ON A DYNAMICAL STABILITY OF THIN PERIODIC PLATES

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In this contribution a dynamical stability of thin periodic plates is considered. For this purpose *the tolerance averaging*, developed for periodic composites and structures in the book [8], is applied to the known Kirchhoff-type plate equation. This method leads to averaged models taking into account *the length-scale effect* on the overall plate behaviour. It was presented for thin periodic plates in [3]. Here, applications to a dynamical stability of those plates will be shown.

1. INTRODUCTION

The main object of this paper is a thin plate whose structure is periodic in planes parallel to the plate midplane. Plates of this kind are composed of many identical small elements, (Fig. 1), treated as thin plates with spans l_1, l_2, and called *periodicity cells*.

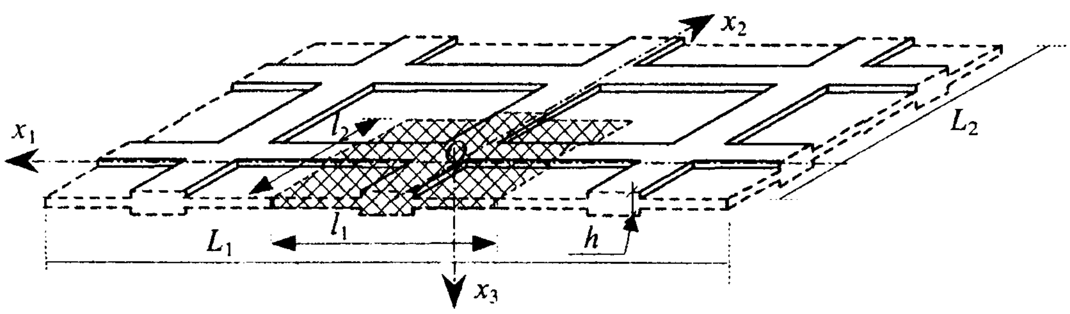

Fig 1. Fragment of thin plate with periodic structure

Analysis of the effect of the periodicity cell size, called *the length-scale effect*, on the overall plate behaviour is very interesting. But in general, the exact equations of the plate theory are too complicated to apply to investigations of engineering problems, because they comprise highly oscillating, non-continuous, periodic coefficients. Thus, many simplified models were proposed. In these averaged models periodic plates are represented by certain homogeneous plate structures with constant homogenised rigidities and averaged mass densities (cf. Caillerie [2], Kohn and Vogelius [5]). Unfortunately, in the most of averaged models the length-scale effect is neglected.

To investigate this effect on the overall plate behaviour the new model of periodic Kirchhoff-type plates was proposed by Jędrysiak [3], where the governing equations of *the length-scale models* were derived. These models are based on general formulations of the tolerance averaging method developed for periodic composites and structures by Woźniak and Wierzbicki in the book [8]. In the aforementioned papers it was presented that the length-scale effect plays a crucial role in dynamic processes and also in certain stationary problems, e. g. in a plate buckling. Similar problems of periodic wavy-plates were analysed by Michalak [6].

The main aim of the contribution is to show that the length-scale effect can not be neglected in a dynamic stability of thin periodic plates. In order to illustrate this thesis a periodic plate band with span L along x_1-axis will be investigated. A similar problem for periodic Reissner-type plates was presented by Baron [1].

