Lviv, 8 October, 2019

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Opinion on the habilitation application and other scientific achievements of Dr. Tomasz Filpczak

The cycle of papers of Dr. T.Filipczak, submitted for habilitation consists of seven papers [H1]–[H7] published in sufficiently high-ranked journals ("Math. Logic Quaterly", "Topology Appl.", "Real Analysis Exchange", "Acta Math. Hungarica", "Tatra Mt. Publ.", "J. Math. Anal. Appl") and also in a collective monograph (published in Lodz in 2011 and dedicated to Prof. W.Wilczyński). All papers but one are published with coauthors (M.Balcerzak, A.Bartoszewicz, M.Filipczak, G.Horbaczewska, A.Rosłanowski, S.Shelah, W.Wilczyński), witnessing about the ability of Dr. T.Filipczak to collaborate fruitfully with other mathematicians.

As follows from the complete list of publications ([H1]–[H7] & [P1]–[P25]), the central topic of interests of Dr. T. Filipczak concerns density topologies, where he obtained many interesting results. Nonetheless, he decided to make habilitation in a wide area including 4 main topics: (i) hull operators, (ii) Steinhaus and Smital properties for measures on topological groups, (iii) f-densities, and (iv) exceptional points of Lebesgue measurable sets. In each of these four topics Dr. T.Filipczak obtained interesting and deep results. Theorems about hull operators (obtained jointly with M.Balcerzak, A.Rosłanowski, S.Shelah), Steinhaus and Smital properties (joint with A.Bartoszewicz and M.Filipczak), and f-densities (joint with M.Filipczak) seem to be of fundamental importance in the respective fields, and the results on the last topic nicely complement and enrich the existing striking results obtained by other mathematicians (Kolyada, Szenes, Csörney, Grahl, O'Neil, Kurka).

Now let me survey the most impressive results of T.Filipczak in each of four topics.

1. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X, $\mathcal{I} \subset \mathcal{A}$ be a σ-ideal and $\mathcal{H} \subset \mathcal{A}$ be a subfamily. An *$\mathcal{H}$-hull operator* is a function $\varphi : \mathcal{P}(X) \to \mathcal{H}$ assigning to each set $A \subset X$ a set $\varphi(A) \in \mathcal{H}$ such that $A \subset \varphi(A)$ and $\varphi(A) \setminus B \in \mathcal{I}$ for any $B \in \mathcal{A}$ such that $A \subset B$. So, φ enlarges any subset $A \subset X$ to a subset $\varphi(A) \in \mathcal{H}$ so that the difference $\varphi(A) \setminus A$ is small with respect to the ideal $\mathcal{I}$. An $\mathcal{H}$-hull operator is called *monotone* if $\varphi(A) \subset \varphi(B)$ for any $A \subset B$. If X is a group, then an $\mathcal{H}$-hull operator is called *translation invariant* if $\varphi(A + x) = \varphi(A) + x$ for any $A \subset X$ and $x \in X$.

Standard examples of triples $(X, \mathcal{A}, \mathcal{I})$ include the triple $(\mathbb{R}, \mathcal{L}, \mathcal{N})$, where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets in the real line and $\mathcal{N}$ is the ideal of Lebesgue null sets. Another important example, is the triple $(\mathbb{R}, \mathcal{B}a, \mathcal{M})$ where $\mathcal{B}a$ is the σ-algebra of sets with the Baire property in $\mathbb{R}$ and $\mathcal{M}$ is the σ-ideal of meager sets in $\mathbb{R}$.

The problem of the existence of monotone hull operators for the triple $(\mathbb{R}, \mathcal{L}, \mathcal{N})$ was first considered by Elekes and Máthé in 2009. In the papers [H2] and [H4], T.Filipczak (with coauthors) obtains many interesting results on hull operators in the general situation. In particular, in [H2] he constructs monotone hull operators for the triple $(\mathbb{R}, \mathcal{B}a, \mathcal{M})$ and some other natural triples $(X, \mathcal{A}, \mathcal{I})$. It should be mentioned that in the construction of such operators, Filipczak applied his deep knowledge of (generalized) density topologies. On the other hand, in [H2] Balcerzak and Filipczak proved that the problem of the existence of a monotone *Borel* hull operator essentially depend on some additional Set-Theoretic Axioms.

