, r+t+s))) \\ &= \sigma^{-1}(\tau(N(r+1, r+t+s))). \end{aligned}$$

Therefore,

$$\rho(N(1, t+s)) = N((r+1)-r, (r+t+s)-r) = N(1, t+s),$$

which means that ρ is a permutation, i.e., $\rho \in S_{t+s}$.

Since the mapping $\sigma|K : K \xrightarrow{1-1} K$ is increasing, we conclude that $(\sigma|K)^{-1} : K \xrightarrow{1-1} K$ is also increasing. Hence, since

$$\tau|N\,(r+1, r+t) \text{ and } \tau|N\,(r+t+1, r+t+s)$$

are increasing, it follows that

$$(\sigma|N(r+1, r+t+s))^{-1} \circ \tau|N(r+1, r+t) - r,$$

$$(\sigma|N(r+1, r+t+s))^{-1} \circ \tau|N(r+t+1, r+t+s) - r$$

are also increasing. From this we deduce that

$$\rho(1) < \cdots < \rho(t) \text{ and } \rho(t+1) < \cdots < \rho(t+s).$$

Consequently, $\rho \in S_{t,s}$.

It is clear that

$$\begin{aligned} (r+\rho)(i) &= \begin{cases} i & \text{if} \quad i \in N(1, r), \\ r+\rho(i-r) & \text{if} \quad i \in N(r+1, r+t+s) \end{cases} \\ &= \begin{cases} i & \text{if} \quad i \in N(1, r), \\ \sigma^{-1}(\tau(i)) & \text{if} \quad i \in N(r+1, r+t+s). \end{cases} \end{aligned}$$

Therefore, $\Xi^{r,s,t}(\sigma, \rho) = \sigma \circ (r+\rho) = \tau$. This shows that $\Xi^{r,s,t}$ is surjective. ■

Corollary 3.1.1. $(t = 2)$ *For $r, s \in \mathbb{N}$ sets $S_{r,s+2} \times S_{2,s}$ and $S_{r,2,s}$ are bijective. Let $(r+\rho) \in S_{r,2,s}$ be given by*

$$(r+\rho)(i) = \begin{cases} i & \textit{if} \quad i \in N(1, r), \\ r+\rho(i-r) & \textit{if} \quad i \in N(r+1, r+s+2). \end{cases}$$

The mapping

$$\Xi^{r,s,2} : S_{r,s+2} \times S_{2,s} \xrightarrow{1-1} S_{r,2,s},$$

$$(\sigma, \rho) \longmapsto \sigma \circ (r+\rho)$$

is bijective.

Corollary 3.1.2. ($r = 2$) *For $s, t \in \mathbb{N}$ the sets $S_{2,s+t} \times S_{t,s}$ and $S_{2,t,s}$ are bijective. The mapping*

$$\Xi^{2,s,t} : S_{2,s+t} \times S_{t,s} \xrightarrow{1-1} S_{2,t,s},$$

$$(\sigma, \rho) \longmapsto \sigma \circ (2 + \rho),$$

where $(2 + \rho) \in S_{2,t,s} \subset S_{2+t+s}$ defined by

$$(2+\rho)(i) = \begin{cases} i & \text{if} \quad i \in N(1,2), \\ r + \rho(i-2) & \text{if} \quad i \in N(3, 2+s+t), \end{cases}$$

is bijective.

Theorem 3.1.2. *For $r, s, t \in \mathbb{N}$ the mapping*

$$\Theta_s^{r,t} : S_{r,t,s} \xrightarrow{1-1} S_{t,r,s},$$

$$\sigma \longmapsto \sigma \circ \sigma_s^{r,t},$$

where $\sigma_s^{r,t} \in S_{t,r,s} \subset S_{r+t+s}$, given by

$$\sigma_s^{r,t}(i) = \begin{cases} r + i & \text{if} \quad i \in N(1,t), \\ i - t & \text{if} \quad i \in N(t+1, t+r), \\ i & \text{if} \quad i \in N(t+r+1, t+r+s), \end{cases}$$

is a bijection between sets $S_{r,t,s}$ and $S_{t,r,s}$. Moreover,

$$\operatorname{sgn} \Theta_s^{r,t}(\sigma) = (-1)^{r \cdot t} \operatorname{sgn} \sigma. \qquad \blacksquare$$

Corollary 3.1.3. *For r, s, $t \in \mathbb{N}$ the sets $S_{r,s+t} \times S_{t,s}$ and $S_{t,r,s}$ are bijective. The mapping*

$$\Theta_s^{r,t} \circ \Xi^{r,s,t} : S_{r,s+t} \times S_{t,s} \xrightarrow{1-1} S_{t,r,s}$$

is bijective. Moreover,

$$\operatorname{sgn}(\Theta_s^{r,t} \circ \Xi^{r,s,t})(\sigma, \rho) = (-1)^{r \cdot t} \operatorname{sgn} \sigma \cdot \operatorname{sgn} \rho$$

for $\sigma \in S_{r,s+t}, \rho \in S_{t,s}$.

Corollary 3.1.4. *Let $r,\ s \in \mathbb{N}$, $\rho \in S_{2,r}$, and let $(\rho + s) \in S_{2,r,s}$ be a permutation defined by*

$$(\rho+s)(j) = \begin{cases} \rho(j) & \textit{if} \quad j \in N(1, r+2), \\ j & \textit{if} \quad j \in N(r+3, r+2+s). \end{cases}$$

Then, the mapping

$$\Upsilon_s^{r,2} : S_{r+2,s} \times S_{2,r} \longrightarrow S_{2,r,s}$$

given by

$$\Upsilon_s^{r,2}(\sigma, \rho) = \sigma \circ (\rho + s)$$

for $\sigma \in S_{r+2,s}$ and $\rho \in S_{2,r}$, is a bijection between sets $S_{r+2,s} \times S_{2,r}$ and $S_{2,r,s}$. Moreover, $\operatorname{sgn}(\Upsilon_s^{r,2}(\sigma,\rho)) = \operatorname{sgn}\sigma \cdot \operatorname{sgn}\rho$ *for $\sigma \in S_{r+2,s}$ and $\rho \in S_{2,r}$.*

Proof. Let $\sigma_0^{p,q} \in S_{q,p}$ be given by

$$\sigma_0^{p,q}(i) = \begin{cases} p+i & \text{if} \quad i \in N(1,q), \\ i-q & \text{if} \quad i \in N(q+1, q+p). \end{cases}$$

The mapping $\Upsilon_s^{r,2}$ is a composition of the bijections:

$$(\tau \mapsto \tau \circ \sigma_{s,r}^2) \circ \Theta_r^{s,2} \circ \Xi^{s,2,r} \circ \left(\tau \mapsto \tau \circ \sigma_0^{r+2,s}\right) \times \mathrm{id}_{S_{2,r}},$$

where $\sigma_{p,q}^k \in S_{k,q,p}$ is given by

$$\sigma_{p,q}^k(i) = \begin{cases} i & \text{if} \quad i \in N(1,k), \\ p+i & \text{if} \quad i \in N(k+1, k+q), \\ i-q & \text{if} \quad i \in N(k+q+1, k+q+p), \end{cases}$$

i.e., the following diagram

$$\begin{tikzcd} S_{r+2,s} \times S_{2,r} \arrow[r, "\Upsilon_s^{r,2}"] \arrow[d, "(\tau \mapsto \tau \circ \sigma_0^{r+2,s}) \times \mathrm{id}_{S_{2,r}}"'] & S_{2,r,s} \\ S_{s,s+2} \times S_{2,r} \arrow[r, "\Xi^{s,2,r}"'] & S_{s,2,r} \arrow[u, "\Theta_r^{s,2}"'] \end{tikzcd}$$

is commutative. In fact, let $(\sigma, \rho) \in S_{r+2,s} \times S_{2,r}$. Then,

$$
\begin{aligned}
&\left(\left(\tau \mapsto \tau \circ \sigma^2_{s,r}\right) \circ \Theta^{s,2}_r \circ \Xi^{s,2,r} \circ \left(\tau \mapsto \tau \circ \sigma_0^{r+2,s}\right) \times \mathrm{id}_{S_{2,r}}\right)(\sigma, \rho) \\
&= \left(\left(\tau \mapsto \tau \circ \sigma^2_{s,r}\right) \circ \Theta^{s,2}_r \circ \Xi^{s,2,r}\right)\left(\sigma \circ \sigma_0^{r+2,s}, \rho\right) \\
&= \left(\left(\tau \mapsto \tau \circ \sigma^2_{s,r}\right) \circ \Theta^{s,2}_r\right)\left(\sigma \circ \sigma_0^{r+2,s} \circ ((s+\rho))\right) \\
&= \left(\left(\tau \mapsto \tau \circ \sigma^2_{s,r}\right)\right)\left(\sigma \circ \sigma_0^{r+2,s} \circ (s+\rho) \circ \sigma^{s,2}_r\right) \\
&= \sigma \circ \sigma_0^{r+2,s} \circ (s+\rho) \circ \sigma^{s,2}_r \circ \sigma^2_{s,r} \\
&= \sigma \circ (\rho + s) \\
&= \Upsilon^{r,2}_s(\sigma, \rho).
\end{aligned}
$$

■

The relation $\overline{\delta}$. Let $\mathbf{g}$ be a vector bundle over a manifold M and let $\mathbf{h}$ be a subbundle of $\mathbf{g}$ such that $\Gamma(\mathbf{h})$ is a real Lie subalgebra of $\Gamma(\mathbf{g})$ and $\mathbf{h}_x$ is a Lie subalgebra of a Lie algebra $\mathbf{g}_x$ for any $x \in M$. Let $[\cdot,\cdot]$ denote the Lie bracket in $\Gamma(\mathbf{g})$. We define a relation

$$\overline{\delta} : \Gamma(\textstyle\bigwedge(\mathbf{g}/\mathbf{h})^*) \longrightarrow \Gamma(\textstyle\bigwedge(\mathbf{g}/\mathbf{h})^*)$$

by

$$
\begin{aligned}
&(\overline{\delta}\Phi)([u_1], \ldots, [u_{n+1}]) \\
&= \sum_{i<j} (-1)^{i+j+1}\, \Phi([[u_i, u_j]], [u_1], \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, [u_{n+1}]) \\
&= \sum_{\sigma \in S_{2,n-1}} \operatorname{sgn}\sigma \cdot \Phi(\left[[u_{\sigma(1)}, u_{\sigma(2)}]\right], \left[u_{\sigma(3)}\right], \ldots, \left[\alpha_{\sigma(n+1)}\right])
\end{aligned}
$$

for $\Phi \in \Gamma(\bigwedge^n(\mathbf{g}/\mathbf{h})^*)$, $u_1, \ldots, u_{n+1} \in \Gamma(\mathbf{g})$.

$\overline{\delta}$ is an antiderivation.

Theorem 3.1.3. $\overline{\delta} : \Gamma(\bigwedge(\mathbf{g}/\mathbf{h})^*) \longrightarrow \Gamma(\bigwedge(\mathbf{g}/\mathbf{h})^*)$ *is an antiderivation, i.e.,*

$$\overline{\delta}(\Psi^1 \wedge \Psi^2) = \overline{\delta}\Psi^1 \wedge \Psi^2 + (-1)^p \Psi^1 \wedge \overline{\delta}\Psi^2$$

for $\Psi^1 \in \Gamma(\bigwedge^p(\mathbf{g}/\mathbf{h})^*)$, $\Psi^2 \in \Gamma(\bigwedge(\mathbf{g}/\mathbf{h})^*)$.

Proof. Let $\Psi^1 \in \Gamma(\bigwedge^p(\mathbf{g}/\mathbf{h})^*)$, $\Psi^2 \in \Gamma(\bigwedge^q(\mathbf{g}/\mathbf{h})^*)$ and $u_1, \ldots, u_{p+q-1} \in \Gamma(\mathbf{g})$. We define $\overline{u}$ to be the class $[u]$ represented by $u \in \Gamma(\mathbf{g})$. Then,

$$
\begin{aligned}
&\overline{\delta}(\Psi^1 \wedge \Psi^2)(\overline{u_1} \wedge \ldots \wedge \overline{u_{p+q+1}}) \\
&= \sum_{i<j} (-1)^{i+j+1} \, (\Psi^1 \wedge \Psi^2)(\overline{[u_i, u_j]}, \overline{u_1}, \ldots, \hat{\imath}, \ldots, \hat{\jmath}, \ldots, \overline{u_{p+q+1}}) \\
&= \sum_{\sigma \in S_{2,p+q-1}} \operatorname{sgn} \sigma (\Psi^1 \wedge \Psi^2)(\overline{[u_{\sigma(1)}, u_{\sigma(2)}]}, \overline{u_{\sigma(3)}}, \ldots, \overline{u_{\sigma(n+1)}}) \\
&= \sum_{\sigma \in S_{2,p+q-1}} \operatorname{sgn} \sigma \sum_{\rho \in S_{p-1,q}} \operatorname{sgn} \rho \; \Psi^1(\overline{[u_{\sigma(1)}, u_{\sigma(2)}]}, \overline{u_{\sigma(2+\rho(1))}}, \ldots, \overline{u_{\sigma(2+\rho(p-1))}}) \\
&\qquad\qquad \cdot \Psi^2(\overline{u_{\sigma(2+\rho(p))}}, \ldots, \overline{u_{\sigma(2+\rho(p+q-1))}}) \\
&\quad + \sum_{\sigma \in S_{2,p+q-1}} \operatorname{sgn} \sigma \sum_{\rho \in S_{p,q-1}} (-1)^p \operatorname{sgn} \rho \, \Psi^1(\overline{u_{\sigma(2+\rho(1))}}, \overline{u_{\sigma(2+\rho(2))}}, \ldots, \overline{u_{\sigma(2+\rho(p))}}) \\
&\qquad\qquad \cdot \Psi^2(\overline{[u_{\sigma(1)}, u_{\sigma(2)}]}, \overline{u_{\sigma(2+\rho(p+1))}}, \ldots, \overline{u_{\sigma(2+\rho(p+q-1))}}) \\
&= (I) + (II) \,,
\end{aligned}
$$

where

$$
\begin{aligned}
(I) = &\sum_{\sigma \in S_{2,p+q-1}} \operatorname{sgn} \sigma \sum_{\rho \in S_{p-1,q}} \operatorname{sgn} \rho \; \Psi^1(\overline{[u_{\sigma(1)}, u_{\sigma(2)}]}, \overline{u_{\sigma(2+\rho(1))}}, \ldots, \overline{u_{\sigma(2+\rho(p-1))}}) \\
&\qquad \cdot \Psi^2(\overline{u_{\sigma(2+\rho(p))}}, \ldots, \overline{u_{\sigma(2+\rho(p+q-1))}})
\end{aligned}
$$

and

$$
\begin{aligned}
(II) = &\sum_{\sigma \in S_{2,p+q-1}} \operatorname{sgn} \sigma \sum_{\rho \in S_{p,q-1}} (-1)^p \operatorname{sgn} \rho \, \Psi^1(\overline{u_{\sigma(2+\rho(1))}}, \overline{u_{\sigma(2+\rho(2))}}, \ldots, \overline{u_{\sigma(2+\rho(p))}}) \\
&\qquad \cdot \Psi^2(\overline{[u_{\sigma(1)}, u_{\sigma(2)}]}, \overline{u_{\sigma(2+\rho(p+1))}}, \ldots, \overline{u_{\sigma(2+\rho(p+q-1))}}).
\end{aligned}
$$