2. FOUNDATIONS

2.1. PRELIMINARIES

Introduce the orthogonal Cartesian co-ordinate system in the physical space denoted by $Ox_1x_2x_3$. Let indices α, β, ... $(i,j,...)$ run over 1, 2 (1, 2, 3); A, B,... run over 1,..., N. Summation convention holds for all aforementioned indices. Setting $\mathbf{x}\equiv(x_1,x_2)$ and $z\equiv x_3$ we assume that the region $\Omega\equiv\{(\mathbf{x},z):-h(\mathbf{x})/2<z<h(\mathbf{x})/2,\ \mathbf{x}\in\Pi\}$ is occupied by the undeformed plate, where Π is the midplane and $h(\mathbf{x})$ is the plate thickness at the point $\mathbf{x}\in\Pi$. Denote by $\Delta\equiv(-l_1/2,l_1/2)\times(-l_2/2,l_2/2)$ the periodicity cell on the $0x_1x_2$ plane, where l_1, l_2 are the cell length dimensions along x_1-, x_2-axis. Define by $l\equiv\sqrt{l_1^2+l_2^2}$ the parameter describing the size of the cell, which is assumed to be sufficiently small compared to the minimum characteristic length dimension of Π and sufficiently large compared to the maximum plate thickness ($h_{\max}<<l<<L_\Pi$). Thus, this parameter is called *the mesostructure parameter*. We shall assume that h is a Δ-periodic function in $\mathbf{x}$ and all material and inertial properties of the plate, e. g. a mass density $\rho=\rho(\mathbf{x},z)$ and elastic moduli $a_{ijkl}=a_{ijkl}(\mathbf{x},z)$, are also Δ-periodic functions in $\mathbf{x}$ and even functions in z, cf. [3]. Periodic plates with the structure will be called *mesoperiodic plates*. Moreover, let w be a plate deflection and p^-, p^+ be loadings in the z-axis direction on the upper and lower plate boundaries. The non-zero terms of the elastic moduli tensor are denoted by $a_{\alpha\beta\gamma\delta}$, $a_{\alpha\beta33}$, a_{3333}; denote also $c_{\alpha\beta\gamma\delta}\equiv a_{\alpha\beta\gamma\delta}-a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$. The considerations are based on the well-known Kirchhoff plate theory assumptions. Introduce the following Δ-periodic functions of the mean plate properties - mass density, bending stiffnesses

$$\mu\equiv\int_{-h/2}^{h/2}\rho dz,\qquad\qquad d_{\alpha\beta\gamma\delta}\equiv\int_{-h/2}^{h/2}z^2c_{\alpha\beta\gamma\delta}dz.$$

The known modelling procedure of Kirchhoff plate theory for mesoperiodic plates leads to the well known fourth order differential equation with highly oscillating Δ-periodic coefficients, which takes the following form

$$(d_{\alpha\beta\gamma\delta}w_{,\gamma\delta})_{,\alpha\beta}-N_{\alpha\beta}w_{,\alpha\beta}+\mu\ddot{w}=p\,, \tag{1}$$

where $p\equiv p^++p^-$; $N_{\alpha\beta}$ (α, β=1, 2) are constant forces in the plate midplane.

2.2. INTRODUCTORY CONCEPTS

In order to take into account the length-scale effect we will adapt *the tolerance averaging method* developed by Woźniak and Wierzbicki [8] for periodic composites, to derive averaged governing equations of periodic plates. In the framework of the method we use some additional concepts as e. g. an averaging operator, a tolerance system, a slowly varying function, a periodic-like function and an oscillating function, which are detaily explained in the book [8].

Define by $\Delta(\mathbf{x})=\Delta+\mathbf{x}$ a periodicity cell at $\mathbf{x}\in\Pi_\Delta$, $\Pi_\Delta=\{\mathbf{x}:\mathbf{x}\in\Pi,\ \Delta(\mathbf{x})\subset\Pi\}$. In the analysis of periodic structures we use the known *averaging operator* (cf. [8, 3]):

$$< \varphi > (\mathbf{x}) \equiv (l_1 l_2)^{-1} \int_{\Delta(\mathbf{x})} \varphi(\mathbf{y}) dy, \qquad \mathbf{x} \in \Pi_\Delta, \tag{2}$$

for an arbitrary integrable function φ defined on Π. If φ is a periodic function in $\mathbf{x}$ its averaged value obtained from (2) is constant. We shall tacitly assume that all functions under consideration satisfy required regularity conditions.

