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LENGTH-SCALE EFFECT IN STABILITY PROBLEMS FOR BIPERIODICALLY STIFFENED CYLINDRICAL SHELLS

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Thin linear-elastic cylindrical shells having a micro-periodic structure along two directions tangent to the shell midsurface (biperiodic shells) are object of considerations. The aim of this paper is to investigate the effect of a periodicity cell size on the stationary stability of such shells. In order to take into account the length-scale effect in special stability problems, a new averaged non-asymptotic model of biperiodic shells, proposed in [Tomczyk B.: Thin cylindrical shells, in: Thermomechanics of Microheterogeneous Solids and Structures. Tolerance Averaging Approach. Ed. by Woźniak C., Michalak B., Jędrysiak J., Lodz Technical University Press, Lodz 2008, pp. 165-175] is applied. In the framework of this model not only the fundamental "classical" critical forces but also the new additional higher-order critical forces depending on the period of heterogeneity will be derived and discussed. These critical forces cannot be obtained from the asymptotic models commonly used for investigations of the shell stability. The differences and similarities between results derived from the aforementioned non-asymptotic biperiodic shell model and a certain asymptotic one as well as from the non-asymptotic model for shells with a micro-periodic structure along one direction tangent to the shell midsurface (uniperiodic shells) will be discussed.

1. Introduction

The object of considerations are thin linear-elastic circular cylindrical shells having a periodically inhomogeneous structure along two directions tangent to the shell midsurface. By periodic inhomogeneity we shall mean periodically variable shell thickness and/or periodically variable inertial and elastic properties of the shell material. Shells of this kind are termed *biperiodic*. As an example we

can mention cylindrical shells with periodically spaced families of thin ribs shown in (Fig. 1). The period of heterogeneity are assumed to be very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface.
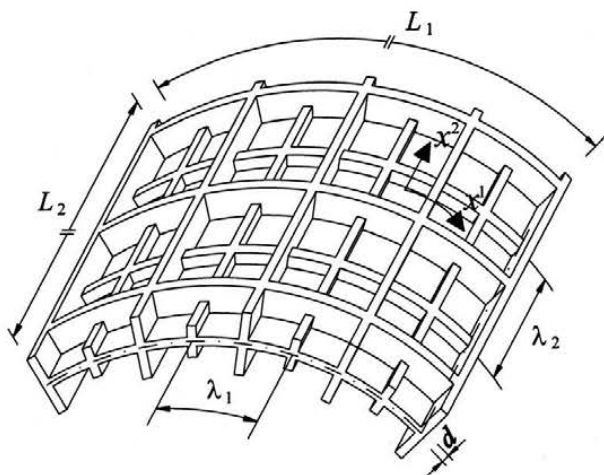

Fig. 1. An example of biperiodically stiffened cylindrical shell

For shells of this kind it is interesting to analyse the effect of the periodicity cell size on the overall shell behaviour (called *the length-scale effect*). However, the exact equations of the shell theory involve highly oscillating, non-continuous, periodic coefficients and hence they are too complicated to apply to investigations of engineering problems. That is why a lot of different approximate modelling methods for shells of this kind have been proposed. Periodic cylindrical shells are usually described using *homogenized models* derived by means of *asymptotic methods*. These models represent certain equivalent structures with constant or slowly varying rigidities and averaged mass densities, cf. [1, 2, 3, 4]. Unfortunately, in models of this kind *the effect of the period lengths* on the overall shell behaviour is neglected in the first approximation which is usually employed.

The periodically densely ribbed shells are also modelled as homogeneous orthotropic structures, cf. [5, 6, 7]. These orthotropic models are also incapable of describing many phenomena (e.g. the dispersion of waves and the existence of higher-order motions and higher free vibration frequencies dependent on a cell size) observed mainly in the dynamics and dynamic stability of periodic structures.