Simple (but exciting and inexpected) results of Filipczak concern *translation invariant* hull operators. Using the classical Steinhaus property of the σ-ideals $\mathcal{M}$ and $\mathcal{N}$, Filipczak (with coauthors) observed in [H4] that the existence of a subgroup $G \subset \mathbb{R}$ of class $\mathcal{M} \setminus \mathcal{N}$ (resp. $\mathcal{N} \setminus \mathcal{M}$) implies the non-existence of a translation hull operator for the pair $(\mathbb{R}, \mathcal{B}a, \mathcal{M})$ (resp. $(\mathbb{R}, \mathcal{L}, \mathcal{N})$). Subgroups in $\mathcal{M} \setminus \mathcal{N}$ and $\mathcal{N} \setminus \mathcal{M}$ exist under CH and some other weaker Set-Theoretic assumptions. It is interesting to remark that the situation with the existence of subgroups of $\mathbb{R}$ in the families $\mathcal{M} \setminus \mathcal{N}$ and $\mathcal{N} \setminus \mathcal{M}$ is not symmetric. According to a recent (2014) result of Rosłanowski and Shelah, the class $\mathcal{N} \setminus \mathcal{M}$ contains a subgroup of $\mathbb{R}$ in ZFC; on the other hand, in some models of ZFC the class $\mathcal{M} \setminus \mathcal{N}$ does not contain subgroups of $\mathbb{R}$. It should be mentioned that the first ZFC-example of a subgroup in the class $\mathcal{N} \setminus \mathcal{M}$ was obtained by Talagrand in 1980 (in terms of non-meager filters). Unfortunately this result of Talagrand was not mentioned in the papers [H2] and [H4].

2. The second topic of the habilitation cycle of Dr. T.Filipczak concerns the Steinhaus and Smital properties of some measures on topological groups. According to the classical results of Steinhaus and Smital, for any subsets $A, B \subset \mathbb{R}$ of positive Lebesgue measure and any dense set $D \subset \mathbb{R}$, the algebraic sum $A+B$ has non-empty interior and the sum $A+D$ has full measure (i.e., the complement $\mathbb{R} \setminus (A+D)$ is Lebesgue null). Analogous results for the category were obtained by Pettis, Piccard, and some other mathematicians. In the papers [H6] and [H7] (written jointly with M.Filipczak and A.Bartoszewicz), T.Filipczak studies the Steinhaus and Smital properties of the Bernoulli measures μ_p on the real line. Those are the images of the countable power of the measure $p\delta_0 + (1-p)\delta_1$ under the standard Cantor map $\gamma : \{0,1\}^{\mathbb{N}} \to [0,1]$, $\gamma : (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} \frac{x_n}{2^n}$. For $p = \frac{1}{2}$ the Bernoulli measure is just the standard Lebesgue measure on $[0,1]$. By the Strong Law of Large Number, the measure μ_p is supported on the set $A_p = \{x \in [0,1] : \lim_{n\to\infty} \frac{1}{n}(x_1 + \cdots + x_n) = p\}$, where $(x_1, x_2, \dots)$ is the binary expansion of x. For $p = \frac{1}{2}$ we know that $A_p - A_p = (-1,1)$ and $A_p + A_p = (0,2)$. In the papers [H6] and [H7] the authors study the properties of the sets $A_p - A_p$ and $A_p + A_p$, $A_p + A_p + A_p$, etc., for numbers $p \in (0,1)$, different from $\frac{1}{2}$. Using some elementary (but nontrivial) tools from algebraic combinatorics of finite cyclic groups, Filipczak (with coauthors) proves that for $p \in [\frac{1}{4}, \frac{3}{4}]$ the difference $A_p - A_p$ still contains the segment $(-1,1)$ but for $p < \frac{1}{3}$ the triple sum $A_p + A_p + A_p$ has empty interior.

Therefore, for $p \in [\frac{1}{4}, \frac{1}{3})$ we obtain a natural Borel subset A_p such that $A_p - A_p = (-1, 1)$ but $A_p + A_p + A_p$ has empty interior. This answers my problem posed in 2003 and also a problem of Roman Ger.