Write

$$(III) \triangleq (\overline{\delta} \Psi^1 \wedge \Psi^2)(\overline{u_1} \wedge \ldots \wedge \overline{u_{p+q+1}})$$

and

$$(IV) \triangleq (\Psi^1 \wedge \overline{\delta} \Psi^2)(\overline{u_1} \wedge \ldots \wedge \overline{u_{p+q+1}}).$$

Then (III) is equal to

$$
\begin{aligned}
&\sum_{\sigma \in S_{p+1,q}} \operatorname{sgn} \sigma \, (\overline{\delta} \Psi^1)(\overline{u_{\sigma(1)}}, \overline{u_{\sigma(2)}}, \ldots, \overline{u_{\sigma(p+1)}}) \cdot \Psi^2 \left(\overline{u_{\sigma(p+2)}}, \ldots, \overline{u_{\sigma(p+q+1)}}\right) \\
&= \sum_{\sigma \in S_{p+1,q}} \operatorname{sgn} \sigma \sum_{\rho \in S_{2,p-1}} \operatorname{sgn} \rho \; \Psi^1(\overline{[u_{\sigma(\rho(1))}, u_{\sigma(\rho(2))}]}, \overline{u_{\sigma(\rho(3))}}, \ldots, \overline{u_{\sigma(\rho(p+1))}}) \\
&\qquad\qquad \cdot \Psi^2(\overline{u_{\sigma(p+2)}}, \ldots, \overline{u_{\sigma(p+q+1)}}).
\end{aligned}
$$

Corollary 3.1.4 yields $\Upsilon_q^{p-1,2}$ is a bijection between sets $S_{p+1,q} \times S_{2,p-1}$ and $S_{2,p-1,q}$. Observe that

$$\Upsilon_q^{p-1,2}(\sigma,\rho)=\begin{pmatrix} 1 & \dots & p+1 & p+2 & \dots & p+q+1 \\ \\ \sigma(\rho(1)) & \dots & \sigma(\rho(p+1)) & \sigma(p+2) & \dots & \sigma(p+q+1) \end{pmatrix}$$

and $\operatorname{sgn}(\Upsilon_q^{p-1,2}(\sigma,\rho))=\operatorname{sgn}\sigma\cdot\operatorname{sgn}\rho$ for $\sigma\in S_{p+1,q}$, $\rho\in S_{2,p-1}$. Therefore, (III) is equal to

$$\sum_{\tau\in S_{2,p-1,q}}\operatorname{sgn}\tau\,\Psi^1(\overline{[u_{\tau(1)},u_{\tau(2)}]},\overline{u_{\tau(3)}},\dots,\overline{u_{\tau(p+1)}})\cdot\Psi^2(\overline{u_{\tau(p+2)}},\dots,\overline{u_{\tau(p+q+1)}}).$$

By Theorem 3.1.1, the mapping $\Xi^{2,q,p-1}$ is a bijection between sets $S_{2,p+q-1}\times S_{p-1,q}$ and $S_{2,p-1,q}$. Moreover, for $\sigma\in S_{2,p+q-1}$ and $\rho\in S_{p-1,q}$, $\Xi^{2,q,p-1}(\sigma,\rho)$ is the permutation $\sigma\circ(2+\rho)$, which we can illustrate as follows

$$\begin{pmatrix} 1 & 2 & 3 & \dots & p+1 & \dots & p+q+1 \\ \\ \sigma(1) & \sigma(2) & \sigma(2+\rho(1)) & \dots & \sigma(2+\rho(p-1)) & \dots & \sigma(2+\rho(p+q-1)) \end{pmatrix}.$$

Hence, (I) is equal to

$$\begin{aligned}(I)=&\sum_{\sigma\in S_{2,p+q-1}}\operatorname{sgn}\sigma\sum_{\rho\in S_{p-1,q}}\operatorname{sgn}\rho\,\Psi^1(\overline{[u_{\sigma(1)},u_{\sigma(2)}]},\overline{u_{\sigma(2+\rho(1))}},\dots,\overline{u_{\sigma(2+\rho(p-1))}})\\ &\qquad\cdot\Psi^2(\overline{u_{\sigma(2+\rho(p))}},\dots,\overline{u_{\sigma(2+\rho(p+q-1))}})\\ =&\sum_{\tau\in S_{2,p-1,q}}\operatorname{sgn}\tau\,\Psi^1(\overline{[u_{\tau(1)},u_{\tau(2)}]},\overline{u_{\tau(3)}},\dots,\overline{u_{\tau(p+1)}})\\ &\qquad\cdot\Psi^2(\overline{u_{\tau(p+2)}},\dots,\overline{u_{\tau(p+q+1)}})\\ =&\,(III).\end{aligned}$$

By definition,

$$\begin{aligned}(IV)=&\sum_{\sigma\in S_{p,q+1}}\operatorname{sgn}\sigma\,\Psi^1(\overline{u_{\sigma(1)}},\overline{u_{\sigma(2)}},\dots,\overline{u_{\sigma(p+1)}})\\ &\qquad\cdot(\overline{\delta}\Psi^2)\left(\overline{u_{\sigma(p+2)}},\dots,\overline{u_{\sigma(p+q+1)}}\right)\\ =&\sum_{\sigma\in S_{p,q+1}}\operatorname{sgn}\sigma\sum_{\rho\in S_{2,q-1}}\operatorname{sgn}\rho\;\Psi^1(\overline{u_{\sigma(1)}},\overline{u_{\sigma(2)}},\dots,\overline{u_{\sigma(p)}})\\ &\qquad\cdot\Psi^2(\overline{[u_{\sigma(p+\rho(1))},u_{\sigma(p+\rho(2))}]},\overline{u_{\sigma(p+\rho(3))}},\dots,\overline{u_{\sigma(p+\rho(q+1))}})\end{aligned}$$

The mapping $\Xi^{p,q-1,2}:S_{p,q+1}\times S_{2,q-1}\xrightarrow{1-1}S_{p,2,q-1}$ given in Theorem

3.1.1 is a bijection between sets $S_{p,q+1} \times S_{2,q-1}$ and $S_{p,2,q-1}$ (cf. $\Xi^{r,s,t}$ in Theorem 3.1.1 for $r = p$, $s = q-1$, and $t = 2$). Observe that

$$\Xi^{r,s,2}(\sigma,\rho) = \begin{pmatrix} 1 & 2 & \cdots & p & p+1 & \cdots & p+q+1 \\ \\ \sigma(1) & \sigma(2) & \cdots & \sigma(p) & \sigma(2+\rho(1)) & \cdots & \sigma(p+\rho(q+1)) \end{pmatrix}$$

for $\sigma \in S_{p,q+1}$ and $\rho \in S_{2,q-1}$. Moreover, form Corollary 3.1.3 (for $r = 2$, $s = q-1$, and $t = p$) we conclude that $\Theta^{2,p}_{q-1} \circ \Xi^{2,q-1,p}$ is a bijection between sets $S_{2,p+q-1} \times S_{p,q-1}$ and $S_{p,2,q-1}$. For $\sigma \in S_{2,p+q-1}$, $\rho \in S_{p,q-1}$ we have:

$$\operatorname{sgn}(\Theta^{2,p}_{q-1} \circ \Xi^{2,q-1,p})(\sigma,\rho) = (-1)^{r \cdot t} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\rho$$

and $(\Theta^{2,p}_{q-1} \circ \Xi^{2,q-1,p})(\sigma,\rho)$ can be illustrate as follows

$$\begin{pmatrix} 1 & \cdots & p & p+1 & p+2 & p+3 & \cdots & p+q+1 \\ \\ \sigma(2+\rho(1)) & \cdots & \sigma(2+\rho(p)) & \sigma(1) & \sigma(2) & \sigma(2+\rho(p+1)) & \cdots & \sigma\left(2+\rho\left(p+q-1\right)\right) \end{pmatrix}.$$

Using bijections $\Xi^{p,q-1,2}$ and $\Theta^{2,p}_{q-1} \circ \Xi^{2,q-1,p}$, we conclude that

$$(-1)^p \cdot (II) = (IV).$$

Combining the equalities $(I) = (II)$ and $(-1)^p \cdot (II) = (IV)$ we get

$$\overline{\delta}\,(\Psi^1 \wedge \Psi^2)(\overline{u_1}, \ldots, \overline{u_{p+q+q}}) = (\overline{\delta}\,\Psi^1 \wedge \Psi^2 + (-1)^p\,\Psi^1 \wedge \overline{\delta}\,\Psi^2)(\overline{u_1}, \ldots, \overline{u_{p+q+q}}).$$

■

Theorem 3.1.4. $\overline{\delta} \circ \overline{\delta} = 0.$

Proof. Let $\Psi \in \Gamma(\bigwedge^n(\mathbf{g}/\mathbf{h})^*)$, $u_0, u_1, \ldots, u_{n+1} \in \Gamma(\mathbf{g})$. Let us denote by $\overline{u}$ the class $[u]$ represented by $u \in \Gamma(\mathbf{g})$. Observe that

$$\begin{aligned} &(\overline{\delta}\,(\,\overline{\delta}\,\Psi))(\overline{u_0}, \overline{u_1}, \ldots, \overline{u_{n+1}}) \\ &= \sum_{0 \le p < q \le n+1} (-1)^{p+q+1}\,(\overline{\delta}\,\Psi)(\overline{[u_p, u_q]}, \overline{u_0}, \ldots, \hat{p}, \ldots, \hat{q}, \ldots, \overline{u_{n+1}}) \\ &= \{A\} + \{B\}\,, \end{aligned}$$

where $\{A\}$ and $\{B\}$ denote the sums of these terms, in which the bracket $[u_p, u_q]$ appears in one and in two arguments, respectively. Therefore,

$$\{A\} =$$

$$= \sum_{r<p<q} (-1)^{(p+q+1)+(r+1)} \Psi(\overline{[[u_p, u_q], u_r]}, \overline{u_0}, ...\hat{r}...\hat{p}...\hat{q}..., \overline{u_{n+1}})$$

$$+ \sum_{p<r<q} (-1)^{(p+q+1)+((r-1)+1)} \Psi(\overline{[[u_p, u_q], u_r]}, \overline{u_0}, ...\hat{p}...\hat{r}...\hat{q}..., \overline{u_{n+1}})$$

$$+ \sum_{p<q<r} (-1)^{(p+q+1)+((r-2)+1)} \Psi(\overline{[[u_p, u_q], u_r]}, \overline{u_0}, ...\hat{p}...\hat{q}...\hat{r}..., \overline{u_{n+1}})$$

$$= \sum_{r<p<q} (-1)^{p+q+r} \ \Psi\left(\overline{[[u_p, u_q], u_r]}, \overline{u_0}, ...\hat{r}...\hat{p}...\hat{q}..., \overline{u_{n+1}}\right)$$

$$+ \sum_{r<p<q} (-1)^{p+q+r+1} \ \Psi\left(\overline{[[u_r, u_q], u_p]}, \overline{u_0}, ...\hat{r}...\hat{p}...\hat{q}..., \overline{u_{n+1}}\right)$$

$$+ \sum_{r<p<q} (-1)^{p+q+r} \ \Psi\left(\overline{[[u_r, u_p], u_q]}, \overline{u_0}, ...\hat{r}...\hat{p}...\hat{q}..., \overline{u_{n+1}}\right)$$

$$= \sum_{r<p<q} (-1)^{p+q+r} \ \Psi\left(\overline{\mathrm{Jac}_{[\cdot,\cdot]}(u_p, u_q, u_r)}, \overline{u_0}, ...\hat{r}...\hat{p}...\hat{q}..., \overline{u_{n+1}}\right)$$

$$= 0,$$

since in the one before last inequality we use the skew-symmetricity of the Lie bracket $[\cdot, \cdot]$, while in the last one the Jacobi identity. Moreover,

$$\{B\} =$$

$$= \sum_{r<s<p<q} (-1)^{(p+q+1)+((s+1)+(r+1)+1)} \Psi(\overline{[u_r, u_s]}, \overline{[u_p, u_q]}, \overline{u_0}, ...\hat{r}...\hat{s}...\hat{p}...\hat{q}..., \overline{u_{n+1}})$$

$$+ \sum_{r<p<s<q} (-1)^{(p+q+1)+((r+1)+s+1)} \Psi(\overline{[u_r, u_s]}, \overline{[u_p, u_q]}, \overline{u_0}, ...\hat{r}...\hat{p}...\hat{s}...\hat{q}..., \overline{u_{n+1}})$$

$$+\sum_{r<p<q<s}(-1)^{(p+q+1)+((r+1)+(s-1)+1)}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{r}...\hat{p}...\hat{q}...\hat{s}...,\overline{u_{n+1}})$$

$$+\sum_{p<r<s<q}(-1)^{(p+q+1)+(r+s+1)}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{p}...\hat{r}...\hat{s}...\hat{q}...,\overline{u_{n+1}})$$

$$+\sum_{p<r<q<s}(-1)^{(p+q+1)+(r+(s-1)+1)}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{p}...\hat{r}...\hat{q}...\hat{s}...,\overline{u_{n+1}})$$

$$+\sum_{p<q<r<s}(-1)^{(p+q+1)+((r-1)+(s-1)+1)}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{p}...\hat{q}...\hat{r}...\hat{s}...,\overline{u_{n+1}})$$

$$=0,$$

because the suitable terms cancel themselves. In fact, the sum of the first and sixth terms is equal to

$$\begin{aligned}
&\sum_{r<s<p<q}(-1)^{p+q+s+r}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{r}...\hat{s}...\hat{p}...\hat{q}...,\overline{u_{n+1}})\\
&\quad+\sum_{p<q<r<s}(-1)^{p+q+r+s}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{p}...\hat{q}...\hat{r}...\hat{s}...,\overline{u_{n+1}})\\
=&\sum_{r<s<p<q}(-1)^{p+q+s+r}\,\Psi(\overline{[u_r,u_s]},\overline{[u_p,u_q]},\overline{u_0},...\hat{r}...\hat{s}...\hat{p}...\hat{q}...,\overline{u_{n+1}})\\
&\sum_{r<s<p<q}(-1)^{p+q+r+s}\,\Psi(\overline{[u_p,u_q]},\overline{[u_r,u_s]},\overline{u_0},...\hat{r}...\hat{s}...\hat{p}...\hat{q}...,\overline{u_{n+1}})\\
=&\,0
\end{aligned}$$

since Ψ is skew-symmetric. Similarly, we show that the sum of the second and fifth terms is equal to zero, as is the sum of the third and fourth terms. ■

Relative Lie algebra cohomology. Let $\mathfrak{g}$ be the Lie algebra of a Lie group G and let $H \subset G$ be a closed Lie subgroup of G with the corresponding Lie algebra $\mathfrak{h}$. We recall [43] that $H^*(\mathfrak{g}, H)$, called the *relative Lie algebra cohomology,* is the cohomology space of the complex $\left(\bigwedge(\mathfrak{g}/\mathfrak{h})^{*H}, d^H\right)$ where $\bigwedge(\mathfrak{g}/\mathfrak{h})^{*H}$ is the space of invariant elements with respect to the adjoint representation of the Lie group H (cf. [43]) and the differential d^H is defined by the formula

$$\left\langle d^H(\psi), [w_0] \wedge ... \wedge [w_k] \right\rangle = \sum_{i<j} (-1)^{i+j} \langle \psi, [[w_i, w_j]] \wedge [w_1] \wedge ...\hat{\imath}...\hat{\jmath}... \wedge [w_k] \rangle$$

for $\psi \in \bigwedge^k (\mathfrak{g}/\mathfrak{h})^{*H}$ and $w_0, \ldots, w_k \in \mathfrak{g}$. We will introduce here a counterpart of this differential for the regular Lie algebroid and its subalgebroid.

