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# SOME REMARKS ON DYNAMIC RESULTS FOR AVERAGED AND EXACT MODELS OF THIN PERIODIC PLATES

Jarosław Jędrysiak
Bohdan Michalak

*Department of Structural Mechanics, Łódź University of Technology*
*e-mail: jarek@p.lodz.pl; bmichala@p.lodz.pl*

The aim of the paper is to show certain justification of the new non-asymptotic model of thin periodic plates, derived using the tolerance averaging (Woźniak and Wierzbicki, 2000). The model describes the effect of periodicity cell size on overall plate behaviour, on the contrary to known homogenised models. Results obtained from those models will be compared to solution to and from the "exact" discrete model. It is shown that for long-wave propagation problems, results obtained for a special case of a periodic plate strip (weightless but covered by a periodically distributed system of two concentrated masses) within the non-asymptotic model are close results calculated from the known "exact" solutions based on the method used to analyse longitudinal vibrations of one-dimensional diatomic lattice (Brillouin, 1953). Similar conformity, taking place in special cases of short waves, is also presented.

*Key words:* periodic plate, effect of periods lengths, travelling wave

## 1. Introduction

In order to show conformity of results by the proposed averaged model and the "exact" model, the problem of a travelling wave, propagating in a homogeneous weightless and unbounded thin plate strip along the $x_1$-axis, with a periodically distributed system of two concentrated masses $M_1$, $M_2$ ($M_1 > M_2$) (Fig. 1; $j = 0, \pm 1, \pm 2, \ldots$) will be analysed in this contribution. The strip is assumed to have constant thickness $h$ and to be made of a material with constant Young's modulus $E$ and Poisson's ratio $\nu$. In the plate strip, a small repeated element can be distinguished, which is called the cell. The cell is treated as a thin plate strip with a span $l$ along the $x_1$-direction. Thus, the plate strip under consideration can be analysed as a plate with an internal periodic structure.

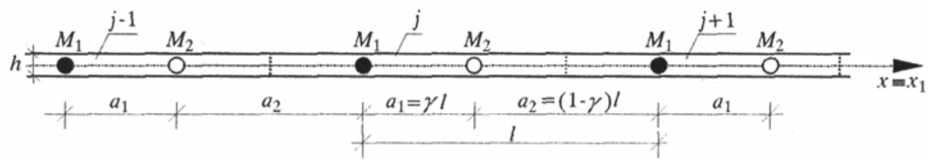


Fig. 1. Plate strip with periodically distributed system of two concentrated masses $M_1$, $M_2$

To analyse periodic plates, many 2D averaged plate models were formulated. Models based on the asymptotic homogenisation can be mentioned, e. g. those using the effective plate stiffnesses, Kohn and Vogelius (1984). However, these models neglect the effect of element length (called *the length-scale effect or the effect of periods lengths*) on the overall plate behaviour in the first approximation which is usually employed.

In this note, an alternative treatment of dynamic phenomena for periodic structures, called *the tolerance averaging technique*, is applied. This approach, presented for periodic composites and structures by Woźniak and Wierzbicki (2000), leads to models taking into account *the effect of periods lengths* on dynamic response of a plate. Those averaged models, called *the length-scale models*, were applied to analyse certain dynamic problems of periodic structures, e. g. for Hencky-Bolle periodic plates by Baron and Woźniak (1995), for Kirchhoff's periodic plates by Jędrysiak (2001a,b), for periodic wavy-plates by Michalak (2000, 2001). A list of papers related to investigations of periodic composites was shown in the book Woźniak and Wierzbicki (2000). Using the tolerance averaging to partial differential equations of periodic structures, the governing equations with constant coefficients are obtained. They describe the effect of period lengths by means of certain extra unknowns.

The main aim of this contribution is to show that results obtained by the new non-asymptotic model are very close to results found from the "exact" solution (the discrete model – Brillouin (1953)) for long-wave propagation problems. Moreover, similar results of both models will also be presented for special short-wave problems. In order to show the justification of the non-asymptotic model, frequencies of the travelling wave in the considered plate strip with an internal periodic structure calculated within the new model, Jędrysiak (2001b), a certain homogenised model and the one obtained from the "exact" solution, Brillouin (1953), will be compared.

## 2. The modelling approach

In our considerations, the governing equations of the non-asymptotic model of periodic plates derived and applied in Jędrysiak (2001b) are used. These equations and an outline of the procedure leading to them will be recalled in this section.

### 2.1. Foundations

Denote an orthogonal Cartesian co-ordinate system by $0x_1x_2x_3$ and define $t$ as the time co-ordinate. The plate midplane is denoted by $\Pi$ and the plate thickness by $h$. *The mesostructure parameter* is defined by $l$, which describes the size of the periodicicity cell, and is large compared to the plate thickness and small compared to the minimum characteristic length dimension of the plate midplane. For the plate strip under consideration the mesostructure parameter $l$ is the length of the cell $(\Delta \equiv [0,l] \times \{0\})$ along the $x_1$-axis. Moreover, denote by $a_{ijkl}$, $\rho$, $p$ components of the elastic moduli tensor, mass density, loading in the $x_3$-axis direction acting on the plate, respectively. The horizontal planes ($x_3 =$ const) are assumed to be the planes of elastic symmetry. Denote by $\boldsymbol{x} \equiv (x_1, x_2)$ a point on $\Pi$ and by $w(\boldsymbol{x}, t)$ a plate midplane deflection at the point $\boldsymbol{x} \in \Pi$. Throughout the paper, the subscripts $\alpha, \beta, \ldots$ run over $1, 2$ and indices $A, B, \ldots$ run over $1, \ldots, N$. The summation convention holds for all aforementioned indices.

The considerations are based on the well-known Kirchhoff plate theory assumptions: *kinematic constraints*, *strain-displacement relations*, *stress-strain relations*, *equations of motion*. We use the following notations for periodic functions

$$\mu \equiv \int\limits_{-h/2}^{h/2} \rho \, dz \qquad\qquad d_{\alpha\beta\gamma\delta} \equiv \int\limits_{-h/2}^{h/2} c_{\alpha\beta\gamma\delta} z^2 \, dz \tag{2.1}$$

describing plate properties as the mass density per unit area, bending stiffnesses, respectively; where $c_{\alpha\beta\gamma\delta} \equiv a_{\alpha\beta\gamma\delta} - a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$. The assumptions of the Kirchhoff plate theory for periodic plates lead to the known differential equation of the fourth order for the plate midplane deflection $w(\cdot, t)$ with highly oscillating periodic coefficients

$$(d_{\alpha\beta\gamma\delta} w_{,\gamma\delta})_{,\alpha\beta} + \mu \ddot{w} = p \tag{2.2}$$