Now, some introductory concepts will be reminded. Denote by T a certain mapping (called a tolerance system), which assigns to every quantity under consideration what is called tolerance parameter (cf. [8]). The continuous function Ψ, defined on $\overline{\Pi}$, will be called *a slowly varying function*, if for every $\mathbf{x}_1$, $\mathbf{x}_2 \in \overline{\Pi}$ such that $\|\mathbf{x}_1-\mathbf{x}_2\| \leq l$ it holds the following condition $F(\mathbf{x}_1) \cong F(\mathbf{x}_2)$. We shall write $F \in SV(T)$ if F and all its derivatives are slowly varying functions. Let for every $\mathbf{x} \in \overline{\Pi}$ a symbol $f_{\mathbf{x}}$ stand for a certain continuous Δ-periodic function. The continuous function f will be called *a periodic-like function* if for every $\mathbf{y} \in \overline{\Pi}$ such that $\|\mathbf{x}-\mathbf{y}\| \leq l$ it holds the condition $f(\mathbf{y}) \cong f_{\mathbf{x}}(\mathbf{y})$. If derivatives of the function f hold similar conditions we will write $f \in PL(T)$. It can be shown ([8]) that averaging (2) of periodic-like function is a slowly varying function. A periodic-like function f will be called *an oscillating function* if it holds the condition $<\mu f>(\mathbf{x}) \cong 0$ for every $\mathbf{x} \in \Pi_\Delta$, where μ is a positive value Δ-periodic function. The set of oscillating periodic-like functions with the weight μ is denoted by $PL^{\mu}(T)$.

The modelling procedure of the tolerance averaging is based on lemmas and assertions, which were formulated and proved in the book [8] using the above concepts.

2.3. MODELLING PROCEDURE

In the modelling procedure we formulate the additional assumption.

The Conformability Assumption (*CA*). It is assumed that the deflection $w(\cdot,t)$ of the plate midplane under consideration is periodic-like function, $w(\cdot,t) \in PL(T)$, i. e. the deflection is conformable to a periodic plate structure. This condition may be violated only near the boundary of a plate.

The modelling procedure of the tolerance averaging can be divided into four steps.

1° The averaging part of deflection is defined by setting $W \equiv <\mu>^{-1}<\mu w>$; where μ is the mass density of the plate. Because of $w \in PL(T)$ we have that $W \in SV(T)$. Thus, the decomposition is obtained $w=W+v$, where $v \in PL^{\mu}(T)$ is called *the deflection disturbance*, and holds the condition $<\mu v>=0$. The averaged part of deflection W will be called *a macrodeflection*.

2° We formulate *the periodic problem* (cf. [8, 3]) on $\Delta(\mathbf{x})$ for $v_{\mathbf{x}}$ being a Δ-periodic approximation of v on a cell $\Delta(\mathbf{x})$ at $\mathbf{x} \in \Pi_\Delta$. Function $v_{\mathbf{x}}$ holds the condition $<\mu v_{\mathbf{x}}>=0$.

3° It is formulated the Galerkin approximation of the above periodic problem by introducing the system of N linear-independent Δ-periodic functions g^A, $A=1,\dots,N$, such that $<\mu g^A>=0$, and by setting $v_{\mathbf{x}}(\mathbf{y},t)=g^A(\mathbf{y})Q^A(\mathbf{x},t)$, where $\mathbf{y} \in \Delta(\mathbf{x})$, $\mathbf{x} \in \Pi_\Delta$; $Q^A \in SV(T)$ are new kinematic unknowns. Functions g^A are called *mode-shape functions* and have to approximate the expected form of the oscillating part of free vibration modes of the Δ-periodic structure of the plate, cf. [8, 3]. Moreover, values of these functions are assumed to satisfy conditions $l^{-1}g^A(\cdot), g^A_{,\alpha}(\cdot), lg^A_{,\alpha\beta}(\cdot) \in O(l)$.

4° After some manipulations we arrive at the equation for the macrodeflection W and equations for kinematic unknowns Q^A.

These equations are derived without the assumption introduced in [3] that in terms $N_{\alpha\beta}w_{,\alpha\beta}$ the deflection w can be replaced by the macrodeflection W.