Let $(A, \varrho_A, [\![\cdot,\cdot]\!])$ be a regular Lie algebroid over a manifold M with the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \longrightarrow A \xrightarrow{\varrho_A} F \longrightarrow 0$$

and let B its subalgebroid with the Atiyah sequence

$$0 \longrightarrow \boldsymbol{h} \longrightarrow B \xrightarrow{\varrho_A|B} F \longrightarrow 0.$$

Consider the adjoint representation of B in $A(\boldsymbol{g}/\boldsymbol{h})$ given by

$$\mathrm{ad}_{B,\boldsymbol{h}} : \Gamma(B) \longrightarrow \Gamma(\mathrm{A}(\boldsymbol{g}/\boldsymbol{h})) = \mathcal{CDO}(\boldsymbol{g}/\boldsymbol{h}),$$

$$\mathrm{ad}_{B,\boldsymbol{h}}(Y)([\nu]) = [[\![Y,\nu]\!]]$$

for $Y \in \Gamma(B)$, $\nu \in \Gamma(\boldsymbol{g})$.

Definition 3.1.4. A section $\Psi \in \Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)$ is *invariant* with respect to the representation

$$\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}} = \mathrm{Hom}(\mathrm{ad}_{B,\boldsymbol{h}}) : \Gamma(B) \longrightarrow \mathcal{CDO}(\mathrm{Hom}(\boldsymbol{g}/\boldsymbol{h}; \mathbb{R}))$$

if

$$\mathrm{Hom}(\mathrm{ad}_{B,\boldsymbol{h}})(Y)(\Psi) = 0$$

for all $Y \in \Gamma(B)$.

By definition we have the following characterization of invariant sections.

Lemma 3.1.1. (cf. [52]) *A section $\Psi \in \Gamma(\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*)$ is invariant with respect to the representation $\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}$ if and only if*

$$\begin{aligned}&(\varrho_B \circ Y)(\langle \Psi, [\nu_1] \wedge \ldots \wedge [\nu_p] \rangle)\\ &= \sum_{j=1}^{p} (-1)^{j-1} \langle \Psi, [[\![Y,\nu_j]\!]] \wedge [\nu_1] \wedge \ldots \hat{\jmath} \ldots \wedge [\nu_p] \rangle\end{aligned}$$

for all $Y \in \Gamma(B)$ and $\nu_1, \ldots, \nu_p \in \Gamma(\boldsymbol{g})$.

Definition 3.1.5. The set of all sections of the bundle $\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*$ invariant with respect to the representation $\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}$ is denoted by

$$(\Gamma(\textstyle\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}.$$

Remark 3.1.1. Since

$$\mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}(Y)(\Psi \wedge \Phi) = \mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}(Y)(\Psi) \wedge \Phi + \Psi \wedge \mathrm{ad}^{\wedge}_{B,\boldsymbol{h}}(Y)(\Phi)$$

for all $Y \in \Gamma(B)$, $\Psi, \Phi \in \Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)$, the exterior multiplication introduces the structure of a $C^\infty(M)$-algebra to the set $(\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$.

In the algebra $(\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$ of all invariant sections

$$\bar{\delta} : (\Gamma(\textstyle\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)} \longrightarrow (\Gamma(\textstyle\bigwedge^{p+1}(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$$

defined by

$$\left\langle \bar{\delta}\Psi, [\nu_0] \wedge ... \wedge [\nu_k] \right\rangle = \sum_{i<j} (-1)^{i+j+1} \langle \Psi, [[\![\nu_i, \nu_j]\!]] \wedge [\nu_1] \wedge ...\hat{\imath}...\hat{\jmath}... \wedge [\nu_k] \rangle$$

for $\Psi \in (\Gamma(\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$, $\nu_0, \ldots, \nu_p \in \Gamma(\boldsymbol{g})$, is a mapping being a differential ($\bar{\delta} \circ \bar{\delta} = 0$), cf. Theorem 3.1.4 and [52], and in this way $\bar{\delta}$ determines the cohomology algebra

$$\mathsf{H}^\bullet(\boldsymbol{g}, B) \triangleq \mathsf{H}^\bullet\left(\left(\Gamma\left(\textstyle\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*\right)\right)^{\Gamma(B)}, \bar{\delta}\right).$$

4. Secondary characteristic homomorphism

In this chapter, we present a construction of the secondary characteristic homomorphism for a pair of regular Lie algebroids (A, B) and a connection which curvature has values in the kernel of the reduction B. In particular, it generalizes the secondary characteristic classes previously formulated for flat connections [52], [10], [11].

In addition to the construction of the secondary characteristic homomorphism, in this chapter we discuss the functorial properties of this homomorphism and note that the characteristic classes from its image generalize other approaches, including those introduced by Kamber and Tondeur in [43] and the exotic classes defined by Crainic and Fernandes [18], [19].

4.1 Construction of the secondary characteristic homomorphism

Let $(A, \varrho_A, [\![\cdot,\cdot]\!])$ be a regular Lie algebroid over a manifold M with the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \xrightarrow{\iota} A \xrightarrow{\varrho_A} F \longrightarrow 0 \tag{4.1}$$

and let B its subalgebroid with the Atiyah sequence

$$0 \longrightarrow \boldsymbol{h} \longrightarrow B \xrightarrow{\varrho_A|B} F \longrightarrow 0. \tag{4.2}$$

Moreover, let $(L, \varrho_L, [\![\cdot,\cdot]\!]_L)$ be a Lie algebroid over M and let

$$\nabla : \Gamma(L) \longrightarrow \Gamma(A)$$

be an L-connection in A with the curvature tensor in values in $\boldsymbol{h}$.

Let $\lambda : F \to B$ be any splitting of (4.2), i.e., λ is a vector bundle homomorphism satisfying

$$\varrho_B \circ \lambda = \mathrm{id}_F, \tag{4.3}$$

where $\varrho_B = \varrho_A | B$. Consequently, λ is an F-connection in B (cf. Theorem 2.1.2). Thus, $j \circ \lambda : F \to A$ is an F-connection in A. Let us denote by $\breve{\lambda}$ the connection form of $j \circ \lambda$, i.e., a homomorphism of vector bundles $\breve{\lambda} : A \to \boldsymbol{g}$ such that

$$\iota \circ \breve{\lambda} + (j \circ \lambda) \circ \varrho_A = \mathrm{id}_A .$$

Thus, we have an L-connection $\nabla : \Gamma(L) \to \Gamma(A)$ in A with the curvature

$$R^\nabla \in \Gamma(\textstyle\bigwedge^2 L^* \otimes \boldsymbol{h})$$

and the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{g} \arrow[r, hook, shift left, "\iota"] & A \arrow[l, shift left, "\breve{\lambda}"] \arrow[r, "\varrho_A"] & F \arrow[r] & 0 \\
0 \arrow[r] & \boldsymbol{h} \arrow[r] \arrow[u, hook] & B \arrow[u, hook, "j"'] \arrow[r, shift left, "\varrho_B"] & F \arrow[u, equal] \arrow[l, shift left, "\lambda"] \arrow[r] & 0.
\end{tikzcd}$$

Lemma 4.1.1. *The homomorphism*

$$\lambda_{B,\nabla} : L \longrightarrow \boldsymbol{g}/\boldsymbol{h}, \qquad \lambda_{B,\nabla}(u) = [-(\breve{\lambda} \circ \nabla)(u)]$$

does not depend on the choice of an auxiliary connection $\lambda : F \to A$, and $\lambda_{B,\nabla} = 0$ if ∇ takes values in B.

Proof. Let $\lambda' : F \to B$ also be a splitting of (4.2). Thus,

$$\varrho_B \circ \lambda' = \mathrm{id}_F$$

and $j \circ \lambda' : F \to A$ is a splitting of the sequence (4.1). Since

$$\varrho_B \circ (\lambda' - \lambda') \circ \varrho_A \circ \nabla = (\mathrm{id}_F - \mathrm{id}_F) \circ \varrho_A \circ \nabla = 0$$

and

$$\iota \circ \left(\breve{\lambda} - \breve{\lambda}'\right) \circ \nabla = j \circ (\lambda' - \lambda) \circ \varrho_A \circ \nabla,$$

it is follows that

$$(\breve{\lambda} \circ \nabla - \breve{\lambda}' \circ \nabla)(u) \in \Gamma(\boldsymbol{h})$$

for all $u \in \Gamma(L)$. Hence the independence of the definition of $\lambda_{B,\nabla}$ from the choice of the connection λ. Thus, indeed, $\lambda_{B,\nabla}$ is a well-defined function. ■

Let us define a homomorphism of algebras

$$\Delta_{(A,B,\nabla)} : (\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)} \longrightarrow \Gamma(\bigwedge L^*)$$

by

$$(\Delta_{(A,B,\nabla)}\Psi)_x(u_1 \wedge \ldots \wedge u_p) = \langle \Psi_x, \lambda_{B,\nabla}(u_1) \wedge \ldots \wedge \lambda_{B,\nabla}(u_p)\rangle$$

for $\Psi \in (\Gamma(\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$, $x \in M$, $u_1, \ldots, u_p \in \Gamma(L)$.

Definition 4.1.1. We use the convention that $\overline{\nu} = [\nu]$ for $\nu \in \Gamma(\boldsymbol{g})$.

Theorem 4.1.1. *The homomorphism $\Delta_{(A,B,\nabla)}$ commutes with the differentials $\bar{\delta}$ and d_L, where $d_L = d^{\varrho_L}$ is the differential in $\Gamma(\bigwedge L^*)$ for the Lie algebroid L.*

Proof. Take any $\Psi \in (\Gamma(\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$, $u_0, u_1, \ldots, u_p \in \Gamma(L)$. Then,

$$\begin{aligned}
&((d_L \circ \Delta_{(A,B,\nabla)})\Psi)(u_0, u_1, \ldots, u_p) \\
&= \sum_{s=0}^{p} (-1)^s \, (\varrho_L \circ u_s)((\Delta_{(A,B,\nabla)}\Psi)(u_0, \ldots \hat{s} \ldots, u_p)) \\
&- \sum_{s<t} (-1)^{s+t} \, (\Delta_{(A,B,\nabla)}\Psi)([\![u_s, u_t]\!]_L, u_0, \ldots \hat{s} \ldots \hat{t} \ldots, u_p) \\
&= \sum_{s=0}^{p} (-1)^s \, (\varrho_L \circ u_s)\Big(\Psi\Big(\overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)\Big) \\
&+ \sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{-\breve{\lambda}\nabla[\![u_s, u_t]\!]_L}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big).
\end{aligned}$$

Using the equalities

$$\varrho_L = \varrho_A \circ \nabla, \qquad \varrho_B \circ \lambda = \mathrm{id}_F, \qquad j \circ \lambda \circ \varrho_A = \mathrm{id}_A - \iota \circ \breve{\lambda},$$

and the invariance of Ψ, we obtain

$$\begin{aligned}
&\sum_{s=0}^{p} (-1)^s \, (\varrho_L \circ u_s)\Big(\Psi\Big(\overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)\Big) \\
&= \sum_{s=0}^{p} (-1)^s \, (\mathrm{id}_F \circ \varrho_A \circ \nabla \circ u_s)\Big(\Psi\Big(\overline{-\breve{\lambda}\nabla u_0}, \ldots\hat{s}\ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)\Big)
\end{aligned}$$

$$= \sum_{s=0}^{p} (-1)^s \, (\varrho_B \circ \lambda \circ \varrho_A \circ \nabla \circ u_s)\Big(\Psi\Big(\overline{-\breve{\lambda}\nabla u_0}, ...\hat{s}..., \overline{-\breve{\lambda}\nabla u_p}\Big)\Big)$$

$$= \sum_{t<s} (-1)^{s+t} \, \Psi\Big(\overline{[\![\lambda\varrho_A\nabla u_s, -\breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, ...\hat{t}...\hat{s}..., \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$+\sum_{s<t} (-1)^{s+t+1} \, \Psi\Big(\overline{[\![\lambda\varrho_A\nabla u_s, -\breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$= \sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{[\![\lambda\varrho_A\nabla u_t, -\breve{\lambda}\nabla u_s]\!]}, \overline{-\breve{\lambda}\nabla u_0}, ...\hat{s}...\hat{t}..., \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$+\sum_{s<t} (-1)^{s+t+1} \, \Psi\Big(\overline{[\![\lambda\varrho_A\nabla u_s, -\breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$= \sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{[\![\breve{\lambda}\nabla u_s, \nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, ...\hat{s}...\hat{t}..., \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$-\sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{[\![\breve{\lambda}\nabla u_s, \breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, ...\hat{s}...\hat{t}..., \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$+\sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{[\![\nabla u_s, \breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$+\sum_{s<t} (-1)^{s+t+1} \, \Psi\Big(\overline{[\![\breve{\lambda}\nabla u_s, \breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$= \sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{[\![\breve{\lambda}\nabla u_s, \nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, ...\hat{s}...\hat{t}..., \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$+\sum_{s<t} (-1)^{s+t} \, \Psi\Big(\overline{[\![\nabla u_s, \breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big)$$

$$+ 2\sum_{s<t} (-1)^{s+t+1} \, \Psi\Big(\overline{[\![\breve{\lambda}\nabla u_s, \breve{\lambda}\nabla u_t]\!]}, \overline{-\breve{\lambda}\nabla u_0}, \ldots \hat{s} \ldots \hat{t} \ldots, \overline{-\breve{\lambda}\nabla u_p}\Big) .$$