The governing equations with constant coefficients can be derived using *the tolerance averaging technique*, Woźniak and Wierzbicki (2000). This procedure for thin periodic plates was shown in Jędrysiak (2001b) and for periodic wavy-plates in Michalak (2001). In the new modelling method, we use some additional concepts as: *an averaging operator* given by

$$\langle\varphi\rangle = \langle\varphi\rangle(\boldsymbol{x}) \equiv l^{-1} \int\limits_{\Delta(\boldsymbol{x})} \varphi(\boldsymbol{y})\, d\boldsymbol{y} \qquad \begin{array}{l} \boldsymbol{x} \in \Pi_\Delta \\ \boldsymbol{y} \in \Delta(\boldsymbol{x}) \end{array}$$

for the one-dimensional cell ($\Delta \equiv [0,l] \times \{0\}$) along the $x_1$-axis and for an arbitrary integrable function $\varphi$ defined on $\Pi$, where $\Delta(\boldsymbol{x}) \equiv \boldsymbol{x} + \Delta$, $\Pi_\Delta = \{\boldsymbol{x} : \ \boldsymbol{x} \in \Pi, \ \Delta(\boldsymbol{x}) \subset \Pi\}$; *tolerance system, slowly varying function, periodic-like function* and *oscillating function*, which were defined in the book by Woźniak and Wierzbicki (2000). Below, they will be reminded. Let $F$, $G$ be functions defined on $\Pi$ and $f$ be a $\Delta$-periodic function. If the approximation

$$\langle fF\rangle(\boldsymbol{x}) \cong \langle f\rangle F(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Pi_\Delta \tag{2.3}$$

holds for every $f$ (with the required accuracy, which depends on $f$) then $F$ is called *a slowly-varying function.* If for every $\boldsymbol{x} \in \Pi_\Delta$ there exists a $\Delta$-periodic function $G_{\boldsymbol{x}}$ such that the approximation

$$\langle fG\rangle(\boldsymbol{x}) \cong \langle fG_{\boldsymbol{x}}\rangle(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Pi_\Delta \tag{2.4}$$

holds as above, then $G$ is referred to as *a periodic-like function.* In this case $G_{\boldsymbol{x}}$ is termed a $\Delta$-periodic approximation of $G$ at $\boldsymbol{x}$.

Formulae (2.3)-(2.4) were called the tolerance averaging approximations in the aforementioned book. The dependency of these formulae on $\Delta$ and on a certain tolerance system $T$ was shown. We shall write $F \in SV(T)$ for the function $F$, which is slowly-varying together with its derivatives and $G \in PL(T)$ for a periodic-like function $G$.

The periodic-like function $G$ with the condition $\langle\mu G\rangle(\boldsymbol{x}) \cong 0$ for every $\boldsymbol{x} \in \Pi_\Delta$, where $\mu(\cdot)$ is a positive-valued $\Delta$-periodic function defined by $(2.1)_1$, is called *an oscillating periodic-like function.* The set of oscillating periodic-like functions with the weight $\mu$ is denoted by $PL^\mu(T)$.

Bearing in mind the assertions given in the book by Woźniak and Wierzbicki (2000), we recall that if $G \in PL(T)$ and $f$ is a $\Delta$-periodic function then $\langle Gf\rangle(\cdot) \in SV(T)$.

Detailed discussion of the above concepts was shown in the aforesaid book, in which the assertions of the tolerance averaging were formulated and proved.

### 2.2. Outline of the modelling procedure

An additional assumption called *the Conformability Assumption* (CA), cf. Woźniak and Wierzbicki (2000), Jędrysiak (2001b), is formulated. It states that the midplane deflection $w(\cdot,t)$ of the periodic plate has to conform to a plate structure, i.e. it can be represented by a periodic-like function, $w(\cdot,t) \in PL(T)$. This condition may be violated only near the plate boundary.

The modelling procedure of the tolerance averaging can be divided into four steps.

1) The deflection $w$ is a periodic-like function, $w(\cdot,t) \in PL(T)$, thus, the decomposition can be obtained

$$w(\cdot,t) = W(\cdot,t) + f(\cdot,t) \tag{2.5}$$

where $W$ is the averaged part of the deflection defined by $W(\cdot,t) \equiv \langle\mu\rangle^{-1}\langle\mu w\rangle(\cdot,t)$ ($\mu$ is the plate mass density); $f(\cdot,t) \in PL^{\mu}(T)$ is *fluctuation of the deflection* and holds the condition $\langle\mu f(\cdot,t)\rangle = 0$. Because $w(\cdot,t) \in PL(T)$ we have that $W(\cdot,t) \in SV(T)$. Hence, it is called *macrodeflection*.

2) *The periodic problem* is formulated on a cell $\Delta(\boldsymbol{x})$ for $f_{\boldsymbol{x}}$, which is a $\Delta$-periodic approximation of the fluctuation $f$ on $\Delta(\boldsymbol{x})$ at $\boldsymbol{x} \in \Pi_\Delta$. This problem is described by the condition $\langle\mu f_{\boldsymbol{x}}\rangle = 0$ and variational equation (cf. Woźniak and Wierzbicki, 2000; Jędrysiak, 2001b)

$$\langle f^*_{,\gamma\delta} d_{\alpha\beta\gamma\delta} f_{\boldsymbol{x},\alpha\beta}\rangle(\boldsymbol{x},t) + \langle\mu f^* \ddot{f}_{\boldsymbol{x}}\rangle(\boldsymbol{x},t) = \langle f^* p\rangle - \langle f^*_{,\gamma\delta} d_{\alpha\beta\gamma\delta}\rangle W_{,\alpha\beta} \tag{2.6}$$

which has to hold for every test function $f^*$, $\langle\mu f^*\rangle = 0$.

3) A solution $f_{\boldsymbol{x}}(\cdot,t)$ to Eq. (2.6) is looked for. For every $\boldsymbol{x} \in \Pi_\Delta$, Eq. (2.6) describes vibrations $f_{\boldsymbol{x}}(\boldsymbol{x},t)$ of the cell $\Delta(\boldsymbol{x})$ with the periodic boundary conditions on the cell edges. The right-hand side of (2.6) can be interpreted as certain time dependent loadings on the cell $\Delta(\boldsymbol{x})$. Hence, general solutions to *periodic problem* (2.6) can be obtained using the method applied to dynamics of structures with an arbitrary time dependent loading. After neglecting the loadings in (2.6), we have a periodic problem on $\Delta(\boldsymbol{x})$ given by

$$(d_{\alpha\beta\gamma\delta} f_{\boldsymbol{x},\alpha\beta})_{,\gamma\delta} + \mu\ddot{f}_{\boldsymbol{x}} = 0 \tag{2.7}$$

Assuming $f_{\boldsymbol{x}}(\boldsymbol{y},t) = g(\boldsymbol{y})\cos(\omega t)$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, from (2.7) we obtain periodic eigenvalue problems of finding $\Delta$-periodic functions $g$ satisfying in $\Delta(\boldsymbol{x})$ the following equation

$$[d_{\alpha\beta\gamma\delta}(\boldsymbol{y}) g_{,\alpha\beta}(\boldsymbol{y})]_{,\gamma\delta} - \mu(\boldsymbol{y})\lambda^2 g(\boldsymbol{y}) = 0 \qquad \begin{array}{l} \boldsymbol{y} \in \Delta(\boldsymbol{x}) \\ \boldsymbol{x} \in \Pi_\Delta \end{array} \tag{2.8}$$