3. GOVERNING EQUATIONS

Applying the above procedure, under the following denotations:

$$D_{\alpha\beta\gamma\delta} \equiv < d_{\alpha\beta\gamma\delta} >, \quad D^{A}_{\alpha\beta} \equiv < d_{\alpha\beta\gamma\delta} g^{A}_{,\gamma\delta} >, \quad D^{AB} \equiv < d_{\alpha\beta\gamma\delta} g^{A}_{,\alpha\beta} g^{B}_{,\gamma\delta} >, \quad h^{AB}_{\alpha\beta} \equiv l^{-2} < g^{A}_{,\alpha} g^{B}_{,\beta} >,$$
$$m \equiv < \mu >, \quad m^{AB} \equiv l^{-4} < \mu g^{A} g^{B} >, \quad P \equiv < p >, \quad P^{A} \equiv l^{-2} < p g^{A} >,$$

we arrive at *the length-scale model* equations:

$$\begin{gathered} (D_{\alpha\beta\gamma\delta} W_{,\gamma\delta} + D^{B}_{\alpha\beta} Q^{B})_{,\alpha\beta} - N_{\alpha\beta} W_{,\alpha\beta} + m\ddot{W} = P, \\ l^4 m^{AB} \ddot{Q}^{B} + D^{A}_{\alpha\beta} W_{,\alpha\beta} + D^{AB} Q^{B} + N_{\alpha\beta} l^2 h^{AB}_{\alpha\beta} Q^{B} = l^2 P^{A}, \end{gathered} \tag{3}$$

where $N_{\alpha\beta}$ is in-plane stress tensor; some terms depend explicitly on the parameter l.

Equations (3) with averaged constant coefficients, make it possible to analyse the length-scale effect in dynamic processes and also in a stability of periodic plates. The basic unknowns W, Q^A, $A=1,\dots,N$, are slowly varying functions. For a rectangular plate with midplane $\Pi=(0,L_1)\times(0,L_2)$ two boundary conditions should be defined on the edges $x_1=0$, L_1 and $x_2=0$, L_2 only for *the macrodeflection W*. Hence, functions Q^A are called *internal variables*. To derive equations (3) we have previously to obtain the mode-shape functions g^A, $A=1,\dots,N$, for every periodic plate under consideration. In the most cases $N=1$ and $g=g^1$ is an approximate solution to the eigenvalue problem on the cell.

At the end of this section we show that a model without the length-scale effect is a special case of the length-scale model. Neglecting terms with the parameter l in (3) and substituting $(3)_2$ into $(3)_1$ we obtain

$$[D_{\alpha\beta\gamma\delta} - D^{A}_{\gamma\delta} D^{B}_{\alpha\beta} (D^{AB})^{-1}] W_{,\alpha\beta\gamma\delta} - N_{\alpha\beta} W_{,\alpha\beta} + m\ddot{W} = P. \tag{4}$$

The above equation describes the averaged model called *the homogenised model*.

4. ANALYSIS OF DYNAMIC STABILITY

Let us consider a simply supported plate band with span L along x_1-axis. It is assumed that the plate is made of an isotropic piece-wise periodically homogeneous material along x_1- and x_2-axis and has the periodic thickness h along x_1- and x_2-axis. Moreover, assume that the plate mass density ρ and Poisson's ratio ν are constant, but Young's modulus E is periodically variable; loadings p are neglected and the plate band is compressed along x_1- and x_2-axis, hence $N_{12}=N_{21}=0$. Let us consider a case with only one mode-shape function g (i. e. $A=N=1$) as the approximate form of solution to the certain eigenvalue problem posed on the cell: $g=g^1=l^2[\cos(2\pi x_1/l_1)\cos(2\pi x_2/l_2)+c]$, where the constant c is derived from the condition $<\mu g>=0$. Moreover, for the assumed symmetric cell and symmetric form of mode-shape function, we have $D^1_{12} = D^1_{21} = 0$. Denote $Q=Q^1$, $x=x_1$, $h_1 \equiv h^{11}_{11}, h_2 \equiv h^{11}_{22}$ and $B=<Eh^3/12(1-\nu^2)>$, and also $N_1\equiv -N_{11}$, $N_2\equiv -N_{22}$. Assume that N_1 is time-dependent function and N_2 is independent of time.