Moreover, the curvature of the connection $j \circ \lambda$ and the curvature of ∇ are in the following relation

$$\mathcal{R}^{j\circ\lambda}(\varrho_A\nabla u_s, \varrho_A\nabla u_t) - \breve{\lambda}(\mathcal{R}^{\nabla}(u_s, u_t))$$

$$= [\![\breve{\lambda}\nabla u_s, \nabla u_t]\!] + [\![\nabla u_s, \breve{\lambda}\nabla u_t]\!] - [\![\breve{\lambda}\nabla u_s, \breve{\lambda}\nabla u_t]\!] - \breve{\lambda}\nabla[\![u_s, u_t]\!]_L .$$

Since

$$\mathcal{R}^{j\circ\lambda} = j \circ \mathcal{R}^{\lambda}$$

and the curvature of ∇ has values in $\boldsymbol{h}$, we obtain that

$$\llbracket\breve{\lambda}\nabla u_s, \nabla u_t\rrbracket + \llbracket\nabla u_s, \breve{\lambda}\nabla u_t\rrbracket - \llbracket\breve{\lambda}\nabla u_s, \breve{\lambda}\nabla u_t\rrbracket - \breve{\lambda}\nabla\llbracket u_s, u_t\rrbracket_L$$

$$= (j \circ \mathcal{R}^{\lambda})(\varrho_A\nabla u_s, \varrho_A\nabla u_t) - \breve{\lambda}(\mathcal{R}^{\nabla}(u_s, u_t)) \in \boldsymbol{h}.$$

From here it follows that

$$\begin{aligned}
&((d_L \circ \Delta_{(A,B,\nabla)})\Psi)(u_0, u_1, \dots, u_p) \\
&= \sum_{s<t} (-1)^{s+t}\, \Psi\Big(\overline{-\breve{\lambda}\nabla\llbracket u_s, u_t\rrbracket_L}, \overline{-\breve{\lambda}\nabla u_0}, \dots \hat{s} \dots \hat{t} \dots, \overline{-\breve{\lambda}\nabla u_p}\Big) \\
&= \sum_{s<t} (-1)^{s+t+1}\, \Psi\Big(\overline{\llbracket-\breve{\lambda}\nabla u_s, -\breve{\lambda}\nabla u_t\rrbracket}, \overline{-\breve{\lambda}\nabla u_0}, \dots \hat{s} \dots \hat{t} \dots, \overline{-\breve{\lambda}\nabla u_p}\Big) \\
&= (\bar{\delta}\Psi)\Big(\overline{-\breve{\lambda}\nabla u_0}, \overline{-\breve{\lambda}\nabla u_1}, \dots, \overline{-\breve{\lambda}\nabla u_p}\Big) \\
&= ((\Delta_{(A,B,\nabla)} \circ \bar{\delta})\Psi)(u_0, u_1, \dots, u_p).
\end{aligned}$$

■

Since the homomorphism $\Delta_{(A,B,\nabla)}$ commutes with the differentials $\bar{\delta}$ and d_L, we obtain the cohomology homomorphism

$$\Delta_{(A,B,\nabla)\#} : \mathsf{H}^{\bullet}(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^{\bullet}(L),$$

$$[\Psi] \longmapsto \left[\Delta_{(A,B,\nabla)}(\Psi)\right].$$

In the case where $L = A$ and $\nabla = \mathrm{id}_A : A \longrightarrow A$ is the identity map (id_A is a flat connection), we have the following particular homomorphism for the pair (A, B):

$$\Delta_{(A,B)} = \Delta_{(A,B,\mathrm{id}_A)} : (\Gamma(\textstyle\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)} \longrightarrow \Gamma(\textstyle\bigwedge A^*),$$

$$(\Delta_{(A,B)}\Psi)_x(u_1 \wedge \ldots \wedge u_p) = \Big\langle \Psi_x, [-\breve{\lambda} \circ u_1] \wedge \ldots \wedge [-\breve{\lambda} \circ u_k] \Big\rangle$$

for $\Psi \in (\Gamma(\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$, $x \in M$, $u_1, \dots, u_p \in \Gamma(A)$.

The homomorphism $\Delta_{(A,B,\nabla)}$ can be written as a composition

$$\Delta_{(A,B,\nabla)} : (\Gamma(\textstyle\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)} \xrightarrow{\Delta_{(A,B)}} \Gamma(\textstyle\bigwedge A^*) \xrightarrow{\nabla^*} \Gamma(\textstyle\bigwedge L^*),$$

where ∇^* is the pullback of differential forms on the Lie algebroids. If ∇ is flat, ∇^* commutes with the differentials $d_A = d^{\varrho_A}$ and $d_L = d^{\varrho_L}$. For this reason, $\Delta_{(A,B)}$ induces the cohomology homomorphism

$$\begin{aligned} \Delta_{(A,B)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) &\longrightarrow \mathsf{H}^\bullet(A), \\ [\Psi] &\longmapsto \left[\Delta_{(A,B)}(\Psi)\right], \end{aligned}$$

which for any flat L-connection $\nabla : \Gamma(L) \longrightarrow \Gamma(A)$ factorizes $\Delta_{(A,B,\nabla)\#}$ in the sense that the following diagram is commutative

$$\begin{tikzcd} \mathsf{H}^\bullet(\boldsymbol{g}, B) \arrow[r, "\Delta_{(A,B)\#}"] \arrow[dr, "\Delta_{(A,B,\nabla)\#}"'] & \mathsf{H}^\bullet(A) \arrow[d, "\nabla^\#"] \\ & H^\bullet(L). \end{tikzcd} \tag{4.4}$$

Definition 4.1.2. The mapping $\Delta_{(A,B,\nabla)\#}$ is called the *secondary characteristic homomorphism of* (A, B, ∇)*.*

Definition 4.1.3. We call elements of a subalgebra $\operatorname{Im} \Delta_{(A,B,\nabla)\#} \subset \mathsf{H}^\bullet(L)$ *the secondary characteristic classes* of (A, B, ∇)*.*

Definition 4.1.4. The homomorphism $\Delta_{(A,B)\#} = \Delta_{(A,B,\mathrm{id}_A)\#}$ we call the *universal secondary characteristic homomorphism.* The characteristic classes from the image of $\Delta_{(A,B)\#}$ we call the *universal secondary characteristic classes* of the pair (A, B).

If connection ∇ takes the value in B, then $\Delta_{(A,B,\nabla)\#}$ is trivial. Thus, the non-triviality of $\Delta_{(A,B,\nabla)\#}$ is an obstruction to the compatibility of ∇ with the structure of the subalgebroid B. In the case of the Lie algebroid of a vector bundle and its reduction with respect to a given Riemann metric discussed below, the non-triviality of the secondary characteristic homomorphism means that ∇ cannot be compatible with the metric.

One can see that for a pair of regular Lie algebroids (A, B), $B \subset A$, both over a manifold M with $\operatorname{Im} \varrho_A = \operatorname{Im} \varrho_B = F$, and for an arbitrary element $\Psi \in \mathsf{H}^\bullet(\boldsymbol{g}, B)$ there exists a (universal) cohomology class $\Delta_{(A,B)\#}(\Psi) \in \mathsf{H}^\bullet(A)$ such that for any Lie algebroid L over M and a flat L-connection $\nabla : \Gamma(L) \longrightarrow \Gamma(A)$ the equality

$$\Delta_{(A,B,\nabla)\#}(\Psi) = \nabla^\#(\Delta_{(A,B)\#}(\Psi))$$

holds. Therefore, no element from the kernel of $\Delta_{(A,B)\#}$ can be used to compare the flat connection ∇ with a reduction $B \subset A$. Hence, it is natural to ask: Is the characteristic homomorphism $\Delta_{(A,B)\#}$ a monomorphism for a given $B \subset A$? The answer yes holds in some cases; see the sections below.

4.2 Functoriality of the secondary characteristic homomorphism

In this section, we discuss some functorial properties of secondary characteristic homomorphisms. The relationships between such characteristic homomorphisms for Lie algebroids over various manifolds are discussed here. Thus, morphisms of Lie algebroids over smooth mappings other than identity take on significance here. We start with the concept of such morphisms.

Definition 4.2.1. (cf. [41], [50]) Let $(A, \varrho, [\![\cdot,\cdot]\!])$ and $(A', \varrho', [\![\cdot,\cdot]\!]')$ be Lie algebroid manifolds M and M', respectively. By a *homomorphism*

$$H : (A', \varrho', [\![\cdot,\cdot]\!]') \longrightarrow (A, \varrho, [\![\cdot,\cdot]\!])$$

of Lie algebroids we mean any homomorphism of vector bundles

$$H : A' \longrightarrow A$$

over a smooth map $f : M' \longrightarrow M$ with the following properties:

1. $\varrho \circ H = f_* \circ \varrho'$,
2. for any $\xi, \xi' \in \Gamma(A')$ with H-decompositions

$$H \circ \xi = \sum_{i=1}^{n} f^i \cdot (\eta_i \circ f), \quad H \circ \xi = \sum_{j=1}^{n} g^j \cdot \big(\eta_j \circ f\big),$$

where f^i, $g^j \in C^\infty(M')$, $\eta_1, ..., \eta_n \in \Gamma(A)$, we have

$$H \circ [\![\xi, \xi']\!]' = \sum_{i,j} f^i g^j ([\![\eta_i, \eta_j]\!] \circ f) + \sum_j (\varrho' \circ \xi)(g^j) \cdot (\eta_j \circ f) - \sum_i (\varrho' \circ \xi')(f^i) \cdot (\eta_i \circ f).$$

Remark 4.2.1. Let $(A, \varrho, [\![\cdot,\cdot]\!])$ and $(A', \varrho', [\![\cdot,\cdot]\!]')$ be regular Lie algebroids over (M, F) and (M', F'), respectively. This means that $\operatorname{Im} \varrho = F$ and $\operatorname{Im} \varrho' = F'$. Let $H : A' \longrightarrow A$ be a homomorphism of these Lie algebroids over smooth map $f : M' \longrightarrow M$. Then, the homomorphism of vector bundles

$$\overline{H} : A' \longrightarrow f^{\wedge}A, \qquad a' \longmapsto (\varrho'(a'), H(a'))$$

is a homomorphism of Lie algebroids (cf. Proposition 1.1.5 [50]). Moreover, H can be write as a composition of the homomorphism of regular Lie algebroids $\overline{H}$ and the projection to the second coordinate $\mathrm{pr}_2 : f^{\wedge}A \longrightarrow A$,

$$H : A' \xrightarrow{\overline{H}} f^{\wedge}A \xrightarrow{\mathrm{pr}_2} A.$$

Remark 4.2.2. Definition of homomorphism of Lie algebroids over the same manifold M (cf. Definition 1.1.6) coincide with Definition 4.2.1 in the case $f = \mathrm{id}_M$. So, Definition 4.2.1 is a generalization of 1.1.6.

Let (A, B) and (A', B') be two pairs of regular Lie algebroids over (M, F) and (M', F'), respectively, where $B \subset A$, $B' \subset A'$, and let $H : A' \longrightarrow A$ be a homomorphism of Lie algebroids over a mapping $f : (M', F') \longrightarrow (M, F)$ of foliated manifolds such that $H(B') \subset B$, which means that $f_*(F) \subset F'$. We write $(H, f) : (A', B') \longrightarrow (A, B)$.

Let $H^{+\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \to \mathsf{H}^\bullet(\boldsymbol{g}', B')$ be the homomorphism of cohomology algebras induced by the pullback $H^{+*} : \Gamma(\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^*) \longrightarrow \Gamma(\bigwedge^k(\boldsymbol{g}'/\boldsymbol{h}')^*)$ — cf. [52, Proposition 4.2].

Theorem 4.2.1 (The functoriality of $\Delta_{(A,B)\#}$). [11] *For a given pair of regular Lie algebroids (A, B), (A', B') and a homomorphism $(H, f) : (A', B') \longrightarrow (A, B)$ we have the commutativity of the following diagram*

$$\begin{tikzcd}
\mathsf{H}^\bullet(\boldsymbol{g}, B) \arrow[r, "\Delta_{(A,B)\#}"] \arrow[d, "H^{+\#}"'] & \mathsf{H}^\bullet(A) \arrow[d, "H^{\#}"] \\
\mathsf{H}^\bullet(\boldsymbol{g}', B') \arrow[r, "\Delta_{(A',B')\#}"'] & \mathsf{H}^\bullet(A').
\end{tikzcd}$$

Definition 4.2.2. The triple (A, B, ∇) is called a *regular FS-Lie algebroid* if A is a regular Lie algebroid over (M, F), $B \subset A$ is a regular subalgebroid of A over (M, F) and $\mathrm{Sec}\,\nabla : \Gamma(L) \longrightarrow \Gamma(A)$ is a flat L-connection in A, where L is a Lie algebroid over a manifold M.

Definition 4.2.3. Let (A', B', ∇') and (A, B, ∇) be two regular FS-Lie algebroids over (M', F') and (M, F), respectively, and where $\nabla : L \longrightarrow A$, $\nabla' : L' \longrightarrow A'$. By a *homomorphism of regular FS-Lie algebroids*

$$H : (A', B', \nabla') \longrightarrow (A, B, \nabla)$$

over $f : (M', F') \longrightarrow (M, F)$ we mean a pair (H, h) such that:

1. $H : A' \longrightarrow A$ is a homomorphism of regular Lie algebroids over f satisfying $H(B') \subset B$,
2. $h : L' \longrightarrow L$ is a homomorphism of Lie algebroids over f,
3. $\nabla \circ h = H \circ \nabla'$.

Since $h^{\#} \circ \nabla^{\#} = \nabla'^{\#} \circ H^{\#}$, from flatness of ∇, the commutativity of the diagram (4.4) and Theorem 4.2.1, it follows the following theorem.

Theorem 4.2.2 (*The functoriality of* $\Delta_{(A,B,\nabla)\#}$). [11] *The following diagram commutes*

$$\begin{array}{ccc}
\mathsf{H}^\bullet(\boldsymbol{g}, B) & \xrightarrow{\Delta_{(A,B,\nabla)\#}} & \mathsf{H}^\bullet(L) \\
{\scriptstyle H^{+\#}}\downarrow & & \downarrow{\scriptstyle h^{\#}} \\
\mathsf{H}^\bullet(\boldsymbol{g}', B') & \xrightarrow[\Delta_{(A',B',\nabla')\#}]{} & \mathsf{H}^\bullet(L').
\end{array}$$

4.3 Homotopy invariance

We recall the definition of homotopy between homomorphisms of Lie algebroids.

Definition 4.3.1. [51] Let H_0, $H_1 : L' \longrightarrow L$ be two homomorphisms of Lie algebroids. By a *homotopy joining H_0 to H_1* we mean a homomorphism of Lie algebroids

$$H : T\mathbb{R} \times L' \longrightarrow L,$$

such that $H(\theta_0, \cdot) = H_0$ and $H(\theta_1, \cdot) = H_1$, where θ_0 and θ_1 are null vectors tangent to $\mathbb{R}$ at 0 and 1, respectively. We then say that H_0 and H_1 are *homotopic* and write $H_0 \sim H_1$.