where $\lambda$ are eigenvalues. From averaging (2.8) on $\Delta(\boldsymbol{x})$ we have $\langle \mu g \rangle = 0$. The functions $g$ have to satisfy the periodic boundary conditions on the edges of $\Delta(\boldsymbol{x})$ and the same regularity conditions as the deflection. An approximate solution $f_{\boldsymbol{x}}(\boldsymbol{y}, t)$ to problem (2.7), which may be obtained by the orthogonalisation method, will be assumed in the form $g^A(\boldsymbol{y})Q^A(\boldsymbol{x}, t)$, where $g^A(\cdot)$, $A = 1, 2, \ldots$, is a sequence of eigenfunctions defined on $\Delta(\boldsymbol{x})$ for the above eigenvalue problem and related to the sequence of eigenvalues $\lambda_A$. *The fluctuation of the deflection* $f(\cdot, t)$, being a solution to the periodic problem on $\Delta(\boldsymbol{x})$ given by variational condition (2.6), as every function can be written in the form of Fourier series. The infinite series can be approximated by a truncated series (cf. Woźniak and Wierzbicki, 2000; Jędrysiak, 2001b) in the following form

$$f(\boldsymbol{y}, t) \simeq g^A(\boldsymbol{y})Q^A(\boldsymbol{x}, t) \tag{2.9}$$

where $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Delta$; $A = 1, \ldots, N$ and $N$ determines different degrees of the approximation; $g^A$ stands for the system of $N$ linear-independent $\Delta$-periodic functions, such that $\langle \mu g^A \rangle = 0$ and $l^{-1}g^A(\cdot)$, $g^A_{,\alpha}(\cdot)$, $lg^A_{,\alpha\beta}(\cdot) \in \mathcal{O}(l)$; $Q^A(\cdot, t) \in SV(T)$ are new kinematic unknowns. Functions $g^A$ are called *mode-shape functions.* These functions approximate the expected form of the oscillating part of free vibration modes of the $\Delta$-periodic structure of the plate, cf. (Woźniak and Wierzbicki, 2000; Jędrysiak, 2001b).

4) After some transformations, using the tolerance averaging approximations, we obtain equations for the macrodeflection $W$ and equations for the kinematic unknowns $Q^A$. These equations are written in the subsequent Section.

### 2.3. Governing equations of the non-asymptotic model

The aforementioned modelling procedure leads to the following equations of *the non-asymptotic model* of plates with an internal periodic structure (Jędrysiak, 2001b)

$$\begin{aligned}
&\langle d_{\alpha\beta\gamma\delta} \rangle W_{,\alpha\beta\gamma\delta} + \langle \mu \rangle \ddot{W} + \langle d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} \rangle Q^A_{,\alpha\beta} = \langle p \rangle \\
&\langle d_{\alpha\beta\gamma\delta} g^A_{,\gamma\delta} \rangle W_{,\alpha\beta} + \langle d_{\alpha\beta\gamma\delta} g^A_{,\alpha\beta} g^B_{,\gamma\delta} \rangle Q^B + \underline{\langle \mu g^A g^B \rangle} \ddot{Q}^B = \underline{\langle p g^A \rangle}
\end{aligned} \tag{2.10}$$

where the underlined terms depend on the mesostructure parameter $l$. All coefficients in brackets $\langle \cdot \rangle$ are constant. The functions $W$, $Q^A$, $A = 1, \ldots, N$, are basic unknowns which have to be slowly varying functions. The function $W$ is called *the plate macrodeflection*; functions $Q^A$ are called *the internal variables*.

Summarizing, *the non-asymptotic model* is defined by:

1° Equations (2.10) for $N+1$ unknowns, $W(\cdot,t)$ and $Q^A(\cdot,t)$, $A=1,\dots,N$

2° Conditions of applicability of the model, i.e. Eqs. (2.10) have physical sense for the unknowns $W$, $Q^A$ being slowly varying functions for every $t$ (it is a certain *a posteriori* criterion of physical reliability for this model of periodic plates)

3° The plate deflection can be approximated by means of the formula

$$w(\cdot,t) \simeq W(\cdot,t) + g^A(\cdot)Q^A(\cdot,t)$$

where the approximation " $\simeq$ " is related to the assumption that the fluctuation of the deflection is defined in the form of the truncated series $g^A(\cdot)Q^A(\cdot,t)$, $A=1,\dots,N$.

It is easy to see that in order to obtain the above equations, we must previously derive the mode-shape functions $g^A$, $A=1,\dots,N$, for every periodic plate under consideration as solutions to a certain eigenvalue problem on the periodicity cell, given by Eq. (2.8). In practice, derivation of these exact solutions is possible only for cells with a structure which is not too complicated. In papers by Jędrysiak (2001a,b), such solutions were shown for a plate strip with periodically varied but piece-wise constant thickness, Young's modulus and mass density of the plate. But in most cases we have to look for an approximate form of these solutions, which is sufficient from the computational point of view. However, in order to obtain exact solutions to that problem for cells with a more complicated structure, the finite element method can be applied. We also restrict our considerations to a small number $N$ of mode shapes. For the analysed structure, which will be presented in Section 3.1.1(b), we have only one mode-shape function, and hence $N=1$ and $g \equiv g^1$. In the subsequent Sections both solutions to the eigenvalue problem, Eq. (2.8), i.e. exact and approximate, will be shown and applied. Equations (2.10) are derived in the paper by Jędrysiak (2001b).

At the end of this Section, let us observe that neglecting the underlined terms in Eqs. (2.10), we obtain governing equations of *the homogenised model* in the form

$$\begin{aligned} &\langle d_{\alpha\beta\gamma\delta}\rangle W_{,\alpha\beta\gamma\delta} + \langle\mu\rangle\ddot{W} + \langle d_{\alpha\beta\gamma\delta}g^A_{,\gamma\delta}\rangle Q^A_{,\alpha\beta} = \langle p\rangle \\ &\langle d_{\alpha\beta\gamma\delta}g^A_{,\alpha\beta}g^B_{,\gamma\delta}\rangle Q^B = -\langle d_{\alpha\beta\gamma\delta}g^A_{,\gamma\delta}\rangle W_{,\alpha\beta} \end{aligned} \tag{2.11}$$

where the effect of mesostructure parameter $l$ is not taken into account.

# 3. A travelling wave in a weightless unbounded plate strip with a periodically distributed system of two concentrated masses

## 3.1. Averaged models

### *3.1.1. Non-asymptotic model*

**(a)** *Frequencies of travelling wave*

Let us consider a homogeneous unbounded plate strip along the $\boldsymbol{x} \equiv x_1$ axis, whose periodicity is related to a periodically distributed system of two concentrated masses $M_1$ and $M_2$ (Fig. 2). The distance between masses $M_1$ and $M_2$ is denoted by $a_1$ and the distance between masses $M_2$ and $M_1$ by $a_2$ (hence $a_2 = l - a_1$). It is assumed that Young's modulus $E$, Poisson's ratio $\nu$ and also thickness $h$ of the plate are constant. Moreover, the plate mass is negligibly small compared with the concentrated masses $M_1$ and $M_2$. Loads $p$ are neglected. In our considerations, only one mode-shape function $g$ is assumed. Denote $Q \equiv Q^1$ as well as

$$
\begin{aligned}
B &\equiv \langle d_{1111} \rangle = \frac{Eh^3}{12(1-\nu^2)} & \qquad D &\equiv \langle d_{1111}(g_{,11})^2 \rangle \\
\widetilde{m} &\equiv \langle \mu \rangle & m^{11} &\equiv l^{-4}\langle \mu g^2 \rangle
\end{aligned}
\tag{3.1}
$$

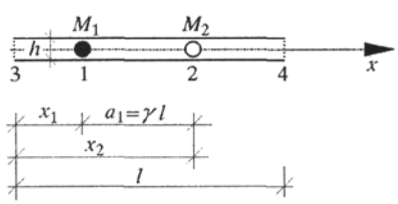


Fig. 2. Periodicity cell of plate strip under consideration

Moreover, for the homogeneous plate is $\langle d_{1111} g_{,11} \rangle = 0$. For $A = N = 1$ from (2.1) we arrive at

$$
BW_{,1111} + \widetilde{m}\ddot{W} = 0 \qquad\qquad DQ + l^4 m^{11}\ddot{Q} = 0 \tag{3.2}
$$

It can be observed that we obtain two independent differential equations of the plate strip under consideration. The first of them describes vibrations of the plate strip for the "macro" scale, and the second equation determines "micro" vibrations related to the periodic system of two concentrated masses.