Moreover, in [H7], T.Filipczak (jointly with A. Bartoszewicz and M. Filipczak) obtained interesting characterizations of singular and absolutely continuous measures on a locally compact Polish group X in terms of Steinhaus and Smital properties:

- a measure μ on X is absolutely continuous with respect to the Haar measure λ on X if and only if μ has the Steinhaus property (which means that $A - A$ has non-empty interior for any μ-positive set $A \subset X$) if and only if μ has the Smital property (which means that for any μ-positive set $A \subset X$ and any dense set $D \subset X$ the set $X \setminus (A + D)$ is μ-null);
- a probability measure μ on X is singular with respect to the Haar measure λ on X if and only if μ is extremely no Steinhaus (which means that $A - A$ has empty interior for some set $A \subset X$ with $\mu(X \setminus A) = 0$) if and only if μ is extremely no Smital (which means that $A + D$ is μ-null for some dense set $D \subset X$ and some $A \subset X$ with $\mu(X \setminus A) = 0$).

The topic of Steinhaus and Smital properties was also considered in the paper [P23] of M.Filipczak, T.Filipczak, and R.Knapik.

3. The third topic of the habilitaion cycle concerns f-density topologies and was elaborated in the papers [H1] and [H3]. Here $f : (0, \infty) \to (0, \infty)$ is a monotone function such that $\liminf_{x \to +0} \frac{f(x)}{x} < \infty$. A point $x \in \mathbb{R}$ is called a *point of f-density* of a measurable subset $A \subset \mathbb{R}$ if $\lim_{h \to +0} \frac{\lambda([x-h,x+h] \setminus A)}{f(h)} = 0$. Let $\Phi_f(A)$ be the set of all points of f-density of the set A. The family $\mathcal{T}_f := \{A \in \mathcal{L} : A \subset \Phi_f(A)\}$ is a topology on $\mathbb{R}$ called the *f-density topology*. If $f(x) = x$, then the f-density topology coincides with the classical density topology $\mathcal{T}_d$. The f-density topologies also generalize ψ-density topologies (for functions $f(x) = x\psi(x)$) and $\langle s \rangle$-density topologies for decreasing vanishing sequences $\langle s \rangle = (s_n)_{n \in \omega}$. The notion of f-density was introduced in 2006 by T.Filipczak and M.Filipczak in their paper [P13] (in the additional list of publications).

The definition of f-density points implies that the f-density topology $\mathcal{T}_f$ is translation invariant. In [P13] the authors presented an example of a function f for which the f-density topology is not invariant with respect to linear transformation of the real line. On the other hand, in the paper [H3], T.Filipczak and M.Filipczak proved that the topology $\mathcal{T}_f$ (with $\mathcal{T}_f \subset \mathcal{T}_d$) is invariant under affine transformations of the real line if (and only if) the function f satisfies the conditions Δ_2: $\lim_{x \to +0} \frac{f(2x)}{f(x)} < \infty$. This condition resembles the doubling condition for metrics, which is important in the fractal and coarse geometries. The paper [H3] also contains many conditions on functions f, g under which the respective density topologies $\mathcal{T}_f$ and $\mathcal{T}_g$ are comparable or equal.

4. The fourth and last topic of the habilitation cycle concerns modifications (for other difference bases) of the striking results of Kolyada, Szenes, Csörney, Grahl, O'Neil, Kurka on exceptional points of measurable sets in the real line. In 1983 Viktor Kolyada (Ukrainian

mathematician from Odesa) proved that there exists a positive real number δ such that for any measurable subset $A \subset [0,1]$ with $0 < \lambda(A) < 1$ there exists a point $x \in [0,1]$ such that

$$\delta \le \liminf_{h\to+0} \frac{\lambda(A\cap[x-h,x+h])}{\lambda([x-h,x+h]} \le \limsup_{h\to+0} \frac{\lambda(A\cap[x-h,x+h])}{\lambda([x-h,x+h])} \le 1-\delta.$$

The largest δ with this property is denoted by $\delta_{\mathcal{H}}$. Kolyada proved that $\frac{1}{4} \le \delta_{\mathcal{H}} \le \frac{\sqrt{17}-3}{4} \approx 0,2808$. These upper and lower bounds were improved by Szenes (2011), Csörney, Grahl, O'Neil (2012) and finally by Kurka (2012) who found the exact value of $\delta_{\mathcal{H}} \approx 0.2685$, equal to the unique solution of the equation $8x^3+8x^2+x-1=0$.

