Mechanical systems with two nonlinear springs connected in series
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Abstract: The aim of the paper is analysis of dynamical regular response of the nonlinear oscillator with two serially connected springs of cubic type nonlinearity. Behaviour of such systems is described by a set of differential-algebraic equations (DAEs). Two examples of systems are solved with the help of the asymptotic multiple scales method in time domain. The classical approach has been appropriately modified to solve the governing DAEs. The analytical approximated solution has been verified by numerical simulations.

1. Introduction

The linear simplification is sometimes too rough to describe the behavior of a physical object with sufficient accuracy. Therefore, models of nonlinear oscillators have been widely considered in physics and engineering. Nonlinear oscillators with serially connected springs were investigated by many authors mostly numerically. Most papers concern a case, when one of the springs is linear and the second one is nonlinear [1, 2, 3].

Telli and Kopmaz [1] showed that the motion of a mass mounted via linear and nonlinear springs in series, is described by a set of differential-algebraic equations. Similar situation occurs in our investigation. It implies a need of a modification of the standard multiple scale method in time domain (MMS). Two examples of the system with two nonlinear springs are presented and analyzed using appropriately adopted MMS. That are one-dimensional oscillator and a spring pendulum.

2. One-dimensional oscillator

Let us consider the one-dimensional motion of a body of mass *m* attached by massless nonlinear springs to an immovable support. The studied system is shown in the Figure 1.

The restoring force of the springs with cubic nonlinearity are

$$F_i = k_i\left(Z_i + \Lambda_i Z_i^3\right), \quad i = 1,2 \tag{1}$$

where Z_i is the elongation of the *i*-th spring, k_i is the constant stiffness and Λ_i is the nonlinearity parameter. Lengths of untensioned springs are L_{10} and L_{20}.

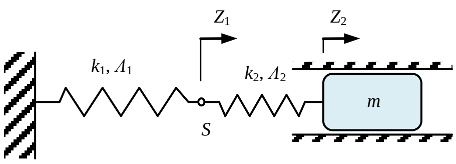

Figure 1. Oscillator with series connection of two nonlinear springs.

Such type of nonlinear elasticity is widely discussed in the papers concerning nonlinear dynamics [1, 4]. When $\Lambda_i > 0$ the characteristics of the spring is called “hard”, while for $\Lambda_i < 0$ the characteristics is called “soft”. Hereafter we consider only the case $\Lambda_i > 0$.

2.1. Mathematical model

Two equations describe behaviour of the system. One of them is the differential equation of the body motion

$$m\left(\ddot{Z}_1 + \ddot{Z}_2\right) + k_2 Z_2\left(1 + \Lambda_2 Z_2{}^2\right) = 0\ . \tag{2}$$

The second one is the algebraic equation describing equilibrium at the massless connection point S and reads

$$k_1 Z_1\left(1 + \Lambda_1 Z_1^2\right) - k_2 Z_2\left(1 + \Lambda_2 Z_2{}^2\right) = 0\ . \tag{3}$$

The above equations are supplemented by the initial conditions

$$Z_2(0) = X_0,\ \dot{Z}_2(0) = V_0\,. \tag{4}$$

After transformation of the governing equations to the more convenient dimensionless form they read

$$\ddot{z}_1 + \ddot{z}_2 + (1+\lambda)z_2\left(1 + \alpha_2 z_2^2\right) = 0\ , \tag{5}$$

$$\lambda z_2\left(1 + \alpha_2 z_2^2\right) - z_1\left(1 + \alpha_1 z_1^2\right) = 0\ , \tag{6}$$

$$z_2(0) + z_1(0) = z_0,\ \dot{z}_2(0) + \dot{z}_1(0) = v_0\ , \tag{7}$$

where z_0 and v_0 are initial displacement and velocity of the body, $z_i = Z_i / L$, $\alpha_i = \Lambda_i L^2$ for $i = 1, 2$, $L = L_{10} + L_{20}$, $\lambda = k_2 / k_1$. Dimensionless time $\tau = t\,\omega_1$ where $\omega_1 = k_e / m$ and $k_e = k_1 k_2 /(k_2 + k_2)$ have been assumed as characteristic quantities.