Introduce now a wave number $k$ (e. g. $k = 2\pi/L$). Solutions to equations (3.2) will be assumed as

$$W(\boldsymbol{x},t) = A_W \exp[\mathrm{i}(kx - \omega t)] \qquad Q(\boldsymbol{x},t) = A_Q \exp[\mathrm{i}(kx - \omega t)] \tag{3.3}$$

where $A_W$, $A_Q$ are amplitudes, $\omega$ is a frequency. After some transformations we arrive at formulae for the lower $\omega_-$ and higher $\omega_+$ *frequency of the travelling wave within the non-asymptotic model* for the plate strip under consideration

$$(\omega_-)^2 = \frac{Bk^4}{\widetilde{m}} \qquad (\omega_+)^2 = \frac{D}{l^4 m^{11}} \tag{3.4}$$

In this case, only the higher frequency $\omega_+$ explicitly depends on the mesostructure parameter $l$. This frequency is related to the internal periodic structure of the plate strip.

(b) *Eigenvalue problem of the periodicity cell*

Denote by $x$ a coordinate on a certain axis of the cell; $\boldsymbol{x} \in [0,l]$; and the derivative by $(\cdot)' \equiv (\cdot)_{,1}$. Eigenfunctions for the periodicity cell will be obtained by solving eigenvalue problem (2.8), which takes the form

$$Bg^{\mathrm{IV}}(x) - \mu(x)\lambda^2 g(x) = 0 \tag{3.5}$$

with periodic boundary conditions on the cell edges; $B$ is the stiffness defined by $(3.1)_1$; $g$ are $l$-periodic functions related to eigenvalues $\lambda \equiv \alpha l$ ($\alpha$ is the wave number); and $\langle \mu g \rangle = 0$. In the case under consideration, after neglecting the plate mass when compared with the concentrated masses, the exact form of eigenfunctions $g(x)$ can be found. Functions $g$ describe forms of free vibrations of the cell. From structural dynamics, it is known that for the considered cell (with two degrees of freedom) we have two eigenfunctions satisfying Eq. (3.5). Under the condition $\langle \mu g \rangle = 0$, we obtain only one eigenfunction $g$.

In the following considerations, the deflection of the cell can be described similarly to deflections of the beam with the stiffness $B$ defined by $(3.1)_1$. In order to find the function $g$, methods of structural mechanics will be applied. Introduce the following functions

$$\begin{aligned} r(\xi) &= 1 - 3\xi^2 + 2\xi^3 & \qquad \overline{r}(\xi) &= 3\xi^2 - 2\xi^3 \\ u(\xi) &= \xi - 2\xi^2 + \xi^3 & \qquad \overline{u}(\xi) &= \xi^2 - \xi^3 \end{aligned} \tag{3.6}$$

where $\xi \in [0,1]$. Joints of masses $M_1$ and $M_2$ are described by 1 and 2, respectively; and joints of the left and right ends of the cell by 3 and 4 (see Fig. 2). Deflections and rotations of the joints with masses $M_1$, $M_2$ (at points

$x_1$, $x_2$) are denoted by $v_1$, $\varphi_1$, $v_2$, $\varphi_2$, respectively, and of the joints of the left and right ends by: $v_3$, $\varphi_3$, and $v_4$, $\varphi_4$. Deflections of parts 3-1, 1-2, 2-4 will be described by

$$\begin{aligned} v_{31}(\xi) &= r(\xi)v_3 + u(\xi)\varphi_3 l_{31} + \overline{r}(\xi)v_1 - \overline{u}(\xi)\varphi_1 l_{31} \\ v_{12}(\xi) &= r(\xi)v_1 + u(\xi)\varphi_1 l_{12} + \overline{r}(\xi)v_2 - \overline{u}(\xi)\varphi_2 l_{12} \\ v_{24}(\xi) &= r(\xi)v_2 + u(\xi)\varphi_2 l_{24} + \overline{r}(\xi)v_4 - \overline{u}(\xi)\varphi_4 l_{24} \end{aligned} \tag{3.7}$$

where $l_{ik} = x_k - x_i$ is the length of the part $i-k$ ($i = 3,\ 1,\ 2,\ 4$; $k = 1,\ 2,\ 4$); $x_i$, $x_k$ are coordinates of the joints $i$, $k$; $x \in [x_i, x_k]$; $\xi = (x - x_i)(l_{ik})^{-1}$. The function $g$ has to satisfy periodic boundary conditions on the cell ends, i.e.

$$\begin{aligned} & g(0) = g(l) = v_3 = v_4 && g'(0) = g'(l) = \varphi_3 = \varphi_4 \\ & g''(0) = g''(l) && g'''(0) = g'''(l) \end{aligned} \tag{3.8}$$

From the equilibrium equations of known from structural mechanics formulae for transversal forces and moments and inertia forces and moments related to the concentrated masses, which are depended on the aforesaid displacements of the joints, and from boundary conditions (3.8) and the normalizing condition $\langle \mu g \rangle = 0$, we arrive at the characteristic equation for eigenvalues $\lambda \equiv \alpha l$ ($\alpha$ is the wave number) in the form of a determinant equaled to zero

$$\det L_{pr} = 0 \qquad p, r = 1, \dots, 4 \tag{3.9}$$

where

$$\begin{aligned} & L_{11} = \zeta\lambda^4 - \frac{12(1+\zeta)l^3}{(x_2 - x_1)^3} - \frac{12 l^3}{x_1^3} + \frac{\zeta\lambda^4}{2x_1^2}\Big\{\frac{2}{l}[(x_1 - l)^3 - (x_2 - l)^3] + \\ & +3[(x_1 - l)^2 - (x_2 - l)^2] + (x_1 - x_2)l\Big\} \\ & L_{12} = \frac{12 l^3}{x_1^3} \qquad L_{13} = 6l^2\Big(\frac{1}{x_1^2} - \frac{1}{(x_2 - x_1)^2}\Big) \qquad L_{14} = -\frac{6l^2}{(x_2 - x_1)^2} \\ & L_2 = \frac{6(1+\zeta)l^2}{(x_2 - x_1)^2} - \frac{6l^2}{x_1^2} + \frac{\zeta\lambda^4}{6x_1}\Big\{\frac{2}{l^2}[(x_1 - l)^3 - (x_2 - l)^3] + \\ & +\frac{3}{l}[(x_1 - l)^2 - (x_2 - l)^2] + x_1 - x_2\Big\} \\ & L_{22} = \frac{6l^2}{x_1^2} \qquad L_{23} = 4l\Big(\frac{1}{x_1} + \frac{1}{x_2 - x_1}\Big) \qquad L_{24} = \frac{2l}{x_2 - x_1} \end{aligned} \tag{3.10}$$