In order to analyse the length-scale effect in dynamic or/and stability problems, the new averaged non-asymptotic models of thin cylindrical shells with a periodic micro-heterogeneity either along two directions tangent to the shell midsurface (*biperiodic structure*) or along one direction (*uniperiodic structure*) have been proposed by Tomczyk in a series of papers, e.g. [8, 9, 10, 11, 12, 13, 14, 15], and also in books [16, 17, 18, 19, 20, 21]. These, co called, *the tolerance models* have been obtained by applying *the non-asymptotic tolerance averaging technique*, proposed and discussed in monographs [22, 23, 24, 25], to the known governing equations of Kirchhoff-Love theory of thin elastic shells (partial differential equations with functional highly oscillating non-continuous periodic coefficients). Contrary to starting equations, the governing equations of the tolerance models have coefficients which are constant or slowly-varying and depend on the period length of inhomogeneity. Hence, these models make it possible to investigate the effect of a cell size on the global shell dynamics and stability. This effect is described by means of certain extra unknowns called *fluctuation amplitudes* and by known *fluctuation shape functions* which represent oscillations inside the periodicity cell. Moreover, the tolerance models describe selected problems of the shell micro-dynamics, cf. [13, 18, 19]. It means that contrary to equations derived by using the asymptotic homogenized methods, the tolerance model equations make it possible to investigate the micro-dynamics of periodic shells independently of their macro-dynamics. In the papers and books, mentioned above, the applications of the proposed models to analysis of special problems dealing with dynamics as well as stationary and dynamical stability of *uniperiodically and biperiodically densely stiffened cylindrical shells* have been presented. *It was shown that the length-scale effect plays an important role in these problems and cannot be neglected.*

It has to be emphasized that the non-asymptotic tolerance models of shells with uni- and biperiodic structure have to be led out independently, because they are based on different modelling assumptions. The governing equations for uniperiodic shells are more complicated. It means that contrary to the asymptotic approach, *the uniperiodic shell is not a special case of biperiodic shell.*

The application of the tolerance averaging technique to the investigations of selected dynamical and/or stability problems for periodic plates can be found in many papers, e.g. [26] and [27], where dynamical stability of Hencky-Bolle-type plates and of Kirchhoff-type plates is analysed, respectively, in [28] and [29], where dynamics of Kirchhoff-type plates and of wavy-type plates is investigated, respectively, in [30], where stationary stability of densely stiffened Kirchhoff-

type plates is discussed, respectively, For review of application of the tolerance approach to the modelling of different periodic and also non-periodic structures the reader is referred to [22, 23, 24, 25].

The main aim of this contribution is to apply the tolerance (non-asymptotic) model equations, derived in [16], and the governing equations of a certain asymptotic shell model, proposed in [20], to investigate the effect of a microstructure size on the critical forces of a circular biperiodically stiffened shell as shown in (Fig. 1). *The new additional higher-order critical forces depending on the cell size will be derived* and discussed. The second aim is to show both the differences and similarities between the critical forces obtained in this paper and the corresponding results presented by Tomczyk in [21], which have been derived for uniperiodically stiffened shell as shown in (Fig.2).
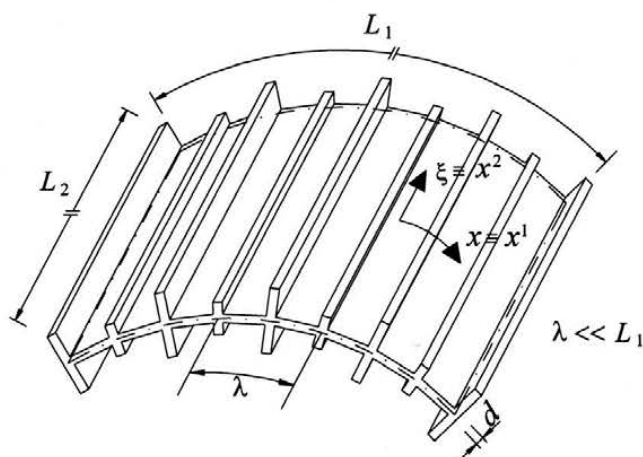

Fig. 2. An example of uniperiodically stiffened cylindrical shell

It should be mentioned that the periodic cylindrical shells, being objects of considerations in this paper, are widely applied in civil engineering, most often as roof girders and bridge girders. They are also widely used as housings of reactors and tanks. Periodic shells having small length dimensions are elements of airplanes, ships and machines.

In the subsequent section the basic denotations, preliminary concepts and starting equations will be presented.

2. Preliminaries

In this paper we investigate linear-elastic thin circular cylindrical shells. The shells are reinforced by families of ribs, which are periodically and densely distributed in circumferential and axial directions. Shells of this kind are termed *biperiodic*. Example of such shell is shown in (Fig. 1).