For the plate band under consideration after some manipulations from equations

(3) we obtain the following equation for the macrodeflection W

$$
\begin{aligned}
&l^4 m^{11}[m\ddot{\ddot{W}} + B\ddot{W}_{,1111} + (N_1 W_{,11})^{\cdot\cdot}] + (D^{11} - N_1 l^2 h_1 - N_2 l^2 h_2)(m\ddot{W} + N_1 W_{,11}) + \\
&+ [B(D^{11} - N_1 l^2 h_1 - N_2 l^2 h_2) - (D^1_{11})^2] W_{,1111} = 0
\end{aligned} \tag{5}
$$

Solutions to (5) satisfying boundary conditions for the simply supported plate band on the edges x=0, L we assume in the form

$$
W(x,t) = \sum_{m=1}^{\infty} a_m \sin(\alpha_m x) T(t), \tag{6}
$$

where $\alpha_m = m\pi/L$, m=1, 2,.... Because the wavelengths of W are sufficiently large compared to l and hence $\alpha_m l \ll 1$ and also $h/l \ll 1$, in the sequel we shall apply the simplified form of (5) in which terms $l^4 m^{11} B\ddot{W}_{,1111}, l^4 m^{11}(N_1 W_{,11})^{\cdot\cdot}$ can be neglected as small compared to $(D^{11} - N_1 l^2 h_1 - N_2 l^2 h_2) m\ddot{W}$. Introducing the following denotations

$$
\begin{aligned}
&N_- \equiv \alpha_m{}^2[B - (D^1_{11})^2 (D^{11})^{-1}], \quad N_+ \equiv (D^{11} l^{-2} - N_2 h_2) h_1^{-1} + \alpha_m{}^2 (D^1_{11})^2 (D^{11})^{-1}, \\
&\omega_-{}^2 \equiv \alpha_m{}^4 [B - (D^1_{11})^2 (D^{11})^{-1}] m^{-1}, \quad \omega_+{}^2 \equiv D^{11} l^{-4} (m^{11})^{-1}, \qquad \widetilde{N}_+ \equiv D^{11} l^{-2} h_1^{-1},
\end{aligned} \tag{7}
$$

the frequency equation (5) within *the length-scale model* can be written as (8)

$$
\ddot{\ddot{T}} + \omega_+{}^2 [1 - (N_1 + N_2 h_2 h_1^{-1}) \widetilde{N}_+{}^{-1}] \ddot{T} + \omega_+{}^2 \omega_-{}^2 (1 - N_1 N_-{}^{-1})(1 - N_1 N_+{}^{-1}) N_+ \widetilde{N}_+{}^{-1} T = 0 \tag{8}
$$

Assuming $N_1 = N_a + N_b \cos pt$, where p is a frequency of oscillations of force N_1, introducing $z = pt$; denoting $T' = \partial T/\partial z$ and also

$$
\begin{aligned}
&\eta_- = \omega_-{}^2 p^{-2}, \quad \eta_+ = \omega_+{}^2 p^{-2}, \quad \zeta = N_+ \widetilde{N}_+{}^{-1}, \quad \chi_- = N_a N_-{}^{-1}, \quad \delta_- = N_b N_-{}^{-1}, \\
&\chi_+ = N_a N_+{}^{-1}, \quad \delta_+ = N_b N_+{}^{-1}, \quad \widetilde{\chi}_+ = N_a \widetilde{N}_+{}^{-1}, \quad \widetilde{\delta}_+ = N_b \widetilde{N}_+{}^{-1}, \quad \widetilde{\chi} = N_2 h_2 (\widetilde{N}_+ h_1)^{-1}, \\
&\xi = \eta_+ (1 - \widetilde{\chi}_+ - \widetilde{\chi}), \qquad \xi_- = \eta_- (1 - \chi_-), \qquad \xi_+ = \eta_+ (1 - \chi_+), \\
&\varphi = \widetilde{\delta}_+ (1 - \widetilde{\chi}_+ - \widetilde{\chi})^{-1}, \quad \varphi_- = \delta_- (1 - \chi_-)^{-1}, \qquad \varphi_+ = \delta_+ (1 - \chi_+)^{-1},
\end{aligned} \tag{9}
$$

we obtain from (8) the following equation

$$
T'''' + \xi(1 - \varphi \cos z) T'' + \xi_- \xi_+ (1 - \varphi_- \cos z)(1 - \varphi_+ \cos z) \zeta T = 0\,. \tag{10}
$$