Definition 4.3.2. We say that homomorphism of Lie algebroids $H : L' \longrightarrow L$ is a *homotopy equivalence* if there is a homomorphism of Lie algebroids $G : L \longrightarrow L'$ such that $G \circ H \sim \mathrm{id}_{L'}$ and $H \circ G \sim \mathrm{id}_L$.

Remark 4.3.1. Let H_0, $H_1 : L' \longrightarrow L$ be two homotopic homomorphisms of Lie algebroids with a homotopy $H : T\mathbb{R} \times L' \longrightarrow L$. The homomorphism H determines a chain homotopy operator (cf. [51], [3]) which yields $H_0^{\#} = H_1^{\#} : \mathsf{H}^\bullet(L) \longrightarrow \mathsf{H}^\bullet(L')$.

Definition 4.3.3. [51] Two Lie subalgebroids B_0, $B_1 \subset A$ of a Lie algebroid A (all over (M, F)) are said to be *homotopic*, if there exists a Lie subalgebroid $B \subset T\mathbb{R} \times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times F)$, such that

$$\upsilon_x \in B_{t|x} \quad \text{if and only if} \quad (\theta_t, \upsilon_x) \in B_{|(t,x)} \tag{4.5}$$

for $t \in \{0, 1\}$. B is then called a *subalgebroid joining* B_0 *with* B_1.

Let B_0, B_1 be two homotopic Lie subalgebroids over (M, F) and let $B \subset T\mathbb{R} \times A$ be a subalgebroid of $T\mathbb{R} \times A$ joining B_0 with B_1. Consider the homomorphism of Lie algebroids

$$F_t^A : A \longrightarrow T\mathbb{R} \times A, \quad \upsilon_x \longmapsto (\theta_t, \upsilon_x)$$

over $f_t : M \longrightarrow \mathbb{R} \times M$, $f_t(x) = (t, x)$, for $t \in \mathbb{R}$. Now, (4.5) shows that $F_t^A(B_t) \subset B$. Moreover, $F_t^{+\#} \equiv \left(F_t^A\right)^{+\#}$ are isomorphisms of algebras. This was proved in [52]. Let $\nabla : L \longrightarrow A$ be a homomorphism of Lie algebroids over id_M. Therefore, $\mathrm{id}_{T\mathbb{R}} \times \nabla$ is a homomorphism of Lie algebroids over $\mathrm{id}_{\mathbb{R}\times M}$. In this way F_t^A defines a homomorphism

$$(A, B_t, \nabla) \longrightarrow (T\mathbb{R} \times A, B, \mathrm{id}_{T\mathbb{R}} \times \nabla)$$

of FS-Lie algebroids over f_t. Now, using the above mentioned properties and Theorem 4.2.1 giving the functoriality of $\Delta_{t\,\#} = \Delta_{(A,B_t)\,\#}$ and $\Delta_{(A,B)\,\#}$, we obtain the following commutative diagram

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathsf{H}^\bullet(\boldsymbol{g}, B_0) \arrow[r, "\Delta_{0\#}"] \arrow[dd, bend right=60, "\alpha"'] & \mathsf{H}^\bullet(A) \arrow[r, "\nabla^{\#}"] & \mathsf{H}^\bullet(L) \\
\mathsf{H}^\bullet(0\times\boldsymbol{g}, B) \arrow[u, "{F_0}^{+\#}", "\simeq"'] \arrow[d, "{F_1}^{+\#}"', "\simeq"] \arrow[r, "\Delta_{(A,B)\#}"'] & \mathsf{H}^\bullet(T\mathbb{R}\times A) \arrow[u, "{F_0}^{A\#}"'] \arrow[d, "{F_1}^{A\#}"] \arrow[r, "(\mathrm{id}\times\nabla)^{\#}"'] & \mathsf{H}^\bullet(T\mathbb{R}\times L) \arrow[u, "{F_0}^{L\#}"'] \arrow[d, "{F_1}^{L\#}"] \\
\mathsf{H}^\bullet(\boldsymbol{g}, B_1) \arrow[r, "\Delta_{1\#}"'] & \mathsf{H}^\bullet(A) \arrow[r, "\nabla^{\#}"'] & \mathsf{H}^\bullet(L)
\end{tikzcd}$$

Remark that the rows of the above diagram are characteristic homomorphisms of regular FS-Lie algebroids. Moreover, $F_0^{L\#} = F_1^{L\#}$, since $F_0^L, F_1^L : L \longrightarrow T\mathbb{R} \times L$ are homotopic homomorphisms. Since $F_0^{L\#}, F_1^{L\#}$ are isomorphisms (cf. [11]), we immediate conclude following theorem on the homotopy independence of the secondary characteristic homomorphism.

Theorem 4.3.1 (The Rigidity Theorem). (cf. [10]) *If B_0, $B_1 \subset A$ are homotopic subalgebroids of A and $\nabla : L \longrightarrow A$ is a flat L-connection in A, characteristic homomorphisms $\Delta_{(A,B_0,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B_0) \longrightarrow \mathsf{H}^\bullet(L)$ and $\Delta_{(A,B_1,\nabla)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B_1) \longrightarrow H_L(M)$ are equivalent in the sense that there exists an isomorphism of algebras*

$$\alpha : \mathsf{H}^\bullet(\boldsymbol{g}, B_0) \xrightarrow{\simeq} \mathsf{H}^\bullet(\boldsymbol{g}, B_1)$$

such that

$$\Delta_{(A,B_1,\nabla)\#} \circ \alpha = \Delta_{(A,B_0,\nabla)\#}.$$

Let E be a vector bundle of rank n over a manifold M and let h be a Riemannian metric in E. The metric h determines the reduction $L_{(E,\{h\})}$ of the principal frames bundle L_E of E and the Lie subalgebroid $A(E,\{h\})$ of the Lie algebroid $A(E)$ [52]. Note that $\xi : \mathbb{R}^n \to E_x$ is an element of $L_{(E,\{h\})}$ if and only if ξ is an isometry. Taking the canonical isomorphism $\Phi_E : A(L_E) \longrightarrow A(E)$ described in Section 1.3 (cf. [50]) we define

$$A(E,\{h\}) = \Phi_E\left(A(L_{(E,\{h\})})\right).$$

We observe that $a \in \Gamma(A(E))$ is an element of $\Gamma(A(E,\{h\}))$ if and only if for any sections $\nu, \mu \in \Gamma(E)$ we have

$$h(a(\nu), \mu) = (\varrho_{A(E)} \circ a)(h(\nu,\mu)) - h(\nu, a(\mu)).$$

We recall that

$$0 \longrightarrow \mathrm{End}(E) \xrightarrow{\iota} A(E) \xrightarrow{\varrho_{A(E)}} TM \longrightarrow 0$$

is the Atiyah sequence of $A(E)$, while the Atiyah sequence of $A(E,\{h\})$ is

$$0 \longrightarrow \mathrm{Sk}(E) \longrightarrow A(E,\{h\}) \longrightarrow TM \longrightarrow 0$$

where $\mathrm{Sk}(E) \subset \mathrm{End}(E)$ is the vector subbundle of skew symmetric endomorphisms with respect to the metric h.

Applying the Rigidity Theorem to a pair of Riemannian reductions of the Lie algebroid $A(E)$ of a vector bundle E, we obtain the independence of the secondary characteristic homomorphism for the pair $A(E,\{h\}) \subset A(E)$ from a metric:

Corollary 4.3.1. [10] *Let E be a vector bundle and $A(E)$ its Lie algebroid. Two Lie subalgebroids $B_0 = A(E,\{h_0\})$, $B_1 = A(E,\{h_1\})$ of the Lie algebroid $A(E)$, corresponding to Riemannian metrics h_0, h_1, are homotopic Lie subalgebroids* [52]*. Therefore, according to the Rigidity Theorem* 4.3.1 *we conclude that*

$$\varDelta_{(A(E),A(E,\{h_0\}))\#} = \varDelta_{(A(E),A(E,\{h_1\}))\#},$$

i.e., the characteristic homomorphism for the pair $(A(E), A(E,\{h\}))$ is an intrinsic notion for $A(E)$ not depending on the metric h.

4.4 Particular cases of the secondary characteristic classes

The secondary characteristic homomorphism for Lie algebroids generalizes flat exotic characteristic classes for Lie algebroids and the following known characteristic classes: for flat regular Lie algebroids (Kubarski, cf. [52]), for flat principal fibre bundles with a reduction (Kamber, Tondeur, cf. [43]) and for representations of Lie algebroids on vector bundles (Crainic, Fernandes, cf. [17], [18], [19]).

4.4.1 Comparison with the case of regular Lie algebroids and classical flat connections

Given a regular Lie algebroid $(A, \varrho_A, [\cdot,\cdot])$ over a manifold M with $F = \mathrm{Im}\,\varrho_A$ and $\boldsymbol{g} = \ker \varrho_A$, consider two geometric structures:

- a flat connection $\omega : \varGamma(F) \longrightarrow \varGamma(A)$, and
- a Lie subalgebroid $j : B \hookrightarrow A$ over (M, F), i.e., with the anchor ϱ_B such that $\mathrm{Im}\,\varrho_B = F$.

Let $\breve{\omega} : A \longrightarrow \boldsymbol{g}$ be the connection form of ω. Fix a connection $\lambda : F \longrightarrow B$, and consider its extension $j \circ \lambda$ to A. Let $\breve{\lambda} : A \longrightarrow \boldsymbol{g}$ be its connection form. Since

$$\iota \circ \breve{\omega} + \omega \circ \varrho_A = \mathrm{id}_A$$

where $\iota : \boldsymbol{g} \hookrightarrow A$ is the inclusion, it follows that

$$\iota \circ \breve{\omega} \circ j \circ \lambda = -\iota \circ \breve{\lambda} \circ \omega.$$

Thus, we see that

$$\begin{aligned}
&\left(\varDelta_{(A,B,\omega)}\varPsi\right)_x (u_1 \wedge \ldots \wedge u_p) \\
&= \left\langle \varPsi_x, \left[-(\breve{\lambda} \circ \omega)(u_1)\right] \wedge \ldots \wedge \left[-(\breve{\lambda} \circ \omega)(u_p)\right]\right\rangle \\
&= \langle \varPsi_x, [\breve{\omega}_x(\widetilde{u}_1)] \wedge \ldots \wedge [\breve{\omega}_x(\widetilde{u}_p)]\rangle
\end{aligned}$$

for $\varPsi \in (\varGamma(\bigwedge^p(\boldsymbol{g}/\boldsymbol{h})^*))^{\varGamma(B)}$, $x \in M$, $u_1, \ldots, u_p \in \varGamma(F)$, and where $\widetilde{u}_i = \lambda(u_i)$ for $i = 1, \ldots, p$. Since $\varrho_B(\widetilde{u}_i) = u_i$, we deduce that

$$\varDelta_{(A,B,\omega)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, B) \longrightarrow \mathsf{H}^\bullet(F)$$

is the characteristic homomorphism for the regular flat Lie algebroids (A, B, ω), which was considered by Kubarski in [52].

4.4.2 Secondary universal characteristic homomorphism of principal fibre subbundles

We recall secondary flat characteristic classes for flat principal bundles [43] and its relationship with the secondary characteristic homomorphism for a suitable pair of Lie algebroids.

Let P be a G-principal fibre bundle on a smooth manifold M, $\omega \subset TP$ a flat connection in P, $H \subset G$ a closed Lie subgroup of G, and $P' \subset P$ a connected H-reduction. Consider Lie algebroids $A(P)$ and $A(P')$ of the principal bundles P and P', respectively. Let $\omega^A : TM \longrightarrow A(P)$ be the induced flat connection in the Lie algebroid $A(P)$. The triple $\left(A(P), A(P'), \omega^A\right)$ defines the secondary characteristic homomorphism

$$\varDelta_{(A(P),A(P'),\omega^A)\#} : \mathsf{H}^\bullet(\boldsymbol{g}, A(P')) \longrightarrow \mathsf{H}^\bullet_{dR}(M).$$

Moreover, let

$$\varDelta_{(P,P',\omega)\#} : \mathsf{H}^\bullet(\mathfrak{g}, H) \longrightarrow \mathsf{H}^\bullet_{dR}(M) \tag{4.6}$$

denote the classical homomorphism on principal fibre bundles (Kamber, Tondeur [43]), where $\mathsf{H}^\bullet(\mathfrak{g}, H)$ is the relative Lie algebra cohomology of $(\mathfrak{g}, H)$ (see [43], [14], and Section 3.1). The characteristic homomorphism $\varDelta_{(P,P',\omega)\#}$ is constructed as follows: Let $\breve{\omega} : TP \longrightarrow \mathfrak{g}$ denote the connection form of ω. There exists a homomorphism of G-DG-algebras $\breve{\omega}_\wedge : \bigwedge \mathfrak{g}^* \longrightarrow \varOmega(P)$ induced by the algebraic connection $\breve{\omega} : \mathfrak{g}^* \longrightarrow \varOmega(P)$, $\alpha \longmapsto \alpha\breve{\omega} = \langle \alpha, \breve{\omega}\rangle$ (flatness of ω is essential). The homomorphism $\breve{\omega}_\wedge$ can be restricted to H-basic elements $\breve{\omega}_H : (\bigwedge \mathfrak{g}^*)_H \longrightarrow \varOmega(P)_H$, and according to the isomorphisms $(\bigwedge \mathfrak{g}^*)_H \cong \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H}$ and $\varOmega(P)_H \cong \varOmega(P/H)$, gives a DG-homomorphism of algebras $\breve{\omega}_H : \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H} \longrightarrow \varOmega(P/H)$, which composed with $s^* : \varOmega(P/H) \longrightarrow \varOmega(M)$ where $s : M \longrightarrow P/H$ is the section determined by the H-reduction P', leads to a homomorphism of DG-algebras

$$\varDelta_{P,P',\omega} : \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H} \xrightarrow{\breve{\omega}_H} \varOmega(P/H) \xrightarrow{s^*} \varOmega(M).$$

Explicitly,

$$\left(\varDelta_{P,P',\omega}(\psi)\right)_x (w_1 \wedge \ldots \wedge w_k) = \langle \psi, [\breve{\omega}_z(\widetilde{w}_1)] \wedge \ldots \wedge [\breve{\omega}_z(\widetilde{w}_k)]\rangle$$

where $z \in P'_{|x}$, $w_i \in T_xM$, $\widetilde{w}_i \in T_zP'$, $\pi'_*\widetilde{w}_i = w_i$ where $\pi' : P' \longrightarrow M$ is the projection in P'. Now, passing to cohomology we have the characteristic homomorphism (4.6).

Since (4.6) is an invariant of homotopic H-reductions and measures the incompatibility of the flat structure ω with a given H-reduction, the nontriviality of $\varDelta_{(P,P',\omega)\#}$ implies that there is no homotopic changing of P' such that TP' contains ω. If $K \subset H \subset G$ where K is a maximal compact subgroup of G and H is closed, then any two H-reductions are homotopic, so (4.6) is independent on the H-reduction P'.

The relation between $\varDelta_{(A(P),A(P'),\omega^A)\#}$ and the classical characteristic homomorphism for a principal bundle $\varDelta_{(P,P',\omega)\#}$ is described by the following theorem [52, Theorem 6.1].