$$L_{31} = -\zeta\lambda^4 + \frac{12(1+\zeta)l^3}{(x_2-x_1)^3} + \frac{12l^3}{(l-x_2)^3} + \frac{\zeta\lambda^4}{2(l-x_2)^2}\cdot$$
$$\cdot\Big\{\frac{2}{l}[(x_2-l)^3-(x_1-l)^3] + 3[(x_2-l)^2-(x_1-l)^2] + (x_2-x_1)l\Big\}$$
$$L_{32} = \frac{12l^3}{(l-x_2)^3} \qquad L_{33} = \frac{6l^2}{(x_2-x_1)^2}$$
$$L_{34} = -6l^2\Big(\frac{1}{(l-x_2)^2} - \frac{1}{(x_2-x_1)^2}\Big)$$
$$L_{41} = \frac{6(1+\zeta)l^2}{(x_2-x_1)^2} - \frac{6l^2}{(l-x_2)^2} + \frac{\zeta\lambda^4}{6(l-x_2)}\Big\{\frac{2}{l^2}[(x_1-l)^3-(x_2-l)^3] +$$
$$+\frac{3}{l}[(x_1-l)^2-(x_2-l)^2] + x_1 - x_2\Big\}$$
$$L_{42} = -\frac{6l^2}{(l-x_2)^2} \qquad L_{43} = \frac{2l}{x_2-x_1}$$
$$L_{44} = 4l\Big(\frac{1}{l-x_2} + \frac{1}{x_2-x_1}\Big)$$

and $\zeta \geqslant 0$ is defined by

$$\zeta \equiv \frac{M_1}{M_2} \tag{3.11}$$

From Eq. (3.9), we can derive one eigenvalue $\lambda$ (hence $A = N = 1$) dependent of the quotient $\zeta$. Introduce notations

$$\Psi_1 = L_{22}(L_{33}L_{44} - L_{43}L_{34}) + L_{32}(L_{43}L_{24} - L_{23}L_{44}) + L_{42}(L_{23}L_{34} - L_{33}L_{24})$$
$$\Psi_2 = L_{22}(L_{41}L_{34} - L_{31}L_{44}) + L_{32}(L_{21}L_{44} - L_{41}L_{24}) + L_{42}(L_{31}L_{24} - L_{21}L_{34})$$
$$\Psi_3 = L_{22}(L_{31}L_{43} - L_{41}L_{33}) + L_{32}(L_{41}L_{23} - L_{21}L_{43}) + L_{42}(L_{21}L_{33} - L_{31}L_{23})$$
$$\Xi = L_{21}(L_{43}L_{34} - L_{33}L_{44}) + L_{31}(L_{23}L_{44} - L_{43}L_{24}) + L_{41}(L_{33}L_{24} - L_{23}L_{34})$$

Formulae for needed deflections and rotations of joints 1, 2, 3, 4 take the following form

$$\begin{aligned} &v_1 = l\Psi_1\Xi^{-1} \qquad && \varphi_1 = \Psi_2\Xi^{-1} \\ &v_2 = -\zeta v_1 && \varphi_2 = \Psi_3\Xi^{-1} \\ &\varphi_3 = \varphi_4 = v_1\frac{\zeta\lambda^4}{12l^2}\Big\{\frac{2}{l^2}[(x_1-l)^3-(x_2-l)^3] + \\ &\qquad +\frac{3}{l}[(x_1-l)^2-(x_2-l)^2] + x_1 - x_2\Big\} \end{aligned} \tag{3.12}$$

Bearing in mind formulae (3.12), (3.10), (3.6)-(3.7), the exact form of *the mode-shape function* $g$, related to the eigenvalue $\lambda$ obtained from (3.9), can be written as

$$g(x) = \begin{cases} Al^2 B_0 & \text{if} \quad x \in [0, x_1] \\ Al^2 B_1 & \text{if} \quad x \in (x_1, x_2] \\ Al^2 B_2 & \text{if} \quad x \in (x_2, l] \end{cases} \tag{3.13}$$

where $A$ is a constant and

$$B_0 = r\Big(\frac{x}{x_1}\Big)\frac{v_3}{l} + u\Big(\frac{x}{x_1}\Big)\varphi_3\frac{x_1}{l} + \overline{r}\Big(\frac{x}{x_1}\Big)\frac{v_1}{l} - \overline{u}\Big(\frac{x}{x_1}\Big)\varphi_1\frac{x_1}{l}$$

$$B_1 = r\Big(\frac{x-x_1}{x_2-x_1}\Big)\frac{v_1}{l} + u\Big(\frac{x-x_1}{x_2-x_1}\Big)\varphi_1\frac{x_2-x_1}{l} + \overline{r}\Big(\frac{x-x_1}{x_2-x_1}\Big)\frac{v_2}{l} + \\ -\overline{u}\Big(\frac{x-x_1}{x_2-x_1}\Big)\varphi_2\frac{x_2-x_1}{l}$$

$$B_2 = r\Big(\frac{x-x_2}{l-x_2}\Big)\frac{v_2}{l} + u\Big(\frac{x-x_2}{l-x_2}\Big)\varphi_2\frac{l-x_2}{l} + \overline{r}\Big(\frac{x-x_2}{l-x_2}\Big) - \overline{u}\Big(\frac{x-x_2}{l-x_2}\Big)\varphi_3\frac{l-x_2}{l}$$

#### *3.1.2. Homogenised model*

In order to compare the calculated results, *the homogenised model* will also be applied. Equations (2.11) of the considered plate strip, under assumption $A = N = 1$, take the form

$$BW_{,1111} + \widetilde{m}\ddot{W} = 0 \tag{3.14}$$

Assuming the solution to Eq. (3.14) in the form of $(3.3)_1$, after some transformations we arrive at the following formula of only one frequency of the travelling wave within *the homogenised model*

$$\omega^2 = \frac{Bk^4}{\widetilde{m}} \tag{3.15}$$

The above formula is identical with the lower frequency, $(3.4)_1$ for the non-asymptotic model.

### 3.2. "Exact" solution – discrete model

To verify the obtained results, the above problem of the plate strip will be analysed within *the discrete model*, which is similar to that applied by Brillouin (1953) to investigate longitudinal vibrations of a one-dimensional structure in which a repeatable element was related to a system of two concentrated masses

$M_1$ and $M_2$ (Fig. 1). In the aforementioned paper, this model was treated as the "exact" solution to the problem under consideration.