2.2. Approximated analytical solution

The problem (5)–(7) can be solved using MSM [4], although the approach requires some significant modification. The assumptions of smallness of the nonlinearity parameters are proposed in the form

$$\alpha_1 = \tilde{\alpha}_1 \varepsilon, \alpha_2 = \tilde{\alpha}_2 \varepsilon\,, \tag{8}$$

where ε is a small perturbation parameter.

The solution is searched in the form of series with respect to the small parameter

$$z_2(\tau;\varepsilon) = \sum_{k=0}^{k=1} \varepsilon^k z_{2k}(\tau_0, \tau_1), \quad z_1(\tau;\varepsilon) = \sum_{k=0}^{k=1} \varepsilon^k z_{1k}(\tau_0, \tau_1)\,. \tag{9}$$

Introducing (8) and (9) into (5) – (6) we obtain two equations in which the small parameter ε appears. These equations should be satisfied for any value of the small parameter, so after sorting them with respect to the powers of ε we get the differential equations of the first and second order. After eliminating secular terms and solving the equations, the approximate solution takes the form:

$$\begin{aligned} z_1(\tau) &= b_0 \lambda \cos(\mathrm{B}\tau + \psi_0) + \frac{b_0^3 \lambda \left(3\alpha_2(5+8\lambda) - \alpha_1 \lambda^2 (16+25\lambda)\right)}{32(1+\lambda)} \cos(\mathrm{B}\tau + \psi_0) \\ &+ \frac{2 b_0^3 \lambda \left(9\alpha_2 + \alpha_1(\lambda - 8)\lambda^2\right)}{32(1+\lambda)} \cos(\mathrm{B}\tau + \psi_0) \cos(2\mathrm{B}\tau + 2\psi_0), \end{aligned} \tag{10}$$

$$z_2(\tau) = b_0 \cos(\mathrm{B}\tau + \psi_0) + \frac{b_0^3 \left(\alpha_2 - 8\alpha_2 \lambda + 9\alpha_1 \lambda^3\right)}{32(1+\lambda)} \cos(3\mathrm{B}\tau + 3\psi_0)\,, \tag{11}$$

where $\mathrm{B} = 1 + 3b_0^2(\alpha_2 + \alpha_1 \lambda^3)/8(1+\lambda)$, while b_0 and ψ_0 are the initial amplitude and phase.

2.3. Results

In Figure 2 the time history of the generalized co-ordinates and their sum are presented.

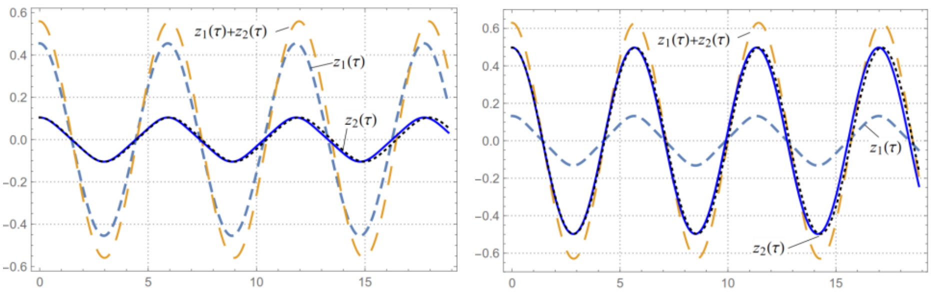

Figure 2. Time history of the motion of the system for $\alpha_1 = 0.8$ $\alpha_2 = 1.4$; a) $\lambda = 8$, $b_0 = 0.05$; $\alpha_2 = 1.4$; b) $\lambda = 0.1$, $b_0 = 0.5$; dotted line – numerical solution.