Theorem 4.4.1. [52, Theorem 6.1] *If P' is a connected H-reduction in a G-principal bundle P, $\mathfrak{g}$ is the Lie algebra of G, then there exists an isomorphism of algebras*

$$\kappa : \mathsf{H}^\bullet(\mathfrak{g}, H) \xrightarrow{\simeq} \mathsf{H}^\bullet(\boldsymbol{g}, A(P'))$$

such that

$$\varDelta_{(A(P),A(P'),\omega)\#} \circ \kappa = \varDelta_{(P,P',\omega)\#}.$$

We recall from [52] that the isomorphism κ at the level of cochains is defined via the isomorphism

$$\tilde{\kappa} : \bigwedge(\mathfrak{g}/\mathfrak{h})^{*H} \longrightarrow (\Gamma(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*))^{\Gamma(B)}$$

given by

$$\tilde{\kappa}(\psi)(x) = \mathrm{Ad}^{\wedge}_{P',\boldsymbol{g}}(z)(\psi), \quad z \in P'_x,$$

where the representation $\mathrm{Ad}^{\wedge}_{P',\boldsymbol{g}}$ of P' on the vector bundle $\bigwedge^k(\boldsymbol{g}/\boldsymbol{h})^*$ is induced by $\mathrm{Ad}_{P',\boldsymbol{g}} : P' \longrightarrow L(\boldsymbol{g}/\boldsymbol{h})$, $z \longmapsto [\hat{z}]$, and $\hat{z} : \mathfrak{g} \xrightarrow{\cong} \boldsymbol{g}_x$, $v \longmapsto [A_{z*v}]$ ($A_z : G \longrightarrow P$, $a \longmapsto za$).

Hence, characteristic classes induced by $\Delta_{(P,P',\omega)\#}$ and $\Delta_{(A(P),A(P'),\omega^A)\#}$ are identical. In [10] we showed that the homomorphism

$$\Delta_{(P,P')\#} = \Delta_{(A(P),A(P'))\#} \circ \kappa : \mathsf{H}^\bullet(\mathfrak{g}, H) \longrightarrow \mathsf{H}^\bullet(A(P)) \longrightarrow \mathsf{H}^{r\bullet}_{dR}(P)$$

factorizes $\Delta_{(P,P',\omega)\#}$ for any flat connection ω in P, i.e., the following diagram commutes

$$\begin{tikzcd} & \mathsf{H}^\bullet_{dR}(P) \arrow[dr, "\omega^\#"] & \\ \mathsf{H}^\bullet(\mathfrak{g}, H) \arrow[ur, "\Delta_{(P,P')\#}"] \arrow[rr, "\Delta_{(P,P',\omega)\#}"'] & & \mathsf{H}^\bullet_{dR}(M), \end{tikzcd}$$

where $\omega^\#$ at the level of right-invariant differential forms $\Omega^r(P)$ is given as the pullback of differential forms, i.e.,

$$\omega^* : \Omega^r(P) \longrightarrow \Omega(M), \quad \omega^*(\eta)_x(u_1, \ldots, u_n) = \eta_p(\widetilde{u}_1, \ldots, \widetilde{u}_n),$$

where $p \in P_x$, $\widetilde{u}_i$ is ω-horizontal lift of $u_i \in T_xM$. If G is a compact, connected Lie group and P' is a connected H-reduction in a G-principal bundle P, $H \subset G$, then there exists a homomorphism of algebras

$$\Delta_{(P,P')\#} : \mathsf{H}^\bullet(\mathfrak{g}, H) \longrightarrow \mathsf{H}^\bullet_{dR}(P),$$

which is called a *universal exotic characteristic homomorphism for the pair* (P', P) such that for arbitrary flat connection ω in P, the characteristic homomorphism $\Delta_{(P,P',\omega)\#} : \mathsf{H}^\bullet(\mathfrak{g}, H) \longrightarrow \mathsf{H}^\bullet_{dR}(M)$ is factorized by $\Delta_{(P,P')\#}$, i.e., the diagram

$$\begin{tikzcd}
& \mathsf{H}^{\bullet}_{dR}(P) \arrow[dr, "\omega^{\#}"] & \\
\mathsf{H}^{\bullet}(\mathfrak{g}, H) \arrow[ur, "\Delta_{(P,P')\#}"] \arrow[rr, "\Delta_{(P,P',\omega)\#}"'] & & \mathsf{H}^{\bullet}_{dR}(M)
\end{tikzcd}$$

is commutative.

4.4.3 Comparison with the Crainic exotic characteristic classes

The Crainic Approach. We briefly explain the Crainic theory of flat characteristic classes [17], [18], [19]. Let L be a Lie algebroid over a manifold M. The Crainic classes of a flat L-connection ∇ in a vector bundle E are in the cohomology algebra $\mathsf{H}^{\bullet}(L)$ of L. For the trivial vector bundle $E = M \times V$ with $\dim V = n$ these classes are constructed as follows: Let $\{e_1, \ldots, e_n\}$ be a frame of E and let $\omega = [\omega^i_j] \in M_{n\times n}(\Gamma(L^*))$ be the matrix of 1-forms on L such that $\nabla_u e_j = \sum_i \omega^i_j(u)\cdot e_i$ for any $u \in \Gamma(L)$. It is evident that $\mathrm{tr}(\omega) = \mathrm{tr}(\tilde{\omega})$, where $\tilde{\omega} = \frac{1}{2}(\omega + \omega^T)$ is the symmetrization of ω, and the flatness condition implies that $\mathrm{tr}(\tilde{\omega}^{2k-1})$ is closed on L for all $k \in \mathbb{N}$. Moreover, their cohomology classes are independent of the choice of frames. These classes vanish if ∇ is a Riemannian connection with respect to some Riemannian metric h in the vector bundle E. A Riemannian connection is a connection in a Riemannian reduction $L_{(E,\{h\})}$ of the frame bundle L_E. For any vector bundle E Crainic uses a local construction (a suitable cocycle) and the Čech double complex $\check{C}^*(\mathcal{U}, C^*(L))$ together with the Mayer-Vietoris argument. For $L = TM$ the usual exotic characteristic classes of flat vector bundles are received. An explicit formula for an arbitrary L-flat real vector bundle (E, ∇) is based on the observation that in a local orthonormal frame $\{e_1, \ldots, e_n\}$ of $(E, \{h\})$ the symmetrization $\tilde{\omega}$ of ω is equal to the matrix of the symmetric-values form $\omega(E, h) = \frac{1}{2}\left(\nabla - \nabla^h\right)$, where ∇^h is the dual L-connection induced by the metric h. The connection ∇^h is also flat since

$$R^{\nabla^h}_{u,\nu} = -\left(R^{\nabla}_{u,\nu}\right)^* \quad \text{for} \quad u, \nu \in \Gamma(L).$$

The appropriate classes are obtained by replacing $\tilde{\omega}$ with $\omega(E, h)$. One explicit formula up to a constant (see [19]) uses the Chern-Simons transgression differential forms cs_k for suitable two connections and is given by

$$u_{2k-1}(E) = [u_{2k-1}(E, \nabla)] \in \mathsf{H}^{2k-1}(L), \quad k \in \mathbb{N},$$

where

$$u_{2k-1}(E, \nabla) = (-1)^{\frac{k+1}{2}} \operatorname{cs}_k(\nabla, \nabla^h) \tag{4.7}$$

if k is an odd natural, $u_{2k-1}(E, \nabla)$ is trivial if k is even, and

$$\operatorname{cs}_k(\nabla, \nabla^h) = \int_0^1 \operatorname{ch}_k(\nabla^{\mathrm{aff}}) \in \Gamma(\textstyle\bigwedge^{2k-1} L^*)$$

for the affine combination

$$\nabla^{\mathrm{aff}} = (1-t) \cdot \tilde{\nabla} + t \cdot \tilde{\nabla}^h : T\mathbb{R} \times A \longrightarrow A(\operatorname{pr}_2^* E)$$

is defined by the formula

$$\left(\int_0^1 \operatorname{ch}_k(\nabla^{\mathrm{aff}})\right)_{u_1,\ldots,u_{2k-1}} = \int_0^1 \operatorname{ch}_k(\nabla^{\mathrm{aff}})_{\frac{\partial}{\partial t}, u_1, \ldots, u_{2k-1}}|_{(t,\bullet)} dt$$

for $u_1, \ldots, u_{2k-1} \in \Gamma(L)$. We denote here the lift of an arbitrary L-connection $\nabla : \Gamma(L) \longrightarrow \mathcal{CDO}(E)$ through the projection $\operatorname{pr}_2 : \mathbb{R} \times M \longrightarrow M$ by $\tilde{\nabla}$, i.e.,

$$\tilde{\nabla} : T\mathbb{R} \times L \longrightarrow A(\operatorname{pr}_2^* E), \qquad \tilde{\nabla}_{(v_t, u_x)}(\nu \circ \operatorname{pr}_2) = \nabla_{u_x}(\nu),$$

where $T\mathbb{R} \times L$ is the Cartesian product of Lie algebroids (cf. Section 1.6). One can check that if ∇ is flat, $\tilde{\nabla}$ is also flat.

The Crainic secondary characteristic classes in terminology of Riemannian reductions. Let E denote a vector bundle of the rank n over a manifold M with a Riemannian metric h. We recall that the metric h determines the Lie subalgebroid $B = A(E, \{h\})$ of the Lie algebroid $A(E)$ described in Section 4.3. The Atiyah sequence of $A(E, \{h\})$ is

$$0 \longrightarrow \operatorname{Sk}(E) \longrightarrow A(E, \{h\}) \longrightarrow TM \longrightarrow 0$$

where $\operatorname{Sk}(E) \subset \operatorname{End}(E)$ is the vector subbundle of skew symmetric endomorphisms with respect to the metric h.

Let L be a Lie algebroid over M and $\nabla : \Gamma(L) \longrightarrow \mathcal{CDO}(E)$ be any flat L-connection in $A(E)$. Consider for

$$(A(E), A(E, \{h\}), \nabla) \;\text{ and }\; (A(E), A(E, \{h\}), \operatorname{id}_{A(E)})$$

theirs secondary characteristic homomorphisms denoted by

$$\Delta_{\#} : \mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h\})) \longrightarrow \mathsf{H}^\bullet(L),$$

$$\Delta_{o\#} : \mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h\})) \longrightarrow \mathsf{H}^\bullet(A(E)),$$

respectively. Recall from Section 4.4.2 the isomorphism (cf. Theorem 4.4.1)

$$\kappa : \mathsf{H}^\bullet(\mathfrak{gl}(n,\mathbb{R}), O(n)) \xrightarrow{\cong} \mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h\})).$$

If the vector bundle E is nonorientable, then

$$\mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h\})) \overset{\kappa}{\cong} \mathsf{H}^\bullet(\mathfrak{gl}(n,\mathbb{R}), O(n)) \cong \bigwedge(y_1, y_3, \ldots, y_m),$$

where m is the largest odd integer $\leq n$, i.e., $m = 2[\frac{n+1}{2}]-1$, and $y_{2k-1} \in \mathsf{H}^{4k-3}(\operatorname{End} E, A(E,\{h\}))$ for $k \in \{1, 2, \ldots, [\frac{n+1}{2}]\}$ are represented by the multilinear trace forms $\widetilde{y}_{2k-1} \in \Gamma(\bigwedge^{4k-3}(\operatorname{End} E/\operatorname{Sk})^*)$,

$$\widetilde{y}_{2k-1}([A_1], ..., [A_{4k-3}]) = \sum_{\sigma \in S_{4k-3}} \operatorname{sgn}\sigma \cdot \operatorname{tr}(\widetilde{A}_{\sigma_1} \circ \cdots \circ \widetilde{A}_{\sigma_{4k-3}}) \quad (4.8)$$

for $A_i \in \Gamma(\operatorname{End} E)$, and where $\widetilde{A}_i = \frac{1}{2}(A_i + A_i^*)$ is the symmetrization with respect to h (see [43, p. 142]).

In the case of an oriented vector bundle E with a volume form v, the metric h and v induce an $SO(n,\mathbb{R})$-reduction $L_{(E,\{h,\mathrm{v}\})}$ of the frames bundle L_E. Note that $\xi : \mathbb{R}^n \to E_x$ is an element of $L_{(E,\{h,\mathrm{v}\})}$ if and only if ξ is an isometry keeping the orientations. Since $A(E,\{h,\mathrm{v}\}) = A(L_{(E,\{h,\mathrm{v}\})}) = A(E,\{h\})$, it follows that

$$\mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h,\mathrm{v}\})) \cong \mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h\})).$$

If E, is orientable of an odd rank, cf. [27],

$$\mathsf{H}^\bullet\,(\mathfrak{gl}(n,\mathbb{R}), SO(n)) \cong \mathsf{H}^\bullet(\mathfrak{gl}(n,\mathbb{R}), O(n)). \quad (4.9)$$

Using the isomorphism κ and (4.9), we get

$$\mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h,\mathrm{v}\})) \cong \mathsf{H}^\bullet(\mathfrak{gl}(n,\mathbb{R}), O(n)) \cong \bigwedge(y_1, y_3, \ldots, y_n).$$

If the vector bundle E is orientable of the even rank $n = 2m$, then

$$\begin{aligned}\mathsf{H}^\bullet(\operatorname{End} E, A(E,\{h,\mathrm{v}\})) &\cong \mathsf{H}^\bullet(\mathfrak{gl}(2m,\mathbb{R}), SO(2m)) \\ &\cong \bigwedge(y_1, y_3, \ldots, y_{2m-1}, y_{2m}),\end{aligned}$$

where $y_{2i-1} \in \mathsf{H}^{4k-3}\mathsf{H}^{\bullet}(\operatorname{End} E, A(E,\{h,\mathrm{v}\}))$ are defined as in (4.8), and $y_{2m} \in \mathsf{H}^{2m}(\mathfrak{gl}(n,\mathbb{R}), SO(n)) \cong \mathsf{H}^{\bullet}(\operatorname{End} E, A(E,\{h,\mathrm{v}\}))$ is represented by

$$\widetilde{y}_{2m} \in \Gamma\left(\bigwedge^{2m}((\operatorname{End} E/\operatorname{Sk})^*\right),$$

$$\widetilde{y}_{2m}([A_1], \ldots, [A_{2m}]) = d(z_{2m-1})(\widetilde{A}_1, \ldots, \widetilde{A}_{2m})$$

for $A_1, \ldots, A_{2m} \in \Gamma(\operatorname{End} E)$, and where d is the usual differential on the algebra $\bigwedge(\operatorname{End} E)^*$, but

$$z_{2m-1} \in \Gamma\big(\bigwedge^{2m-1}(\operatorname{End} E)^*\big)$$

is defined by

$$\begin{aligned} &z_{2m-1}(A_1, \ldots, A_{2m-1}) \\ &= c_m \sum_{\sigma \in S_{2m-1}} \operatorname{sgn}\sigma\big(e, \alpha A_{\sigma(1)} \wedge \alpha[A_{\sigma(2)}, A_{\sigma(3)}] \wedge \ldots \wedge \alpha[A_{\sigma(2m-2)}, A_{\sigma(2m-1)}]\big) \end{aligned}$$

where $c_m = \frac{(-1)^{m-1}(m-1)!}{2^{m-1}(2m-1)!} \in \mathbb{R}$, e is a non-zero section of $\bigwedge^{2m}(\operatorname{End} E)^*$, and

$$\alpha : \operatorname{End} E \longrightarrow \bigwedge^2 E$$

is given by

$$(\alpha(A), \nu \wedge \mu) = \tfrac{1}{2}\left((A\nu, \mu) - (\nu, A\mu)\right)$$

for $A \in \Gamma(\operatorname{End} E)$, $\nu, \mu \in \Gamma(E)$. The form z_{2m-1} is the image of the Pfaffian for a pair (E, e) by the Cartan map for $\operatorname{End} E$ (for the Cartan map we refer for example [36, Ch. VI, 6.7, 6.8]).