Let us now consider cells of the periodic structure numbered by $j-1, j$, $j+1$. Denote by $w_s^{M_1}$, $\varphi_s^{M_1}$ and $w_s^{M_2}$, $\varphi_s^{M_2}$ deflections and rotations of the concentrated masses $M_1$ and $M_2$, respectively, at cell $s = j-1, j, j+1$. Using the known formulae from structural mechanics for forces and moments of the weightless unbounded plate strip (treated as a beam with stiffness $B$ – $(3.1)_1$) and taking into account inertia forces and moments for the concentrated masses, the equilibrium equations could be written for both masses. After some transformations, we arrive at the following system of equations for deflections and rotations

$$\begin{aligned}
&M_1\ddot{w}_j^{M_1} + 6B[2(a_1^{-3}+a_2^{-3})w_j^{M_1} - 2(a_1^{-3}w_j^{M_2} + a_2^{-3}w_{j-1}^{M_2}) + \\
&+(a_1^{-2}-a_2^{-2})\varphi_j^{M_1} + (a_1^{-2}\varphi_j^{M_2} - a_2^{-2}\varphi_{j-1}^{M_2})] = 0 \\
&M_2\ddot{w}_j^{m} - 6B[2(a_1^{-3}w_j^{M_1} + a_2^{-3}w_{j+1}^{M_1}) - 2(a_1^{-3}+a_2^{-3})w_j^{M_2} + \\
&+(a_1^{-2}\varphi_j^{M_1} - a_2^{-2}\varphi_{j+1}^{M_1}) + (a_1^{-2}-a_2^{-2})\varphi_j^{M_2}] = 0 \\
&2B[3(a_1^{-2}-a_2^{-2})w_j^{M_1} - 3(a_1^{-2}w_j^{M_2} - a_2^{-2}w_{j-1}^{M_2}) + 2(a_1^{-1}+a_2^{-1})\varphi_j^{M_1} + \\
&+(a_1^{-1}\varphi_j^{M_2} + a_2^{-1}\varphi_{j-1}^{M_2})] = 0 \\
&2B[3(a_1^{-2}w_j^{M_1} - a_2^{-2}w_{j+1}^{M_1}) - 3(a_1^{-2}-a_2^{-2})w_j^{M_2} + (a_1^{-1}\varphi_j^{M_1} + a_2^{-1}\varphi_{j+1}^{M_1}) + \\
&+2(a_1^{-1}+a_2^{-1})\varphi_j^{M_2}] = 0
\end{aligned} \tag{3.16}$$

where $a_1$ and $a_2$ are distances between the masses $M_1$, $M_2$ and $M_2$, $M_1$. Solutions to the above equations will be found in the form

$$\begin{aligned}
w_j^{M_1} &= A_{w_{M_1}} \exp[\mathrm{i}(kjl - \omega t)] \\
\varphi_j^{M_1} &= A_{\varphi_{M_1}} \exp[\mathrm{i}(kjl - \omega t)] \\
w_j^{M_2} &= A_{w_{M_2}} \exp\Big\{\mathrm{i}\Big[k\Big(j + \frac{a_1}{l}\Big)l - \omega t\Big]\Big\} \\
\varphi_j^{M_2} &= A_{\varphi_{M_2}} \exp\Big\{\mathrm{i}\Big[k\Big(j + \frac{a_1}{l}\Big)l - \omega t\Big]\Big\}
\end{aligned} \tag{3.17}$$

where $A_{w_{M_1}}$, $A_{w_{M_2}}$, $A_{\varphi_{M_1}}$, $A_{\varphi_{M_2}}$ are amplitudes, $k$ is the wave number (e.g. $k = 2\pi/L$, and $L$ is the wave length), $l$ is the mesostructure parameter (cell length), $\omega$ is a frequency. Substituting the right-hand sides of (3.17) to Eqs. (3.16), we obtain a system of four algebraic homogeneous equations for amplitudes which has non-trivial solutions under the condition that its determinant is equal to zero. From this condition, we obtain a characteristic equation for the frequency $\omega$ in the form

$$\alpha l^6 \varpi^4 - \beta l^3 \varpi^2 + \delta = 0 \tag{3.18}$$

with coefficients defined by

$$\begin{aligned}
&\alpha \equiv \zeta (M_2)^2 \gamma^2 (1-\gamma)^2 \{3 + 2\gamma[1 - \cos(kl)](1-\gamma)\} \\
&\beta \equiv 12 B M_2 (\zeta + 1)[2 + \cos(kl)] \\
&\delta \equiv 72 B^2 [3 + \cos(2kl) - 4\cos(kl)]
\end{aligned} \tag{3.19}$$

where $B$ is stiffness of the plate strip $(3.1)_1$; $\gamma = a_1/l$. Solutions to (3.18), being the "exact" formulae for frequencies of the travelling wave in the framework of *the exact discrete model*, have the following forms

$$\varpi_1^2 = \frac{1}{2\alpha l^3}(\beta - \sqrt{\beta^2 - 4\alpha\delta}) \qquad \varpi_2^2 = \frac{1}{2\alpha l^3}(\beta + \sqrt{\beta^2 - 4\alpha\delta}) \tag{3.20}$$

where $\varpi_1$ and $\varpi_2$ are *the lower* and *higher frequencies*, respectively.

It can be observed that only the *exact discrete* and the *new length-scale model* make it possible to investigate higher frequencies related to the periodic structure of the plate strip. The *homogenised model* is not able to describe higher vibrations.

## 4. Special case – short-wave propagation problem

Applications of the new length-scale model (and also the homogenised model) are related to long-wave propagation problems (in which $kl \ll 1$). This restriction is caused by conditions of applicability of the proposed model, i.e. unknowns $W$, $Q^A$ which have to be slowly varying functions. However, the "exact" solution (the discrete model) can be used to analyse not only long waves but also short waves ($kl > 1$).

In this Section, it will be presented that special cases of short-wave problems can be investigated also in the framework of the non-asymptotic model. In order to show such application, we will consider a travelling wave propagating in the plate strip, in which the periodicity cell has two pairs of concentrated masses $M_1$, $M_2$ and span $2l$. To analyse that problem, the mode-shape function in the exact form, derived as in Subsection 3.1.1b, can be used, but this procedure leads to determinant (3.9) in which $p, r = 1, \ldots, 8$. Hence, simplifying our investigations, we will apply only approximate forms of mode-shape functions.

Let us consider a special plate strip with equal distances between the masses $M_1$, $M_2$, $a_1 = a_2 = l/2$, where $l$ is the mesostructure parameter (cf. Fig. 2). For the long-wave propagation problem, it will be shown that the application of approximate forms of mode-shape functions is sufficient from the calculational point of view. Analysing free vibrations of that plate strip (the cell with span $l$), we will apply the exact form, (3.13), of the mode-shape function $g$ and also the following approximate form

$$g(x) = l^2\Big(c + \sin\frac{2\pi x}{l}\Big) \qquad x \in [0,l] \tag{4.1}$$

where the constant $c$ derived from $\langle \mu g\rangle = 0$ is

$$c = -\frac{1}{1+\zeta}\Big(\zeta \sin\frac{2\pi x_1}{l} + \sin\frac{2\pi x_2}{l}\Big)$$

and $x_1$, $x_2$ are coordinates of the masses $M_1$, $M_2$; $\zeta$ is defined by (3.11). Function (4.1) stands for an approximate solution to eigenvalue problem (3.5) for the cell with periodic boundary conditions. Assuming $x_1 = l/4$, $x_2 = 3l/4$, we have

$$c = \frac{1-\zeta}{1+\zeta} \tag{4.2}$$

Correctness of applications of the approximate forms of mode-shape functions for cells with a simple structure was shown in Jędrysiak (2001a,b) for plate strips, whose masses were not negligibly small compared to concentrated masses.