In order to describe the shell geometry define $\Omega = (0, L_1) \times (0, L_2)$ as a set of points $\mathbf{x} \equiv (x^1, x^2)$ in R^2; x^1, x^2 being the Cartesian orthogonal coordinates parametrizing region $\Omega \subset R^2$. Let $O\bar{x}^1\bar{x}^2\bar{x}^3$ stand for a Cartesian orthogonal coordinate system in the physical space E^3. Points of E^3 will be denoted by $\bar{\mathbf{x}} \equiv (\bar{x}^1, \bar{x}^2, \bar{x}^3)$. A cylindrical shell midsurface M is given by its parametric representation $M \equiv \{\bar{\mathbf{x}} \in E^3 : \bar{\mathbf{x}} = \bar{\mathbf{r}}(x^1, x^2), (x^1, x^2) \in \Omega\}$, where $\bar{\mathbf{r}}(\cdot)$ is the smooth function such that $\partial \bar{\mathbf{r}}/\partial x^1 \cdot \partial \bar{\mathbf{r}}/\partial x^2 = 0$, $\partial \bar{\mathbf{r}}/\partial x^1 \cdot \partial \bar{\mathbf{r}}/\partial x^1 = 1$, $\partial \bar{\mathbf{r}}/\partial x^2 \cdot \partial \bar{\mathbf{r}}/\partial x^2 = 1$. It means that on M we have introduced the orthonormal parametrization and hence L_1, L_2 are length dimensions of M. It is assumed that x^1 and x^2 are coordinates parametrizing the shell midsurface along the lines of its principal curvature and along its generatrix, respectively, cf. (Fig. 1).

Subsequently, sub- and superscripts $\alpha, \beta, \ldots$ run over sequence *1, 2* and are related to midsurface parameters x^1, x^2; summation convention holds. The partial differentiation related to x^α is represented by ∂_α. Moreover, it is denoted $\partial_{\alpha \ldots \delta} \equiv \partial_\alpha \ldots \partial_\delta$. Differentiation with respect to time coordinate $t \in [t_0, t_1]$ is represented by the overdot. Denote by $a_{\alpha\beta}$ and $a^{\alpha\beta}$ the covariant and contravariant midsurface first metric tensors; respectively. For the introduced parametrization $a_{\alpha\beta} = a^{\alpha\beta} = \delta^{\alpha\beta}$ are the unit tensors.

Let $d(\mathbf{x})$ and r stand for the shell thickness and the constant midsurface curvature radius, respectively.

Denote by $b_{\alpha\beta}$ the covariant midsurface second metric tensor. For the introduced parametrization $b_{22} = b_{12} = b_{21} = 0$ and $b_{11} = -r^{-1}$.

Let λ_1 and λ_2 be the period lengths of the stiffened shell structure respectively in x^1- and x^2- directions, cf. (Fig. 1). Define *the basic cell* Δ and *the cell distribution* (Ω, Δ) *assigned to* $\Omega = (0, L_1) \times (0, L_2) \subset R^2$ by means of:

$$\Delta \equiv [-\lambda_1/2, \lambda_1/2] \times [-\lambda_2/2, \lambda_2/2],$$

$$(\Omega, \Delta) \equiv \{\Delta(x^1, x^2) \equiv (x^1, x^2) + \Delta, (x^1, x^2) \in \overline{\Omega}\},$$

where point (x^1, x^2) is a centre of a cell $\Delta(x^1, x^2)$ and $\overline{\Omega}$ is a closure of Ω.

The diameter $\lambda \equiv \sqrt{(\lambda_1)^2 + (\lambda_2)^2}$ of Δ is assumed to satisfy conditions: $\lambda/d_{max} \gg 1$, $\lambda/r \ll 1$ and $\lambda/\min(L_1, L_2) \ll 1$. Hence, the diameter will be called *the microstructure length parameter*. In every cell $\Delta(\mathbf{x})$ we introduce local coordinates z^1, z^2 along the x^1- and x^2-directions, respectively, with the 0-point at the centre of the cell. It means that the cell Δ has two symmetry axes:

for $z^1 = 0$ and $z^2 = 0$. Thus, inside the cell, the geometrical, elastic and inertial properties of the stiffened shell are described by symmetric (i.e. even) functions of $\mathbf{z} \equiv (z^1, z^2) \in [-\lambda_1/2, \lambda_1/2] \times [-\lambda_2/2, \lambda_2/2]$.