In the paper [H5] (joint with M.Filipczak, G.Horbaczewska and W.Wilczyński) and [P25] (joint with G. Horbaczewska), T.Filipczak consider two modifications $\delta_{\mathcal{B}_r}$ and $\delta_{\mathcal{B}_{\langle s\rangle}}$ of the number $\delta_{\mathcal{H}}$, where r is a positive real number and $\langle s\rangle = (s_n)_{n\in\omega}$ is a decreasing to zero sequence of real numbers. Replacing in the definition of $\delta_{\mathcal{H}}$ the interval $[x-h,x+h]$ by the interval $[x-h,x+rh]$ (resp. by $[x-s_n,x+s_n]$), we obtain the definition of the number $\delta_{\mathcal{B}_r}$ (resp. $\delta_{\mathcal{B}_{\langle s\rangle}}$). The paper [H5] contains many lower and upper bounds for these numbers. In particular, it was proved that $(\frac{1}{r+1})^2 \le \delta_{\mathcal{B}_r} \le \frac{1}{r+1}$ for any $r \ge 1$, and $\delta_{\mathcal{B}_{\langle s\rangle}} = \delta_{\mathcal{H}}$ if $\lim_{n\to\infty}\frac{s_{n+1}}{s_n} = \delta_{\mathcal{H}}$. In the paper [P25] it is proved that $\delta_{\mathcal{B}_{\langle s\rangle}} \le \frac{33-\sqrt{577}}{32} \approx 0.2806$ for any sequence $\langle s\rangle$.

This completes my survey of the main publications [H1]–[H7] for the habilitation. The remaining publications [P1]–[P25] concern mainly density topologies, porosity. The papers [P4]–[P11] study so-called local systems which allow to formulate some interesting general conditions guaranteeing that a function that has a weak continuity property with respect to a local system is a function of the first Baire class. All these additional publications [P1]–[P25] also make a good impression and contribute essentially to the knowledge in various subareas of Real Analysis. Some of these results were mentioned above during the discussion of the main publications [H1]–[H7]. The papers [P1]–[P25] are published in typical journals for Real Analysis: "Real Anal. Exchange", "Acta Univ. Lodz. Folia Math.", "Demonstratio Math.", "Tatra Mt. Math. Publ.", "Comment Math.", "J. Math. Anal. Appl.".

To my opinion the principal results presented to habilitation are the following:

- Construction of monotone hull operators for various σ-algebras and σ-ideals under suitable Set-Theoretic assumptions in [H2];
- The proof of non-existence of translation-invariant hull operators in [H4];
- Establishing algebraic properties of supports of the Bernoulli measures μ_p for various p in [H7];
- The characterizations of absolutely continuous measures and singular measures in terms of Steinhaus and Smital properties in [H7];
- The introducing and studying basic properties of f-densities and respective topologies in [H1], [H3], [P1], [P3], [P13].

Those are important achievements of fundamental importance in the field of Real Analysis, covering all requirements to the scientific degree of Dr.Hab. **So, without any doubts, I support this application.**

Concerning the citations of the publications of Dr. T.Filipczak, the situation is not very impressive: 37 in MathSciNet, 21 in ZblMath, 15 in Scopus, and 12 citations in WoS, and without self-citations: 18, 13, 9, 8, respectively. The h-index is also not very high: 2 according to Scopus and WoS. To improve these numbers, I would recommend to Dr. T.Filipczak to put his (published) papers as preprints to open public sources (arXiv, ReserchGate, etc.) In fact, it is very difficult to find open publications of T.Filipczak in the Internet. I understand that the commercial publishers have their own interests restricting the access to publications behind paywalls, but the interests of the scientific community should not be identical to interests of publishing corporations, so mathematicians should invent some (legal) ways to be more visible and accessible.

Besides the scientific achievements, Dr. T. Filipczak made also important didactic work at Łódź University of Technology: in 2009 he delivered a cycle of popular lectures on graph theory for pupils, supervised writing 33 Master Theses – 2 in 1998-1999 and 31 in 2005-2018, was a (pomocniczy) supervisor of Ph.D. Thesis by Piotr Nowakowski, served as a referee for many journals. Many times he obtained awards from the Rectors of the Łódź University and Łódź University of Technology for scientific and didactic achievements.

Now the necessary sentence in Polish.

Konkluzja:

Uważam, że recenzowana rozprawa habilitacyjna, dorobek naukowy oraz pozostałe osiągnięcia wypełniają wymagania Ustawy z dnia 18 marca 2011 r. o zmianie ustawy o stopniach naukowych i tytule naukowym oraz o stopniach i tytule w zakresie sztuki oraz zmianie niektórych innych ustaw (Dz. U. Nr 84, poz. 455), i popieram wniosek o nadanie stopnia doktora habilitowanego dr. Tomaszowi Filipczakowi.