The comparison between numerical and analytical solutions confirms correctness of the asymptotic calculations. The explicit form of the solution allows for deeper analysis of the motion of the body.

From solution (10) the period of the primary vibration can be derived

$$T = \frac{16\pi(\lambda+1)}{3\alpha_1 b_0^2 \lambda^3 + 3\alpha_2 b_0^2 + 8(\lambda+1)}. \tag{12}$$

Expression (12) quantitatively describes dependence of the period with respect to amplitude, involved nonlinearities and the parameter λ. When the springs are nonlinear with hard characteristics ($\alpha_i > 0$), the period is smaller than 2π. In Figure 3 the value of period of the vibration versus nonlinearity parameters α_1 and α_2 is shown.

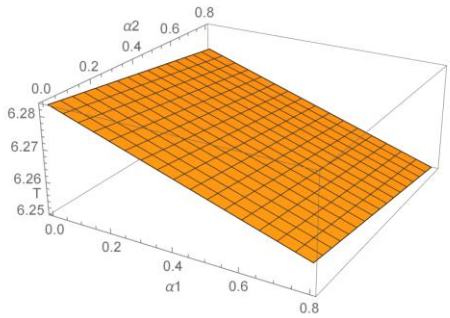

Figure 3. Period vs. nonlinearity parameters; $\lambda = 1.5$, $b_0 = 0.1$.

The dependences of the vibration period and amplitude versus λ obtained from (12) are presented in Figure 4.

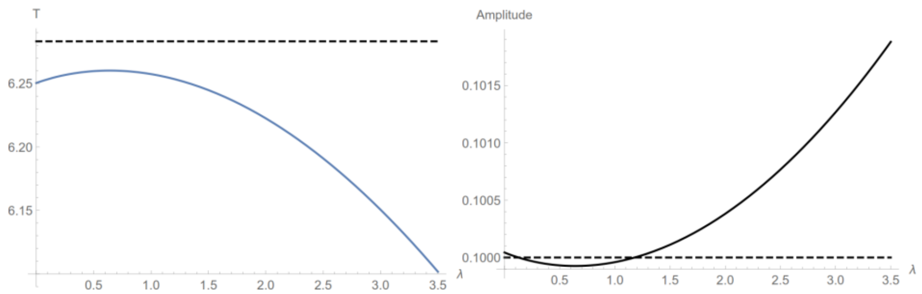

Figure 4. Period and amplitude vs. λ; $\alpha_1 = 0.8$, $\alpha_2 = 1.4$; dashed line – linear case $\alpha_1 = 0$, $\alpha_2 = 0$.

The position of the extremum value in Figure 4 depends on nonlinearity of springs α_1, α_2.

3. Spring pendulum

The dynamics of the nonlinear spring pendulum presented in Figure 5 is investigated in this point. Such quite simple and intuitive system serves as a very good example of a study of non-linear phenomena exhibited by two degrees-of-freedom mechanical systems.

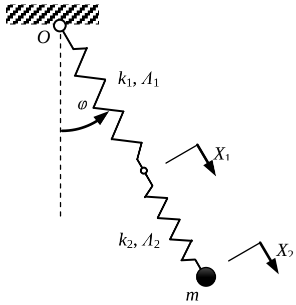

Figure 5. The pendulum with two nonlinear springs connected in series.

The investigated pendulum-type system consists of the small body of mass *m* suspended at a fixed point on the two nonlinear springs of the length L_{01}, and L_{02} whose elastic constants are denoted by k_1, Λ_1 and k_2, Λ_2. Due to the introduced constraints, the body can move only in the fixed vertical plane. Moreover we assume that the springs are straight and collinear. We are interested in free motion of the system, thus no external force or damping are admitted. The total springs elongation Z_1, Z_2 and the angle φ describe unambiguously the position of the system.