Denote by $u_\alpha = u_\alpha(\mathbf{x},t)$, $w = w(\mathbf{x},t)$, $\mathbf{x} \in \Omega$, $t \in (t_0, t_1)$, the midsurface shell displacements in directions tangent and normal to M, respectively. Elastic properties of the shell are described by shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$. Let $\mu(\mathbf{x})$ stand for a shell mass density per midsurface unit area. Let $f^\alpha(\mathbf{x},t)$, $f(\mathbf{x},t)$ be external forces per midsurface unit area, respectively tangent and normal to M. We denote by $\overline{N}^{\alpha\beta}(t)$ the time-dependent compressive membrane forces.

Functions $\mu(\mathbf{x})$, $D^{\alpha\beta\gamma\delta}(\mathbf{x})$, $B^{\alpha\beta\gamma\delta}(\mathbf{x})$ and $d(\mathbf{x})$, $\mathbf{x} \in \Omega$, are assumed to be Δ-periodic with respect to arguments x^1, x^2.

It is assumed that the in the general case the behaviour of the stiffened shell under consideration is described by the action functional

$$A(u_\alpha, w) = \int_0^{L_1} \int_0^{L_2} \int_{t_0}^{t_1} L(\mathbf{x}, \partial_\beta u_\alpha, u_\alpha, \dot{u}_\alpha, \partial_{\alpha\beta} w, \partial_\alpha w, w, \dot{w}) dt dx^2 dx^1, \quad (1)$$

where lagrangian $L(\mathbf{x}, \partial_\beta u_\alpha, u_\alpha, \dot{u}_\alpha, \partial_{\alpha\beta} w, \partial_\alpha w, w, \dot{w})$ is highly oscillating function with respect to $\mathbf{x}$ and has the well-known form, cf. [7, 31],

$$L = \frac{1}{2}(D^{\alpha\beta\gamma\delta}\partial_\beta u_\alpha \partial_\delta u_\gamma + 2r^{-1}D^{\alpha\beta11}w\partial_\beta u_\alpha + r^{-2}D^{1111}ww + \\ + B^{\alpha\beta\gamma\delta}\partial_{\alpha\beta}w\partial_{\gamma\delta}w + \overline{N}^{\alpha\beta}(t)\partial_\alpha w\partial_\beta w - \mu a^{\alpha\beta}\dot{u}_\alpha\dot{u}_\beta - \mu\dot{w}^2) - f^\alpha u_\alpha - fw. \quad (2)$$

Obviously, in the above formula it has been taken into account that $b_{11} = -r^{-1}$.

The principle of stationary action applied to A leads to the following system of Euler-Lagrange equations

$$\partial_\beta \frac{\partial L}{\partial(\partial_\beta u_\alpha)} - \frac{\partial L}{\partial u_\alpha} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{u}_\alpha} = 0,$$

$$-\partial_{\alpha\beta}\frac{\partial L}{\partial(\partial_{\alpha\beta}w)} + \partial_\alpha \frac{\partial L}{\partial(\partial_\alpha w)} - \frac{\partial L}{\partial w} + \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{w}} = 0. \quad (3)$$

After combining (3) with (2) the above system can be written in the form

$$\partial_\beta(D^{\alpha\beta\gamma\delta}\partial_\delta u_\gamma) + r^{-1}\partial_\beta(D^{\alpha\beta11}w) - \mu a^{\alpha\beta}\ddot{u}_\beta + f^\alpha = 0,$$

$$r^{-1}D^{\alpha\beta11}\partial_\beta u_\alpha + \partial_{\alpha\beta}(B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) + r^{-2}D^{1111}w - \overline{N}^{\alpha\beta}\partial_{\alpha\beta}w + \mu\ddot{w} - f = 0. \quad (4)$$

It can be observed that equations (4) coincide with the well-known governing equations of simplified Kirchhoff-Love second-order theory of thin elastic shells, cf. [6, 31]. In the above equations the displacements $u_\alpha = u_\alpha(\mathbf{x}, t)$, $w = w(\mathbf{x}, t)$ are the basic unknowns. For periodic shells coefficients of lagrangian L and hence also of equations (4) are highly oscillating non-continuous functions depending on $\mathbf{x}$ with a period λ. That is why equations (3) (or their explicit form (4)) cannot be directly applied to investigations of engineering problems. Using *the tolerance modelling technique*, cf. [23], to equations (3), the *tolerance non-asymptotic model equations* with constant coefficients depending on microstructure length parameter λ have been derived by Tomczyk in [16]. Here, this model will be applied to investigate a length-scale effect in a certain stationary stability problem of the biperiodic shells under consideration. Obviously, for stationary problems argument t and the terms involving time derivatives (i.e. inertial forces) will drop out from the tolerance model equations. At the same time forces $\overline{N}^{\alpha\beta}$ will be assumed as the time-independent constant compressive membrane forces.