We will show that $\Delta_{\#}(y_{2j-1})$ is the Crainic class for all $j \in \{1, 2, \ldots, [\frac{n+1}{2}]\}$. Let $d^{\nabla} : \Gamma(\bigwedge L^* \otimes E) \longrightarrow \Gamma(\bigwedge L^* \otimes E)$ be the differential determined by $\nabla : \Gamma(L) \longrightarrow \mathcal{CDO}(E)$. Fix two L-connections ∇_0, $\nabla_1 : \Gamma(L) \longrightarrow \mathcal{CDO}(E)$. Write their affine combination by

$$\nabla^{\mathrm{aff}} = (1-t)\,\tilde{\nabla}_0 + t\tilde{\nabla}_1 : T\mathbb{R} \times L \to A(\operatorname{pr}_2^* E).$$

One can observe that

$$R^{\nabla_1} = R^{\nabla_0} + d^{\nabla_0}\theta + [\theta, \theta], \tag{4.10}$$

where

$$\theta = \nabla_1 - \nabla_0 \in \Gamma(L^* \otimes \operatorname{End} E) \tag{4.11}$$

and

$$[\theta, \theta] = \theta^2 = \theta \wedge \theta \in \Gamma(\textstyle\bigwedge^2 L^* \otimes \operatorname{End} E),$$

$$[\theta, \theta](u, \nu) = [\theta(u), \theta(\nu)] \quad \text{for} \quad u, \nu \in \Gamma(L).$$

We can lift any 1-form $\omega \in \Gamma(L^* \otimes \operatorname{End} E)$ to $\tilde{\omega} \in \Gamma((T\mathbb{R} \times L)^* \otimes \operatorname{End} E)$ defining $\tilde{\omega}_{(v_t, \xi_x)} = \omega_{\xi_x}$. Let $\tilde{\theta}$ denote the lifting of (4.11). We follow the convention that the section $(0, u)$ of $T\mathbb{R} \times L$ we will be briefly denoted by u and $(\frac{\partial}{\partial t}, 0)$ by $\frac{\partial}{\partial t}$. Observe that $\nabla^{\mathrm{aff}} = \tilde{\nabla}_0 + \Xi$, where $\Xi_{(t,x)} = t \cdot \tilde{\theta}_x$ and $(d^{\tilde{\nabla}_1} \tilde{\theta})_{u,\nu}(w \circ \mathrm{pr}_2) = (d^{\nabla_1} \theta)_{u,\nu}(w) \circ \mathrm{pr}_2$ for any $u, \nu \in \Gamma(L)$, $w \in \Gamma(E)$, $x \in M$, $t \in \mathbb{R}$. From (4.10) we have

$$R^{\nabla^{\mathrm{aff}}} = d^{\tilde{\nabla}_0} \Xi + [\Xi, \Xi]\,. \tag{4.12}$$

Hence, we see that ∇^{aff} need not be flat even if ∇_0 is flat.

Lemma 4.4.1. [10] *The curvature tensor $R^{\nabla^{\mathrm{aff}}}$ of the affine combination ∇^{aff} of two flat L-connections ∇_0, ∇_1 satisfies the conditions:*

$$(R^{\nabla^{\mathrm{aff}}})_{\frac{\partial}{\partial t}, u}(w \circ \mathrm{pr}_2) = \theta_u(w),$$
$$(R^{\nabla^{\mathrm{aff}}})_{u,\nu}(w \circ \mathrm{pr}_2)_{|(t,\cdot)} = \left(t^2 - t\right) \cdot (\theta \wedge \theta)_{u,\nu}\,(w),$$
$$((R^{\nabla^{\mathrm{aff}}})^k_{\frac{\partial}{\partial t}, u_1, \ldots, u_{2k-1}})_{|(t,\cdot)} = k \cdot t^{k-1} \cdot (t-1)^{k-1} \cdot \theta^{2k-1}_{u_1, \ldots, u_{2k-1}}.$$

From the above we obtain the following theorem.

Theorem 4.4.2. [10] *For all $k \in \mathbb{N}$, we have*

$$\mathrm{cs}_k(\nabla_0, \nabla_1) = (-1)^{k+1} \, \frac{k! \cdot (k-1)!}{(2k-1)!} \cdot \operatorname{tr} \theta^{2k-1}, \tag{4.13}$$

where θ is defined by (4.11).

The above formula is well known in the classical cases (see, for example, the papers by S.-S. Chern and J. Simons [13] or J. Heitsch and B. Lawson Jr. [37]).

Let

$$\widetilde{\ } : \operatorname{End} E \longrightarrow \operatorname{End} E, \quad v \longmapsto \widetilde{v} := \frac{1}{2}(v + v^*)$$

denote the symmetrization. Let us consider $\mathrm{id}_{A(E)}$ as an $A(E)$-connection and take its adjoint $\mathrm{id}^h_{A(E)}$ defined by the metric h. Let $\lambda : TM \longrightarrow$

$A(E)$ be any h-Riemannian connection, i.e., a connection such that $\operatorname{Im}\lambda \subset A(E,\{h\})$. For the connection form $\breve{\lambda} : A(E) \longrightarrow \operatorname{End} E$ of λ, we have $\iota \circ \breve{\lambda} + \lambda \circ \varrho_{A(E)} = \mathrm{id}_{A(E)}$. Therefore,

$$-\widetilde{\breve{\lambda}\,(a)} = \tfrac{1}{2}(\mathrm{id}^h_{A(E)} - \mathrm{id}_{A(E)})(a) \tag{4.14}$$

for $a \in \Gamma(A(E))$. Compare with Corollary 2.6.5, which shows that such affine combinations of connections appearing in (4.14) are compatible with the metric. (4.14), (4.13) and (4.7) now become

$$\Delta_o(\widetilde{y}_{2k-1}) = (-1)^k \cdot 2^{3-4k} \cdot \frac{(4k-3)!}{(2k-1)! \cdot (2k-2)!} \cdot u_{4k-3}(E, \mathrm{id}_{A(E)}).$$

From this and the equalities

$$\mathrm{cs}_k(\nabla, \nabla^h) = \nabla^*(\mathrm{cs}_k(\mathrm{id}_{A(E)}, \mathrm{id}^h_{A(E)}))$$

and

$$u_{4k-3}(E, \nabla) = \nabla^{\#} u_{4k-3}(E, \mathrm{id}_{A(E)}),$$

we obtain

$$\Delta_{\#}(y_{2k-1}) = [\nabla^* \Delta_o\,(\widetilde{y}_{2k-1})] = \frac{(-1)^k \cdot (4k-3)!}{2^{4k-3} \cdot (2k-1)! \cdot (2k-2)!} \cdot u_{4k-3}(E).$$

From the above formulae we can explain the relation between the secondary characteristic homomorphism $\Delta_{\#}$ of $(A(E), A(E,\{h\}), \nabla)$ and the family of the Crainic classes $\{u_{4k-3}(E)\}$.

Theorem 4.4.3. [10] *Let E be a real vector bundle over a manifold M and*

$$\Delta_{\#} : \mathsf{H}^{\bullet}(\operatorname{End} E, A(E,\{h\})) \longrightarrow \mathsf{H}^{\bullet}(L)$$

the secondary characteristic homomorphism corresponding to the triple $(A(E), A(E,\{h\}), \nabla)$, in which $\nabla : \Gamma(L) \longrightarrow \mathcal{CDO}(E)$ is a flat L-connection in $A(E)$.

- *If the vector bundle E is nonorientable or orientable and of odd rank, then the image of $\Delta_{\#}$ is generated by $u_1(E)$, $u_5(E)$,…,$u_{4\left[\frac{n+3}{4}\right]-3}(E)$.*
- *If the vector bundle E is orientable and of even rank $n = 2m$, then the image of $\Delta_{\#}$ is generated by $u_1(E)$, $u_5(E)$,…,$u_{4\left[\frac{n+3}{4}\right]-3}(E)$ and additionally by $\Delta_{\#}(y_{2m})$, where y_{2m} is the form generated by the Pfaffian.*

4.4.4 Secondary characteristic homomorphism for a pair of Lie algebras

In this section, we will consider the characteristic homomorphism $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ for a pair of finite dimensional Lie algebras $(\mathfrak{g},\mathfrak{h})$, $\mathfrak{h}\subset\mathfrak{g}$, and give a class of such pairs for which $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is a monomorphism.

An arbitrary Lie algebra is a Lie algebroid over a point with the zero map as an anchor. Consider the homomorphism of pairs of Lie algebras $(\mathrm{id}_{\mathfrak{g}},0) : (\mathfrak{g},0) \longrightarrow (\mathfrak{g},\mathfrak{h})$, $\mathfrak{h}\subset\mathfrak{g}$. By the definition of the universal exotic characteristic homomorphism, observe that

$$\Delta_{(\mathfrak{g},0)\#} : \mathsf{H}^\bullet(\mathfrak{g},0) = \mathsf{H}^\bullet(\mathfrak{g}) \xrightarrow{(-\mathrm{id}_{\mathfrak{g}})_\#} \mathsf{H}^\bullet(\mathfrak{g}).$$

Now, the functoriality of $(\mathrm{id}_{\mathfrak{g}},0)$ described in Theorem 4.2.1 implies that

$$\Delta_{(\mathfrak{g},\mathfrak{h})\#} = \Delta_{(\mathfrak{g},0)\#}\circ(\mathrm{id}_{\mathfrak{g}})^{+\#} = (-\mathrm{id}_{\mathfrak{g}})^{+\#} : \mathsf{H}^\bullet(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathsf{H}^\bullet(\mathfrak{g}).$$

Let $(\bigwedge\mathfrak{g}^*)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}$ be the basic subalgebra of $\bigwedge\mathfrak{g}^*$, i.e., the subalgebra of invariant and horizontal elements of $\bigwedge\mathfrak{g}^*$ with respect to the Lie subalgebra $\mathfrak{h}$. Denote by k the inclusion from $(\bigwedge\mathfrak{g}^*)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}$ into $\bigwedge\mathfrak{g}^*$ (cf. [47], [36, p. 412]). Moreover, consider the projection $s:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ and the map

$$(-s)^* : \left(\bigwedge(\mathfrak{g}/\mathfrak{h})^*\right)^{\mathfrak{h}} \longrightarrow \left(\bigwedge\mathfrak{g}^*\right)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0}$$

given by

$$((-s)^*\Psi)(x_1\wedge\ldots\wedge x_k) = \langle\Psi,(-s(x_1))\wedge\ldots\wedge(-s(x_k))\rangle$$

for $\Psi\in(\bigwedge^k(\mathfrak{g}/\mathfrak{h})^*)^{\mathfrak{h}}$, $x_1,\ldots,x_k\in\mathfrak{g}$. One can see that $(-s)^*$ is an isomorphism of algebras and

$$\Delta_{(\mathfrak{g},\mathfrak{h})} = \mathrm{k}\circ(-s)^*.$$

Therefore, the exotic characteristic homomorphism $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ for the pair $(\mathfrak{g},\mathfrak{h})$ can be written as the composition

$$\Delta_{(\mathfrak{g},\mathfrak{h})\#} : \mathsf{H}^\bullet(\mathfrak{g},\mathfrak{h}) \underset{\cong}{\xrightarrow{(-s)^\#}} \mathsf{H}^\bullet(\mathfrak{g}/\mathfrak{h}) \xrightarrow{\mathrm{k}^\#} \mathsf{H}^\bullet(\mathfrak{g}),$$

where $\mathsf{H}^\bullet(\mathfrak{g}/\mathfrak{h})$ denotes the cohomology algebra $\mathsf{H}^\bullet((\bigwedge\mathfrak{g}^*)_{i_{\mathfrak{h}}=0,\theta_{\mathfrak{h}}=0},d_{\mathfrak{g}})$.

Example 4.4.1. [11] Let $\mathfrak{g}$, $\mathfrak{h}$ be finite dimensional Lie algebras and $\mathfrak{g}\oplus\mathfrak{h}$ their direct product. The secondary characteristic homomorphism of the pair $(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ is given by

$$\Delta_{(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\#} : \mathsf{H}^\bullet(\mathfrak{g}) \longrightarrow \mathsf{H}^\bullet(\mathfrak{g})\otimes\mathsf{H}^\bullet(\mathfrak{h}), \quad \Delta_{\#(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})}([\Phi]) = [(-1)^{|\Phi|}\cdot\Phi]\otimes 1,$$

and $\Delta_{(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})\#}$ is a monomorphism.

Remark 4.4.1. Let $(\mathfrak{g},\mathfrak{h})$ be a reductive pair of finite dimensional Lie algebras (here $\mathfrak{h}\subset\mathfrak{g}$), and let $s:\mathfrak{g}\to\mathfrak{g}/\mathfrak{h}$ be the projection. Theorems IX and X from [36, sections 10.18, 10.19] imply that $\mathrm{k}^\#$ is injective if and only if the algebra $\mathsf{H}^\bullet(\mathfrak{g}/\mathfrak{h})$ is generated by 1 and odd-degree elements. Thus, $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is injective if and only if $\mathsf{H}^\bullet(\mathfrak{g},\mathfrak{h})$ is generated by 1 and odd-degree elements since $(-s)^\#$ is an isomorphism of algebras. If $\mathfrak{h}$ is reductive in $\mathfrak{g}$, the homomorphism $\mathrm{k}^\#$ is injective if and only if $\mathfrak{h}$ is noncohomologous to zero (briefly: n.c.z.) in $\mathfrak{g}$, i.e., if the homomorphism of algebras $\mathsf{H}^\bullet(\mathfrak{g})\to\mathsf{H}^\bullet(\mathfrak{h})$ induced by the inclusion $\mathfrak{h}\hookrightarrow\mathfrak{g}$ is surjective. Tables I, II and III at the end of Chapter XI of [36] contain many n.c.z. pairs including for example:

$$(\mathfrak{gl}(n),\mathfrak{so}(n)) \text{ for odd } n, \qquad (\mathfrak{so}(n,\mathbb{C}),\mathfrak{so}(k,\mathbb{C})) \text{ for } k<n,$$

$$(\mathfrak{so}(2m+1),\mathfrak{so}(2k+1)), \qquad (\mathfrak{so}(2m),\mathfrak{so}(2k+1)) \text{ for } k<m.$$

Example 4.4.2. We consider a reductive pair $(\mathrm{End}(V),\mathrm{Sk}(V))$ of Lie algebras where V is a vector space of the positive dimension.