Equation (10) is a starting point of analysis of dynamic stability of the plate band under consideration in the framework of the length-scale model.

Now, let us consider the above problem in the framework of *the homogenised model*. From equation (4) we arrive at

$$
[B - (D^1_{11})^2 (D^{11})^{-1}] W_{,1111} - N_{11} W_{,11} + m\ddot{W} = 0.
$$

Denoting $N_1 \equiv -N_{11}$ and assuming solutions to the above equation as (6), after some manipulations, we obtain the frequency equation for the macrodeflection W, which can be written in the form under denotations (7)

$$\ddot{T} + \omega_-^2(1 - N_1 N_-^{-1})T = 0 . \tag{11}$$

Assuming $N_1 = N_a + N_b \cos pt$, introducing $z = pt$, using (9) equation (11) takes the form

$$T'' + \xi_-(1 - \varphi_- \cos z)T = 0 . \tag{12}$$

It can be observed that the above equation (12) for *the homogenised model* of periodic plates has a form of the known Mathieu's equation, which describes dynamic stability or parametric vibrations of different structures (e. g. bars, plates, etc., cf. [7, 4]). Using this equation we can determine for parameters φ_-, ξ_- regions of parametric resonance [4] neglecting the effect of the cell size.

However, in the framework of the length-scale model we derive the fourth order equation (10) which can be treated as a certain generalization of the Mathieu's equation. This equation makes it possible to investigate the effect of the mesostructure parameter l on a shape of boundaries of regions of stable and unstable vibrations.

5. REMARKS

It is necessary to emphasized that the presented modelling approach, different from the known homogenisation methods for periodic plates, leads to the model, which makes it possible to investigate the length-scale effect on the overall behaviour of these plates (cf. [3]). The main advantage of this model is that analysed problems are described by relatively simple differential equations with constant coefficients. Thus, the length-scale model can be used to analyse many engineering problems.

Using this model we can investigate the length-scale effect on dynamic stability problems, parametric vibrations and boundaries of regions of stable and unstable vibrations for Kirchhoff-type plates with periodic structure.

REFERENCES

[1] Baron E., Dynamika i stateczność średniej grubości płyt o strukturze periodycznej, in: Proceedings of IXTH Symposium „Stability of Structures", Zakopane 2000, 7-14, (in Polish).

[2] Caillerie D., Thin elastic and periodic plates, Math. Meth. in the Appl. Sci., 6, 1984, 159-191.

[3] Jędrysiak J., Modele dyspersyjne cienkich płyt periodycznych. Teoria i zastosowania, Zeszyty Naukowe Politechniki Łódzkiej Nr 872, series: Rozprawy naukowe z. 289, Łódź 2001, (in Polish).

[4] Kaliski S. [ed.], Vibrations, PWN, Warsaw; ELSEVIER, Amsterdam 1992.

[5] Kohn R.V., Vogelius M., A new model for thin plates with rapidly varying thickness, Int. J. Solids Structures, 20, 1984, 333-350.

[6] Michalak B., Dynamika i stateczność płyt pofałdowanych, Zeszyty Naukowe Politechniki Łódzkiej Nr 881, series: Rozprawy naukowe z. 295, Łódź 2001, (in Polish).

[7] Timoshenko S., Gere J., Theory of elastic stability, McGraw-Hill, New York 1961.

[8] Woźniak C., Wierzbicki E., Averaging techniques in thermomechanics of composite solids, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2000.