3.1. Mathematical model

The equations of motion are obtained with the help of the Lagrange equations of second kind. Similarly as in previous section, the differential equations are supplemented by the algebraic one, which describes equilibrium of the spring's connecting point *S*. The restoring forces in the springs are of the same type as previously. They are described by eq. (1). The dimensionless form of the mathematical model is as follows

$$\begin{aligned}&\ddot{z}_1 + \ddot{z}_2 + \left(1 + 3z_{2r}^2\alpha_2\right)\left(1+\lambda\right)z_2 + 3z_{2r}\alpha_2\left(1+\lambda\right)z_2^2 + \alpha_2\left(1+\lambda\right)z_2^3 - w^2\left(\cos\varphi - 1\right)\\ &-\left(1 + z_1 + z_2\right)\dot{\varphi}^2 = 0,\end{aligned} \tag{13}$$

$$\left(1 + z_1 + z_2\right)\left(\left(1 + z_1 + z_2\right)\ddot{\varphi} + 2\left(\dot{z}_1 + \dot{z}_2\right)\dot{\varphi} + w^2\sin\varphi\right) = 0\ , \tag{14}$$

$$\lambda\, z_2\left(\alpha_2 z_2\left(z_2 + 3z_{2r}\right) + 3\alpha_2 z_{2r}^2 + 1\right) - z_1\left(\alpha_1 z_1\left(z_1 + 3z_{1r}\right) + 3\alpha_1 z_{1r}^2 + 1\right) = 0\ . \tag{15}$$

The initial conditions reads

$$z_2(0) + z_1(0) = z_0,\ \dot{z}_2(0) + \dot{z}_1(0) = v_0,\ \varphi(0) = \varphi_0,\ \dot{\varphi}(0) = \omega_0\,. \tag{16}$$

The dimensionless parameters are defined in the same way as above, in section 2. The elongations of the springs at the static equilibrium position z_{1r} and z_{2r} fulfill the following additional conditions

$$z_{2r}\left(1+z_{2r}^{2}\alpha_{2}\right)=w^{2}/(1+\lambda) \text{ and } z_{1r}\left(1+z_{1r}^{2}\alpha_{2}\right)=\lambda w^{2}/(1+\lambda)\ . \tag{17}$$

The trivial solution of Eq. (14), which fulfills $z_1+z_2+1=0$, is omitted.

3.2. Approximated analytical solution

The problem (13)–(16) can be solved analytically using the multiple scale method [4], although the approach requires some significant modification. The assumptions of smallness of the nonlinearity parameters are proposed now in the form

$$\alpha_1=\tilde{\alpha}_1\varepsilon^2,\ \alpha_2=\tilde{\alpha}_2\varepsilon^2\ , \tag{18}$$

In this problem three time scales should be used, so the solutions are searched in the form

$$z_2(\tau;\varepsilon)=\sum_{k=1}^{k=3}\varepsilon^k z_{2k}(\tau_0,\tau_1,\tau_2),\quad z_1(\tau;\varepsilon)=\sum_{k=1}^{k=3}\varepsilon^k z_{1k}(\tau_0,\tau_1,\tau_2),\quad \varphi(\tau;\varepsilon)=\sum_{k=1}^{k=3}\varepsilon^k z_{1k}(\tau_0,\tau_1,\tau_2)\,. \tag{19}$$

Adopting the MSM we obtain the asymptotic analytical solution in the form:

$$\begin{aligned} z_1 &= a_{10}\lambda\left(1-3\left(z_{1r}^2\alpha_1-z_{2r}^2\alpha_2\right)\right)\cos\left((\Gamma_1+\Gamma_2)\tau+\psi_{10}\right)+a_{20}^2w^2\lambda\left(3\cos\left(2(\Gamma_5+\Gamma_6)\tau+2\psi_{20}\right)\right.\\ &\left.+4w^2-1\right)/4\left(4w^2-1\right)(1+\lambda)-3a_{10}a_{20}^2w\lambda\left(\left(1+w-2w^2\right)\cos\left((\Gamma_2+\Gamma_3+2\Gamma_4)\tau+\psi_{10}-2\psi_{20}\right)\right.\\ &\left.+\left(-1+w+2w^2\right)\cos\left((\Gamma_2+\Gamma_3-2\Gamma_4)\tau+\psi_{10}+2\psi_{20}\right)\right)/\left(16\left(4w^2-1\right)\right) \end{aligned} \tag{20}$$