To make the analysis more clear, in the subsequent section we will recall the governing equations of the tolerance model proposed in [16] and shortly outline the tolerance modelling procedure leading to them. We will also remind the governing equations of a certain asymptotic model derived in [20].

3. Modelling approach

Following monographs [23, 24, 25], we outline below the basic concepts and assumptions which is used in the course of the tolerance modelling procedure.

3.1. Basic concepts and modelling assumption

The fundamental concepts of the tolerance modelling are those of *tolerance determined by tolerance parameter, cell distribution, tolerance periodic function* and its two special cases: *slowly-varying* and *highly-oscillating functions*. The tolerance approach is based on the notion of *the averaging of tolerance periodic function*.

The main statement of the modelling procedure is that every measurement as well as numerical calculation can be realized in practice only within a certain accuracy defined by *tolerance parameter* δ being a positive constant.

The concept of *cell distribution* (Ω, Δ) assigned to $\Omega = (0, L_1) \times (0, L_2)$ has been introduced in the previous Section.

A bounded integrable function $f(\cdot)$ defined on $\overline{\Omega} = [0, L_1] \times [0, L_2]$ (which can also depend on t as a parameter) is called *tolerance periodic* with respect to cell Δ and tolerance parameter δ, if roughly speaking, its values in an arbitrary cell $\Delta(x)$ can be approximated, with sufficient accuracy, by the corresponding values of a certain Δ-periodic function $f_x(z), z \in \Delta(x), x \in \overline{\Omega}$. Function f_x is a Δ-*periodic approximation* of f in $\Delta(x)$. This condition has to be fulfilled by all derivatives of f up to the R-th order, i.e. by *all its derivatives which occur in the problem under consideration*; in the problem analysed here R is equal either 1 or 2. In this case we shall write $f \in TP_\delta^R(\Omega, \Delta)$. It has to be emphasized that for periodic structures being object of considerations in this paper function $f_x(z), z \in \Delta(x), x \in \overline{\Omega}$ has the same analytical form in every cell $\Delta(x), x \in \overline{\Omega}$. Hence, $f_x(\cdot)$ is independent of x. In the general case, i.e. for tolerance periodic structures (i.e. structures which in small neighbourhoods of $\Delta(x)$ can be approximately regarded as periodic), $f_x = f_x(x, z), z \in \Delta(x), x \in \overline{\Omega}$.

Subsequently we will denote by $\partial \equiv (\partial_1, \partial_2)$ the gradient operator in Ω and by $\partial^k f(\cdot)$, $k = 0, 1, .., R$, the k-th gradient of function $f(\cdot)$ defined in Ω, where $\partial^0 f(\cdot) \equiv f(\cdot)$. Let $f_x^{(k)}(z)$, $z \in \Delta(x)$ be a periodic approximation of $\partial^k f \in TP_\delta^R(\Omega, \Delta)$ in cell $\Delta(x)$, $x \in \overline{\Omega}$, $k = 0, 1, .., R$, $f_x^0(\cdot) \equiv f_x(\cdot)$.

A continuous bounded differentiable function $v(x)$ defined on $\overline{\Omega} = [0, L_1] \times [0, L_2]$ (which can also depend on t as a parameter) is called *slowly-varying* with respect to cell Δ and tolerance parameter δ, if

$$v(x) \in TP_\delta^R(\Omega, \Delta),$$
$$v_x^{(k)}(z) = \partial^k v(x), \quad k = 0, 1, ..., R, \quad \text{for every } z \in \Delta(x), x \in \overline{\Omega}. \tag{5}$$

It means that periodic approximation $v_x^{(k)}$ of $\partial^k v(\cdot)$ in $\Delta(x)$ is *a constant function* for every $x \in \overline{\Omega}$. Under the above conditions we shall write $v \in SV_\delta^R(\Omega, \Delta)$.