(1) If the dimension of V is odd, $\dim V=2m-1$, we have

$$\begin{aligned}\mathsf{H}^\bullet(\mathrm{End}(V),\mathrm{Sk}(V)) &\cong \mathsf{H}^\bullet(\mathfrak{gl}(2m-1,\mathbb{R}),O(2m-1))\\ &\cong \textstyle\bigwedge(y_1,y_3,\ldots,y_{2m-1}),\end{aligned}$$

where $y_{2k-1}\in\mathsf{H}^{4k-3}(\mathrm{End}(V),\mathrm{Sk}(V))$ are represented by the multilinear trace forms ([27], [43]). Since $\mathsf{H}^\bullet(\mathrm{End}(V),\mathrm{Sk}(V))$ is generated by odd-degree elements. Remark 4.4.1 now shows that $\Delta_{(\mathrm{End}(V),\mathrm{Sk}(V))\#}$ is injective.

(2) If the dimension of V is even, $\dim V=2m$, we have:

$$\begin{aligned}\mathsf{H}^\bullet(\mathrm{End}(V),\mathrm{Sk}(V)) &\cong \mathsf{H}^\bullet(\mathfrak{gl}(2m,\mathbb{R}),SO(2m))\\ &\cong \textstyle\bigwedge(y_1,y_3,\ldots,y_{2m-1},y_{2m}),\end{aligned}$$

where y_{2k-1} are as in item (1) of this example above, and $y_{2m} \in \mathsf{H}^{2m}(\mathrm{End}(V), \mathrm{Sk}(V))$ is a class determined by the Pfaffian (cf. Section 4.4.3). Since in the case of even $\dim V$, an even-degree elements is in generating set of $\mathsf{H}^{\bullet}(\mathrm{End}(V), \mathrm{Sk}(V))$, it follows that then $\Delta_{(\mathrm{End}(V),\mathrm{Sk}(V))\#}$ is not a monomorphism.

4.5 Secondary characteristic homomorphism for a pair of transitive Lie algebroids

Consider a pair (A, B) of transitive Lie algebroids on a manifold M, $B \subset A$, $x \in M$, and a pair of corresponding isotropy Lie algebras $(\boldsymbol{g}_x, \boldsymbol{h}_x)$. Obviously, the inclusion $\iota_x : (\boldsymbol{g}_x, \boldsymbol{h}_x) \to (A, B)$ is a homomorphism of pairs of Lie algebroids over $\{x\} \hookrightarrow M$. Theorem 4.2.1 implies the commutativity of the following diagram

$$\begin{CD}
\mathsf{H}^{\bullet}(\boldsymbol{g}_x, B) @>{\Delta_{(A,B)\#}}>> \mathsf{H}^{\bullet}(A) \\
@V{\iota_x^{+\#}}VV @VV{\iota_x^{\#}}V \\
\mathsf{H}^{\bullet}(\boldsymbol{g}_x, \boldsymbol{h}_x) @>>{\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}}> \mathsf{H}^{\bullet}(\boldsymbol{g}_x).
\end{CD} \tag{4.16}$$

Obviously, if the left and bottom homomorphisms in (4.16) are monomorphisms, then so is $\Delta_{(A,B)\#}$. The homomorphism $\iota_x^{+\#}$ is a monomorphism if each invariant element $v \in (\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\boldsymbol{h}_x}$ can be extended to a global invariant section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$. Consequently, we get the following theorem linking the Koszul homomorphism for a pair of Lie algebras (discussed in the previous section) with secondary characteristic classes.

Theorem 4.5.1. [11] *Let (A, B) be a pair of transitive Lie algebroids over a manifold M, $B \subset A$, $x \in M$, $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ be a pair of the isotropy Lie algebras at x, and suppose that any element of $(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\boldsymbol{h}_x}$ can be extend to an invariant section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$. If the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}$ for the pair $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ is a monomorphism, then $\Delta_{(A,B)\#}$ is a monomorphism.*

Remark 4.5.1. Theorem 6.5.15 of [63] yields $\zeta \in (\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\boldsymbol{h}_x}$ can be extended to an invariant section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$ if and only if it is invariant with respect to the $\pi_1(M)$-action on $(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*)^{\boldsymbol{h}_x}$ by the holonomy morphism of the (flat) B-connection $\nabla^{\mathrm{ad}^\wedge_{B,\boldsymbol{h}}} = \overline{\mathrm{ad}^\wedge_{B,\,\boldsymbol{h}}} \circ \lambda$ in

$$\left(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*\right)^{\boldsymbol{h}} = \bigsqcup_{x\in M}\left(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*\right)^{\boldsymbol{h}_x}$$

where $\lambda : TM \to B$ is any TM-connection in B and $\overline{\mathrm{ad}^\wedge_{B,\boldsymbol{h}}}$ is the representation of B in $(\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*)^{\boldsymbol{h}}$ defined by $\mathrm{ad}^\wedge_{B,\boldsymbol{h}}$.

We will show examples of pairs of Lie algebroids satisfying the assumptions of the last theorem, including integrable and nonintegrable Lie algebroids.

The case of integrable Lie algebroids. Let P be a principal G-bundle and P' some of its reduction with a connected structural Lie group $H \subset G$. According Theorem 1.1 from [48], we observe that for any transitive Lie subalgebroid $B \subset A(P)$ there exists a connected reduction P' of P such that $B = A(P')$. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. The representation $\mathrm{ad}_{B,\boldsymbol{h}}$ is integrable, because it is the differential of the representation $\mathrm{Ad}_{P',\boldsymbol{h}} : P' \to L(\boldsymbol{g}/\boldsymbol{h})$ of the principal fibre bundle P' defined by $z \mapsto [\hat{z}]$ (see [52, p. 218]). For every $z \in P'$, the isomorphism $\hat{z} : \mathfrak{g} \to \boldsymbol{g}_x$, $v \mapsto [(A_z)_{*e}\, v]$ maps $\mathfrak{h}$ onto $\boldsymbol{h}_x$ (see [50, Section 5.1]) and defines an isomorphism $[\hat{z}] : \mathfrak{g}/\mathfrak{h} \to \boldsymbol{g}_x/\boldsymbol{h}_x$. Thus, we have a natural isomorphism (see also [52, Proposition 5.5.2-3])

$$\kappa : \left(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*\right)^{H} \xrightarrow{\cong} \left(\Gamma\left(\bigwedge(\boldsymbol{g}/\boldsymbol{h}^*)\right)\right)^{\Gamma(B)}.$$

On account of the connectedness of H, we have

$$\left(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*\right)^{H} = \left(\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*\right)^{\boldsymbol{h}_x}.$$

It follows that $\iota_x^{+\#}$ is an isomorphism. Thus, we conclude that the assumptions of Theorem 4.5.1 hold for any pair of integrable Lie algebroids (A, B), i.e., if A is a Lie algebroid of some principal bundle P and B is its Lie subalgebroid of some reduction of P. Theorem 4.5.1 now gives the following theorem.

Theorem 4.5.2. [11] *Let A be a Lie algebroid of some principal bundle $P(M, G)$, $B = A(P')$ its Lie subalgebroid for some reduction P' of P, $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ be a pair of adjoint Lie algebras at $x \in M$. If the Koszul homomorphism $\Delta_{(\boldsymbol{g}_x,\boldsymbol{h}_x)\#}$ for the pair $(\boldsymbol{g}_x, \boldsymbol{h}_x)$ is a monomorphism for any $x \in M$, then the universal exotic characteristic homomorphism $\Delta_{(A,B)\#}$ is a monomorphism.*

The case of non-integrable Lie algebroids of TC-foliations . We discuss non-integrable Lie algebroids (i.e., not isomorphic to any Lie algebroid of a principal bundle) satisfying the assumptions of Theorem 4.5.1. We consider Lie algebroids $A(G,H)$ of transversely complete foliations (TC-foliations) studied in [1], [50]. These TC-foliations play an essential role in the theory of Riemannian foliations [68].

Let G be a Lie group and let H be its connected nonclosed Lie subgroup. Let us recall that $A(G,H)$ is the Lie algebroid of left cosets of H in G (see [1], [50]). Remark that if G is connected and simply connected, the Lie algebroid $A(G,H)$ is non-integrable (see [1]). There are examples of non-integrable Lie algebroids $A(G,H)$ for which the Chern-Weil homomorphism studied in [50] is nontrivial.

Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Let $\overline{H}$ denote the closure of H and let $\mathfrak{s}$ be the Lie algebra of $\overline{H}$. Using trivializations of $TG = G \times \mathfrak{g}$ given by left-invariant vector fields one can check that $A(G,H)$ is a vector bundle over $G/\overline{H}$ which is the quotient space $(G \times (\mathfrak{g}/\mathfrak{h}))_{\overline{H}}$ with respect to the right action of $\overline{H}$ on G and the adjoint action of $\overline{H}$ on $\mathfrak{g}/\mathfrak{h}$. Moreover, there exists an isomorphism

$$c : l(G,H) \xrightarrow{\cong} \Gamma(A(G,H))$$

of the module $l(G,H)$ of transverse fields onto the module $\Gamma(A(G,H))$, which is also an isomorphism of real Lie algebras.

Every right-invariant vector field $\overline{Y}_w$ generated by $w \in \mathfrak{g}$ and every left-invariant vector field $\overline{X}_w$ generated by $w \in \mathfrak{s}$ is a transverse field [50]. The Lie algebra bundle $\boldsymbol{g}$ associated with $A(G,H)$ is a trivial vector bundle of abelian Lie algebras with the trivialization $G/\overline{H} \times \mathfrak{s}/\mathfrak{h} \to \boldsymbol{g}$ given by $(x,[w]) \mapsto (c(\overline{X}_w))(x)$.

Let H_1 and H_2 be Lie subgroups of G such that

$$H_1 \subsetneq H_2 \subsetneq \overline{H_1} = \overline{H_2} \subsetneq G.$$

Denote by $\mathfrak{h}_1$, $\mathfrak{h}_2$ and $\mathfrak{t}$ the Lie algebras of H_1, H_2 and $T = \overline{H_1} = \overline{H_2}$, respectively. Write

$$A = A(G,H_1)$$

and

$$B = A(G,H_2).$$

The Lie algebra bundles of A and B are $\boldsymbol{g} = G/T \times \mathfrak{t}/\mathfrak{h}_1$ and $\boldsymbol{h} = G/T \times \mathfrak{t}/\mathfrak{h}_2$, respectively. Fix

$$\sigma_o \in \bigwedge^k (\boldsymbol{g}_x/\boldsymbol{h}_x)^* \cong \bigwedge^k ((\mathfrak{t}/\mathfrak{h}_1) \,/\, (\mathfrak{t}/\mathfrak{h}_2))^*.$$

The section Π of $\bigwedge^k (\boldsymbol{g}/\boldsymbol{h})^*$ which is equal to σ_o at all points of G/T is invariant (see [11]). This means that any element of $\bigwedge(\boldsymbol{g}_x/\boldsymbol{h}_x)^*$ can be extended to an invariant section of $\bigwedge(\boldsymbol{g}/\boldsymbol{h})^*$. We now apply Theorem 4.5.1 to obtain the following one.

Theorem 4.5.3. [11] *Let G be a Lie group, and H_1 and H_2 be its Lie subgroups such that $H_1 \subsetneq H_2 \subsetneq \overline{H_1} = \overline{H_2} \subsetneq G$. Then the universal secondary characteristic homomorphism for the pair of Lie algebroids $A(G, H_2) \subset A(G, H_1)$ is a monomorphism. Moreover, if G is connected and simply connected, $A(G, H_2)$ and $A(G, H_1)$ are non-integrable Lie algebroids.*

4.6 Example of a nontrivial secondary characteristic class defined by the Pfaffian

Finally, we demonstrate another an example of Lie algebroid and its reduction, for which secondary characteristic homomorphism is nontrivial. We note that the even dimension of the vector bundle and the fact that the manifold is orientable are important for the example here.

Let M be an oriented, connected manifold, $\dim M \geq 1$, and $\mathfrak{g} = \mathrm{End}(\mathbb{R}^2)$. Consider a transitive Lie algebroid $(A, \varrho_A, [\![\cdot,\cdot]\!])$ over M, where

$$A = TM \oplus \mathrm{End}(\mathbb{R}^2) \cong A(M \times \mathbb{R}^2),$$

$\varrho_A = \mathrm{pr}_1$ is a projection on the first coordinate, and

$$[\![(X_1, \sigma_1), (X_2, \sigma_2)]\!] = ([X_1, X_2], X_1(\sigma_2) - X_2(\sigma_1) + [\sigma_1, \sigma_2])$$

for all $X_1, X_2 \in \Gamma(TM)$, $\sigma_1, \sigma_2 \in C^\infty(M; \mathrm{End}(\mathbb{R}^2))$. Observe that (cf. Section 1.5) that

$$0 \longrightarrow M \times \mathrm{End}(\mathbb{R}^2) \cong \mathrm{End}(M \times \mathbb{R}^2) \overset{\iota}{\hookrightarrow} A \xrightarrow{\mathrm{pr}_1} TM \longrightarrow 0$$

is the Atiyah sequence of A.

Let $B \subset A$ be the Riemannian reduction of A, i.e.,

$$B = TM \oplus \mathrm{Sk}(\mathbb{R}^2)$$

is a transitive subalgebroid of A. Observe that in the domain of the universal characteristic homomorphism

$$\Delta_{(A,B)\#} : \mathsf{H}^\bullet(M \times \mathfrak{g}, B) \longrightarrow \mathsf{H}^\bullet(A)$$

is a class

$$[\widetilde{y}_2] \in \mathsf{H}^2(M \times \mathfrak{g}, B)$$

defined by

$$\widetilde{y}_2([\sigma_1], [\sigma_2]) = \mathrm{Pf}([[\widetilde{\sigma}_1, \widetilde{\sigma}_2]])$$

for all $\sigma_1, \sigma_2 \in \Gamma(\ker \varrho_A) \cong C^\infty(M; \mathfrak{g})$. One can check that $\Delta_{(A,B)\#}([\widetilde{y}_2]) \in \mathsf{H}^2(A)$, represented by $\Delta_{(A,B)}(\widetilde{y}_2) \in \Gamma(A^*)$, is a nontrivial secondary characteristic class for the pair of Lie algebroids

$$\big(TM \oplus \mathrm{End}(\mathbb{R}^2), TM \oplus \mathrm{Sk}(\mathbb{R}^2)\big)$$

of even rank. For details of the proof of the non-triviality of this class, please refer to [10].

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