$$\begin{aligned} z_2 &= a_{10}\cos\left((\Gamma_1+\Gamma_2)\tau+\psi_{10}\right)+\frac{a_{20}^2w^2\left(4w^2-1+3\cos\left(2(\Gamma_5+\Gamma_6)\tau+2\psi_{20}\right)\right)}{4\left(4w^2-1\right)(1+\lambda)}\\ &-3a_{10}a_{20}^2w\left(\left(1+w-2w^2\right)\cos\left((\Gamma_2+\Gamma_3+2\Gamma_4)\tau+\psi_{10}-2\psi_{20}\right)\right.\\ &\left.+\left(-1+w+2w^2\right)\cos\left((\Gamma_2+\Gamma_3-2\Gamma_4)\tau+\psi_{10}+2\psi_{20}\right)\right)/\left(16\left(4w^2-1\right)\right) \end{aligned} \tag{21}$$

$$\begin{aligned} \varphi &= a_{20}\cos\left((\Gamma_5+\Gamma_6)\tau+\psi_{20}\right)+a_{10}a_{20}w(1+\lambda)\left(\left(-2-3w+2w^2\right)\cos\left((-\Gamma_1-\Gamma_2+\Gamma_5+\Gamma_6)\tau-\psi_{10}+\psi_{20}\right)\right.\\ &\left.+\left(2-3w-2w^2\right)\cos\left((\Gamma_1+\Gamma_2+\Gamma_5+\Gamma_6)\tau+\psi_{10}+\psi_{20}\right)\right)/\left(2\left(4w^2-1\right)\right)\\ &+a_{20}a_{10}^2w\left(6-5w+w^2\right)(1+\lambda)^2\cos\left((-2(\Gamma_2+\Gamma_3)-\Gamma_4/2)\tau-2\psi_{10}+\psi_{20}\right)/16(2w-1)\\ &+a_{20}a_{10}^2w\left(-6+7w+9w^2+2w^3\right)(1+\lambda)^2\cos\left((2(\Gamma_2+\Gamma_3)-\Gamma_4/2)\tau+2\psi_{10}+\psi_{20}\right)/192\left(4w^2-1\right)\\ &+a_{20}^3\left(1-49w^2\right)\cos\left(3(\Gamma_5+\Gamma_6)\tau+3\psi_{20}\right)/192\left(4w^2-1\right) \end{aligned} \tag{22}$$

where $a_{10}, a_{20}, \psi_{10}, \psi_{20}$ are the initial values of amplitudes and phases of z_2 and φ respectively. They are related to the initial values z_0, v_0, φ_0 and ω_0 by the conditions (16) and eqs.(20) – (22) at instant $\tau=0$. The shortening denotations $\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4,\Gamma_5,\Gamma_6$ have the following form

$$\Gamma_1 = 1 + \frac{3\left(z_{2r}^2\alpha_2 + z_{1r}^2\alpha_1\lambda\right)}{2(1+\lambda)}, \ \Gamma_2 = \frac{3a_{20}^2 w^2\left(w^2-1\right)}{4\left(4w^2-1\right)}, \ \Gamma_3 = \frac{2 + 3z_{2r}^2\alpha_2 + 2\lambda + 3z_{1r}^2\alpha_1\lambda}{2(1+\lambda)},$$

$$\Gamma_4 = \frac{12a_{10}^2(1+\lambda)^2\left(w-w^3\right) + w\left(8a_{20}^2w^4 + \left(16-a_{20}^2\right) - w^2\left(64+7a_{20}^2\right)\right)}{8\left(4w^2-1\right)},$$

$$\Gamma_5 = \frac{12wa_{10}^2(1+\lambda)^2\left(w^2-1\right)}{64w^2-16}, \ \Gamma_6 = \frac{-8a_{20}^2w^5 - w\left(16-a_{20}^2\right) + w^3\left(64+7a_{20}^2\right)}{64w^2-16}.$$