Function $h(x)$ defined in $\overline{\Omega} = [0, L_1] \times [0, L_2]$ is called *the highly oscillating function* with respect to cell Δ and tolerance parameter δ, $h \in HO_\delta^R(\Omega, \Delta)$, if

$$h(x) \in TP_\delta^R(\Omega, \Delta),$$
$$(\forall v(x) \in SV_\delta^R(\Omega, \Delta)) (f = hv \in TP_\delta^R(\Omega, \Delta)), \tag{6}$$
$$f_x^{(k)}(z) = \partial^k h_x(z) v(x), \quad k = 0, 1, ..., R, \quad z \in \Delta(x), x \in \overline{\Omega}.$$

In the problem considered here we also deal with *the highly-oscillating functions* which are Δ-*periodic*, i.e. they are special cases of *the highly-oscillating tolerance Δ-periodic functions*, defined above. Set of *the highly-oscillating Δ-periodic functions* is denoted by $h \in HO^R(\Omega, \Delta)$. Let $h(x)$ be a

highly-oscillating λ-periodic function defined in $\overline{\Omega}$ which is continuous together with its gradients $\partial^k h, k = 1,...,R-1$, and has either continuous or a piecewise continuous bounded gradient $\partial^R h$. Periodic function $h(\cdot)$ will be called *the fluctuation shape function*, if it depends on λ as a parameter and satisfies conditions $(6)_2$ and $(6)_3$, (in $(6)_3$ $\partial^k h_x(\mathbf{z})$ has to be replaced by $\partial^k h(\mathbf{z})$), together with conditions:

$$\partial^k h \in O(\lambda^{R-k}), \quad k = 0,1,...,R, \quad \partial^0 h \equiv h,$$

$$\int_{\Delta(\mathbf{x})} \partial^k h(\mathbf{z})d\mathbf{z} = 0, \quad \mathbf{z} \in \Delta(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad k = 1,2,...,R, \qquad (7)$$

$$\int_{\Delta(\mathbf{x})} \mu(\mathbf{z})h(\mathbf{z})d\mathbf{z} = 0, \quad \mathbf{z} \in \Delta(\mathbf{x}),$$

where μ is a certain positive valued λ-periodic function defined in Ω. In stationary problems, condition $(7)_3$ is replaced by $\int_{\Delta(\mathbf{x})} h(\mathbf{z})d\mathbf{z} = 0$.

Let $f(\cdot) \in TP_\delta^R(\Omega,\Delta)$. By *the averaging of tolerance periodic function* $f \equiv \partial^0 f$ and its derivatives $\partial^k f, k = 1,2,...,R$, we shall mean function $< \partial^k f > (\mathbf{x}), \mathbf{x} \in \overline{\Omega}$, defined by

$$< \partial^k f > (\mathbf{x}) \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f_\mathbf{x}^{(k)}(\mathbf{x},\mathbf{z})d\mathbf{z}, \quad k = 0,1,...,R, \quad \mathbf{z} \in \Delta(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}. \quad (8)$$

For periodic media periodic approximation $f_\mathbf{x}^{(k)}$ of $\partial^k f$ in $\Delta(\mathbf{x})$ is independent of argument $\mathbf{x}$ and $< \partial^k f >$ is constant. For tolerance periodic media $< \partial^k f >$ is a smooth slowly-varying function of $\mathbf{x}$.

Let $f(\mathbf{x},\partial^k g(\mathbf{x})), \quad k = 0,1,...,R$ be a composite function defined in Ω such that $f(\mathbf{x},\partial^k g(\mathbf{x})) \in HO_\delta^0(\Omega,\Delta), \quad g(\mathbf{x}) \in TP_\delta^R(\Omega,\Delta)$. The tolerance averaging of this function is defined by

$$< f(\cdot,\partial^k g(\cdot) > (\mathbf{x}) \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f_\mathbf{x}(\mathbf{x},\mathbf{z},g_\mathbf{x}^{(k)}(\mathbf{x},\mathbf{z}))d\mathbf{z}, \quad \mathbf{z} \in \Delta(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}. \quad (9)$$

For periodically microheterogeneous shells under consideration function $f_\mathbf{x}$ is independent of $\mathbf{x}$ and $< f(\cdot,\partial^k g(\cdot) >$ is constant. It can be seen, that definition (8) is a special case of definition (9).

In the tolerance modelling of dynamic problems for periodic shells we also deal with mean (constant) value $< f >$ of Δ - periodic integrable function $f(\cdot)$ defined by

$$< f(\mathbf{z}) > \equiv \frac{1}{|\Delta|} \int_{\Delta(\mathbf{x})} f(\mathbf{z}) d\mathbf{z}, \quad \mathbf{z} \in \Delta(x), \quad \mathbf{x} \in \overline{\Omega}. \tag{10}$$

More general definitions of these concepts are given in [23, 24, 25].