For the short-wave propagation problem, i.e. in analysis of free vibrations of the plate strip in which a cell has two pairs of concentrated masses $M_1$, $M_2$ and span $2l$, the mode-shape function $g$, satisfying equation (3.5), will be assumed only in an approximate form given by

$$g(x) = l^2\Big(c + \sin\frac{\pi(x+d)}{l}\Big) \qquad x \in [0,2l] \tag{4.3}$$

where $d$ is a constant; the constant $c = 0$ from the condition $\langle \mu g\rangle = 0$ for every $d$. Consider two cases of function (4.3):

— the first for $d = l/4$

$$g(x) = l^2 \sin\Big[\pi\Big(\frac{x}{l} + \frac{1}{4}\Big)\Big] \tag{4.4}$$

which has roots at the joints with masses $M_2$, i.e. at $x_2 = 3l/4$, $x_2 = 7l/4$ – the second for $d = -l/4$

$$g(x) = l^2 \sin\Big[\pi\Big(\frac{x}{l} - \frac{1}{4}\Big)\Big] \tag{4.5}$$

which has roots at the joints with masses $M_1$, i.e. at $x_1 = l/4$, $x_1 = 5l/4$.

Analysing the travelling wave, the following formulae for frequencies will be used: (3.4) for the non- asymptotic model, (3.15) for the homogenised model and (3.20) for the discrete ("exact") model.

Now, certain coefficients will be calculated. For the discrete model, for $a_1 = a_2 = l/2$ (hence $\gamma = a_1/l = 1/2$) the coefficients defined by $(3.19)_{1,2}$ take the form

$$\begin{aligned}\alpha &= \frac{1}{16}\zeta(M_2)^2\Big\{3 + \frac{1}{2}[1 - \cos(kl)]\Big\} \\ \beta &= 12BM_2(\zeta+1)[2 + \cos(kl)]\end{aligned} \tag{4.6}$$

Moreover, we can calculate from (3.1) coefficients for each of the three above approximate forms of mode-shape functions. Hence, we obtain for these three cases that

$$B = \frac{Eh^3}{12(1-\nu^2)} \qquad \widetilde{m} = M_2\frac{\zeta+1}{l} \tag{4.7}$$

where by applying $(4.6)_2$ the concentrated masses $M_1$, $M_2$ will be equally "scattered" equally along the cell length $l$. Using (4.1) from $(3.1)_2$ and $(3.1)_4$, we have

$$\widetilde{m}^{11} \equiv m^{11} = 4\frac{M_2}{l}\frac{\zeta}{\zeta+1} \qquad \widetilde{D} \equiv D = 8\pi^4 B \tag{4.8}$$

However, using the mode-shape functions (4.4) or (4.5), the coefficient in $(3.1)_2$ takes the form

$$\widehat{D} \equiv D = \frac{1}{2}\pi^4 B \tag{4.9}$$

and coefficient $(3.1)_4$ for function (4.4) is

$$\widehat{m}^{11} \equiv m^{11} = M_2\frac{\zeta}{l} \tag{4.10}$$

and for function (4.5)

$$\breve{m}^{11} \equiv m^{11} = M_2\frac{1}{l} \tag{4.11}$$

In the subsequent Section, some calculation results of the above problems will be presented.

## 5. Calculation results

In this Section, comparison between the frequencies of the travelling wave obtained for three models: *non-asymptotic model, homogenised model* and *exact discrete model* in the above Sections, will be shown. Introduce a dimensionless wave number: $q \equiv kl$, and dimensionless frequency parameters describing frequencies of the travelling wave

$$
\begin{array}{lll}
\Omega_{-} \equiv \sqrt{\chi}\omega_{-} \qquad & \Omega_{+} \equiv \sqrt{\chi}\omega_{+} \qquad & \Omega \equiv \sqrt{\chi}\omega \\
\Lambda_1 \equiv \sqrt{\chi}\varpi_1 & \Lambda_2 \equiv \sqrt{\chi}\varpi_2 &
\end{array}
$$

where $\chi \equiv 12(1-\nu^2)M_2/E$; $E$ is Young's modulus; $\nu$ is Poisson's ratio; $\omega_{-}$, $\omega_{+}$ are frequencies (3.4) of the non-asymptotic model; $\omega$ is frequency (3.15) of the homogenised model; $\varpi_1$, $\varpi_2$ are frequencies (3.20) of the discrete model; $k$ is the wave number; $l$ is the mesostructure parameter (the cell length).

In calculations, we assume that the thickness of the plate strip is equal $h = 0.1l$, the ratio between masses $M_1$ and $M_2$ is given by the parameter $\zeta = 2$ (defined by (3.11)), i.e. $M_1 = 2M_2$.

Results are shown in Fig. 3 and Fig. 4. In Fig. 3 dispersion curves of parameters $\Omega_{-}$, $\Omega_{+}$, $\Omega$, $\Lambda_1$, $\Lambda_2$ (describing frequencies) versus the dimensionless wave number $q \in [-\pi, \pi]$ are presented (for both the cases: long-wave and short-wave propagation problems). These plots are made, using exact form (3.13) of the mode-shape function $g$, for the concentrated masses $M_1$, $M_2$ located at points: $x_1 = 0.1l$, $x_2 = 0.9l$ – lines 1; $x_1 = 0.25l$, $x_2 = 0.75l$ – lines 2. In this Figure, a part between the vertical dashed lines is related to long waves, $q \in [-0.1\pi, 0.1\pi]$.

The results illustrating the short-wave propagation problem are presented in Fig. 4. Dispersion curves of $\Omega_{-}$, $\Omega_{+}$, $\Omega$, $\widetilde{\Omega}_{+}$, $\widehat{\Omega}_{+}$, $\breve{\Omega}_{+}$, $\Lambda_1$, $\Lambda_2$ versus the dimensionless wave number $q \in [-\pi, \pi]$ for the plate strip with equal distances between the masses $M_1$, $M_2$, i.e. $a_1 = a_2 = l/2$ are shown. By higher frequencies are denoted $\Omega_{+}$, $\widetilde{\Omega}_{+}$, $\widehat{\Omega}_{+}$, $\breve{\Omega}_{+}$. They are obtained, respectively, for exact mode-shape function $g$ (3.13), for approximate form (4.1) of the mode-shape function $g$, for approximate form (4.4) and for approximate form (4.5).