3.3. Results

Time histories of the coordinates describing position of the body are presented in Figure 6. In both graphs, the solid line represents the asymptotic solution according to (20) – (22).

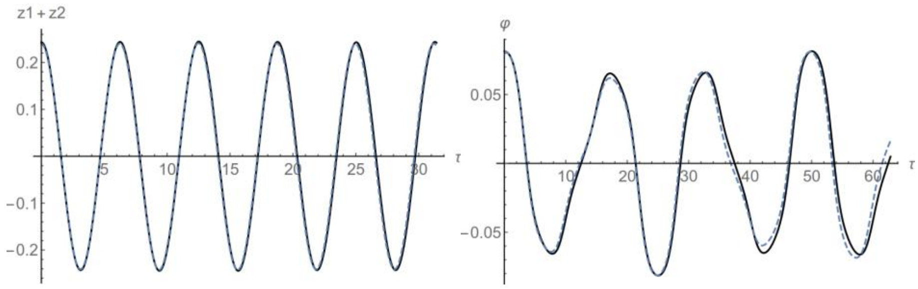

Figure 6. Time history of the body position for $\alpha_1 = 0.35$, $\alpha_2 = 0.25$, $\lambda = 2.5$, $a_{10} = 0.07$, $a_{20} = 0.07$, $\psi_{10} = 0$, $\psi_{20} = 0$; dashed line – numerical solution.

The period of the first term of the asymptotic solution for longitudinal as well as swing vibration as a function of the parameter λ are presented in Figure 7.

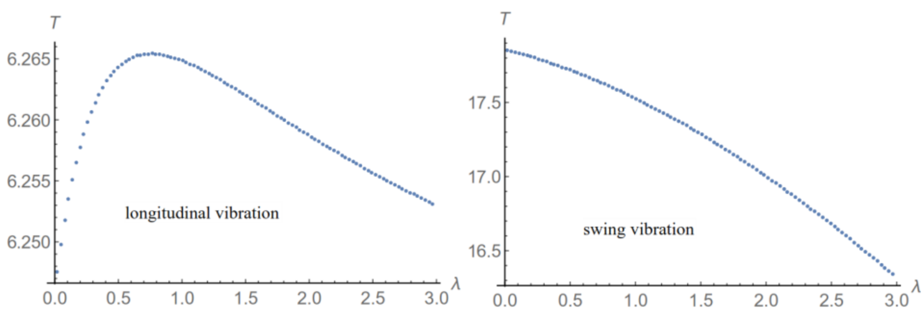

Figure 7. Period vs. λ for longitudinal and swing vibration; for $\alpha_1 = 0.35$, $\alpha_2 = 0.25$, $a_{10} = 0.07$, $a_{20} = 0.07$, $\psi_{10} = 0$, $\psi_{20} = 0$.

4. Conclusions

The mathematical model of the mechanical systems containing two serially connected nonlinear springs consists of the differential and algebraic equations. Properly modified multiple scales method in time domain allows to solve effectively this problem and to obtain the approximate asymptotic solutions. The range of parameters, for which the error is reasonably small, is limited according to the assumptions of the MSM. The correctness of the results has been confirmed by numerical simulation.

The analytical solution allows to analyse the influence of the parameters on the studied system motion. The influence of some parameters on the period and amplitude has been discussed in the case of free vibration.

Finally, we confirmed that the applied software Wolfram Mathematica has been very helpful in the analytical transformations and simplification of the derived and studied DAEs.

Acknowledgments

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