The fundamental assumption imposed on the lagrangian under consideration in the framework of *the tolerance averaging approach* is called *the micro-macro decomposition*. It states that the displacement fields occurring in this lagrangian have to be *the tolerance periodic functions* in **x** . Hence, they can be decomposed into *unknown averaged displacements* being *slowly-varying functions* in **x** and fluctuations represented by *known highly-oscillating functions* called *fluctuation shape functions* and by *unknown fluctuation amplitudes* being *slowly-varying* in **x** .

3.2. Outline of the modelling procedure

The tolerance modelling procedure for Euler-Lagrange equations (3) is realized in two steps.

The first step is *the tolerance averaging of action functional* (1). To this end let us introduce two systems of linear independent *highly-oscillating functions*, called *the fluctuation shape functions*, being λ-periodic in $\mathbf{x} \equiv (x^1, x^2)$: $h^a(\mathbf{x}) \in HO^1(\Omega, \Delta)$, $a = 1,..,n$ and $g^A(\mathbf{x}) \in HO^2(\Omega, \Delta)$, $A = 1,..,N$. These functions are assumed to be known in every problem under consideration. Agree with (7), they have to satisfy conditions:

$h^a \in O(\lambda)$, $\lambda \partial_\alpha h^a \in O(\lambda)$,
$g^A \in O(\lambda^2)$, $\lambda \partial_\alpha g^A \in O(\lambda^2)$, $\lambda^2 \partial_{\alpha\beta} g^A \in O(\lambda^2)$,
$<\mu h^a> = <\mu g^A> = 0$ and $<\mu h^a h^b> = <\mu g^A g^B> = 0$ for $a \neq b, A \neq B$, where $\mu(\cdot)$ is the shell mass density being a λ-periodic function with respect to **x** .

In dynamic problems, functions $h^a(\mathbf{x})$, $g^A(\mathbf{x})$ represent either the principal modes of free periodic vibrations of the cell $\Delta(\mathbf{x})$ or physically reasonable approximation of these modes. Hence, they can be obtained as solutions to certain periodic eigenvalue problems describing free periodic vibrations of the cell, cf. [17]. In stationary problems, these functions can be treated as the shape functions resulting from the finite element periodic discretization of the cell.

Now, we have to introduce *the micro-macro decomposition* of displacements $u_\alpha(\mathbf{x},t) \in TP_\delta^1(\Omega, \Delta)$, $w(\mathbf{x},t) \in TP_\delta^2(\Omega, \Delta)$, $\mathbf{x} \in \Omega$, $t \in (t_0, t_1)$, which in the problem under consideration is assumed in the form

$$u_\alpha(\mathbf{x},t) = u_{h\alpha}(\mathbf{x},t) = u_\alpha^0(\mathbf{x},t) + h^a(\mathbf{x})U_\alpha^a(\mathbf{x},t), \quad a = 1,...,n,$$
$$w(\mathbf{x},t) = w_g(\mathbf{x},t) = w^0(\mathbf{x},t) + g^A(\mathbf{x})W^A(\mathbf{x},t), \quad A = 1,...,N, \tag{11}$$

where

$$u_\alpha^0(\mathbf{x},t), U_\alpha^a(\mathbf{x},t) \in SV_\delta^1(\Omega,\Delta) \subset TP_\delta^1(\Omega,\Delta),$$
$$w^0(\mathbf{x},t), W^A(\mathbf{x},t) \in SV_\delta^2(\Omega,\Delta) \subset TP_\delta^2(\Omega,\Delta), \tag{12}$$

and where summation convention over a and A holds. Functions u_α^0, w^0, called *averaged variables*, and functions U_α^a, W^A, called *fluctuation amplitudes*, are *the new unknowns* being slowly-varying in $\mathbf{x}$.

Since lagrangian (2) is a highly-oscillating function with respect to $\mathbf{x}$, $L \in HO_\delta^0(\Omega,\Delta)$, then there exists its periodic approximation in every $\Delta(\mathbf{x})$. The periodic approximation of L is obtained by replacing displacements u_α, w and their derivatives occurring in (2) by periodic approximations of these functions. These approximations are calculated applying micro-macro decomposition (11) and bearing in mind properties of the slowly-varying and highly-oscillating functions given by means of (5), (6). Then, using tolerance averaging formula (9) we arrive at function $< L_{hg} >$ being *the tolerance averaging of lagrangian (2) in* $\Delta(\mathbf{x})$ *under micro-macro decomposition (11)*. The obtained result has the form