By analysing the obtained calculation results, one can conclude that:

- lower frequencies of the travelling wave of the considered plate strip calculated within the non-asymptotic model are identical with the frequencies from the homogenised model and also with the frequencies from the discrete model not only for long-wave propagation problems

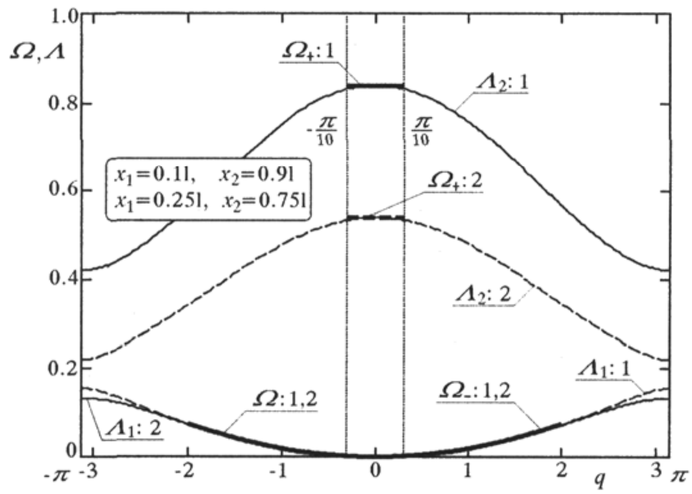


Fig. 3. Dispersion curves of parameters $\Omega_-$, $\Omega_+$, $\Omega$, $\Lambda_1$, $\Lambda_2$; (lines $\Lambda_1$, $\Lambda_2$ – discrete ("exact") model, lines $\Omega_-$, $\Omega_+$ – non-asymptotic model, lines $\Omega$ – for homogenised model); ($h = 0.1l$; $M_1 = 2M_2$; $x_1$, $x_2$ –coordinates of masses $M_1$, $M_2$)

($q \in [-0.1\pi, 0.1\pi]$) but also in a wider range of the dimensionless wave number $q$ (Fig. 3 and Fig. 4);

- higher frequencies of the travelling wave from the non-asymptotic model stand for the upper limit of higher frequencies calculated within the discrete model (Fig. 3 and Fig. 4); the differences between frequencies for both models are less than 2% (for long waves: $q \in [-0.1\pi, 0.1\pi]$, i.e. $k = 2\pi/L$, $l/L \in [-0.05, 0.05]$);

- differences between higher frequencies for long-wave propagation problems, $q \in [-0.1\pi, 0.1\pi]$, obtained from the non-asymptotic model with exact (3.13) or approximate (4.1) form of the mode-shape function $g$, are less than 2% (Fig. 4);

- frequencies of the travelling wave can be investigated in the framework of the proposed length-scale model also for special cases of short-wave propagation problems by using special approximate forms of mode-shape functions, e.g. (4.4) and (4.5); higher frequencies calculated from the non-asymptotic model stand for lower limits of higher frequencies from the discrete model (for mode-shape function (4.4)) or upper limits of lower frequencies from the discrete model (for mode-shape function (4.5)).

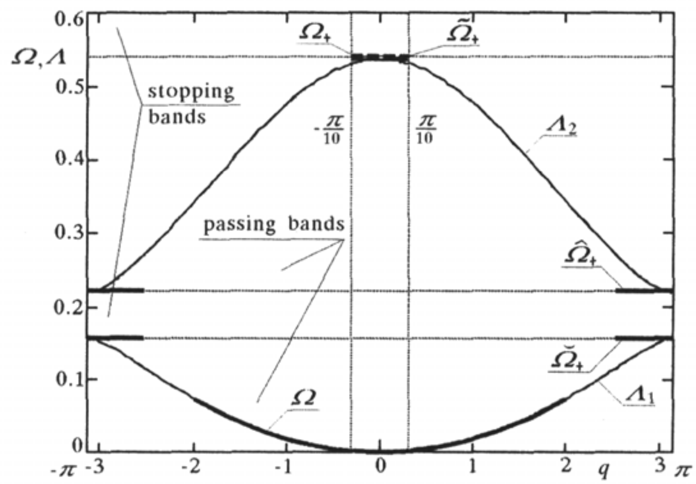


Fig. 4. Dispersion curves of parameters $\Omega_-$, $\Omega_+$, $\Omega$, $\widetilde{\Omega}_+$, $\widehat{\Omega}_+$, $\breve{\Omega}_+$, $\Lambda_1$, $\Lambda_2$; (discrete ("exact") model – lines $\Lambda_1$, $\Lambda_2$; non-asymptotic model: for exact (3.13) mode-shape function $g$ – lines $\Omega_-$, $\Omega_+$, and approximate (4.1) mode-shape function $g$ – $\widetilde{\Omega}_+$; non-asymptotic model for approximate mode-shape function $g$: (4.4) – $\widehat{\Omega}_+$, (4.5) – $\breve{\Omega}_+$; homogenised model – $\Omega$); ($h = 0.1l$, $M_1 = 2M_2$)

## 6. Final remarks

The applied *length-scale model* of periodic plates, taking into account the effect of period lengths on the overall plate behaviour, makes it possible to analyse higher frequencies of the travelling wave related to the internal periodic plate structure. Frequencies of this kind can be investigated only for special structures such as one-dimensional diatomic lattice, Brillouin (1953).

Here, the non-asymptotic model has been applied to investigate frequencies of the travelling wave propagating in a thin weightless unbounded plate strip with a system of two concentrated masses $M_1$, $M_2$ periodically distributed in the plate. For structures of this kind, we can obtain "exact" formulae for two frequencies of the travelling wave – the lower and higher, using *the discrete model* similar to that applied by Brillouin (1953) who analysed longitudinal vibrations of one-dimensional diatomic lattice. In the paper, comparison of the results obtained within the new length-scale model, the homogenised model and the discrete model has been presented.

Summarizing all considerations, one can formulate the following conclusions regarding periodic plates, in particular a weightless unbounded plate strip with a periodically distributed system of two concentrated masses:

- The presented examples confirm that the effect of period lengths plays a crucial role in vibrations of periodic plates, and can also be analysed within the new length-scale model. In the paper it has been that the non-asymptotic model can be applied in under to obtain frequencies of higher order vibrations, similarly as in the exact models.

- The non-asymptotic model can be applied, first of all, to long-wave problems (compared with the mesostructure parameter $l$ describing the cell size), leading to results conformable with those found from "exact" solutions to special problems.

- Some special short waves, whose lengths are of the order of the cell size, can also be investigated within the framework of the non-asymptotic model by using proper forms of mode-shape functions.

- Higher frequencies, calculated within the non-asymptotic model using those mode-shape functions, define so called *passing* and *stopping bands* of lower and higher order vibrations of periodic plates, i.e. ranges of frequencies in which vibrations of plates take place.

- In special cases of periodic plates, approximate forms of mode-shape functions can be used instead of the exact forms of those functions, yielding sufficiently precise results from the calculational point of view.

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## Porównanie wyników analizy dynamicznej według uśrednionych i dokładnych modeli cienkich płyt periodycznych

Streszczenie

Celem pracy jest dokonanie pewnej weryfikacji nowego nieasymptotycznego modelu cienkich płyt periodycznych, wyprowadzonego przy zastosowaniu tolerancyjnego uśredniania (Woźniak i Wierzbicki, 2000). W przeciwieństwie do znanych modeli zhomogenizowanych, model ten opisuje wpływ wymiaru komórki periodyczności na ogólną pracę płyty. Wyniki uzyskane w ramach tych modeli będą porównane z rozwiązaniem według "dokładnego" modelu dyskretnego. Można pokazać, że w zagadnieniach propagacji fal długich wyniki otrzymane w ramach modelu nieasymptotycznego dla szczególnego przypadku periodycznego pasma płytowego (nieważkiego lecz z periodycznie rozłożonym układem dwóch mas skupionych) są bliskie wynikom uzyskanym ze znanych rozwiązań "dokładnych", które wykorzystują metodę stosowaną do analizy drgań podłużnych jednowymiarowej dwuatomowej siatki (Brillouin, 1953). Pokazano także podobną zgodność wyników w szczególnych przypadkach propagacji fal krótkich.

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