$$< L_{hg} > (\partial_\beta u_\alpha^0, u_\alpha^0, \dot{u}_\alpha^0, U_\alpha^a, \dot{U}_\alpha^a, \partial_{\alpha\beta} w^0, \partial_\alpha w^0, w^0, W^A, \dot{w}^0, \dot{W}^A) =$$

$$= \frac{1}{2}[< D^{\alpha\beta\gamma\delta} > \partial_\beta u_\alpha^0 \partial_\delta u_\gamma^0 + 2 < D^{\alpha\beta\gamma\delta} \partial_\delta h^a > \partial_\beta u_\alpha^0 U_\gamma^a +$$

$$+ < D^{\alpha\beta\gamma\delta} \partial_\beta h^a \partial_\delta h^b > U_\gamma^a U_\alpha^b +$$

$$+ 2r^{-1}(< D^{\alpha\beta 11} > \partial_\beta u_\alpha^0 w^0 + < D^{\alpha\beta 11} \partial_\beta h^a > w^0 U_\alpha^a +$$

$$+ \underline{< D^{\alpha\beta 11} g^A >} \partial_\beta u_\alpha^0 W^A + \underline{< D^{\alpha\beta 11} \partial_\beta h^a g^A >} U_\alpha^a W^A) +$$

$$+ r^{-2}(< D^{1111} > w^0 w^0 + 2\underline{< D^{1111} g^A >} w^0 W^A + \tag{13}$$

$$+ \underline{< D^{1111} g^A g^B >} W^A W^B) + < B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta} w^0 \partial_{\gamma\delta} w^0 +$$

$$+ 2 < B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g^A > \partial_{\alpha\beta} w^0 W^A + < B^{\alpha\beta\gamma\delta} \partial_{\alpha\beta} g^A \partial_{\gamma\delta} g^B > W^A W^B +$$

$$+ \overline{N}^{\alpha\beta} \partial_\alpha w^0 \partial_\beta w^0 - \overline{N}^{\alpha\beta} < \partial_\alpha g^A \partial_\beta g^B > W^A W^B - < \mu > a^{\alpha\beta} \dot{u}_\alpha^0 \dot{u}_\beta^0 +$$

$$- < \mu > (\dot{w}^0)^2 - \underline{< \mu h^a h^b >} a^{\alpha\beta} \dot{U}_\alpha^a \dot{U}_\beta^b - \underline{< \mu g^A g^B >} \dot{W}^A \dot{W}^B] +$$

$$- < f^\alpha > u_\alpha^0 - \underline{< f^\alpha h^a >} U_\alpha^a - < f > w^0 - \underline{< f g^A >} W^A.$$

Due to periodic structure of the shell, averages $< \cdot >$ on the right-hand side of (13) are constant and calculated by means of (10). The underlined terms in (13) depend on the microstructure length parameter λ.

Functional

$$A_{hg}(u_\alpha^0, U_\alpha^a, w^0, W^A) = \int_0^{L_1L_2} \int_0^{t_1} \int_{t_0}^{t_1} < L_{hg} > dt dx^2 dx^1, \qquad (14)$$

where $< L_{hg} >$ is given by (13), is called *the tolerance averaging of functional* $A(u_\alpha, w)$ defined by (1) *under decomposition* (11).

The second step in the tolerance modelling of Euler-Lagrange equations (3) is to apply the principle of stationary action to A_{hg} given above.

The principle of stationary action applied to A_{hg} leads to the system of Euler-Lagrange equations for $u_\alpha^0, w^0, U_\alpha^a, W^A$ as the basic unknowns. The explicit form of these equations will be given in the next subsection.

3.3. Tolerance model equations

In the previous subsection, applying the tolerance averaging of the starting lagrangian (2) and then using the principle of stationary action to tolerance averaged action functional (14) defined by means of averaged lagrangian (13), we have arrived at the Euler-Lagrange equations, which explicit form can be written as

constitutive equations

$$\begin{aligned}
N^{\alpha\beta} &= < D^{\alpha\beta\gamma\delta} > \partial_\delta u_\gamma^0 + < D^{\alpha\beta\gamma\delta} \partial_\delta h^b > U_\gamma^b + \\
&\quad + r^{-1}(< D^{\alpha\beta11} > w^0 + \underline{< D^{\alpha\beta11} g^B >} W^B), \\
M^{\alpha\beta} &= < B^{\alpha\beta\gamma\delta} > \partial_{\gamma\delta} w^0 + < B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g^B > W^B, \\
h^{\alpha\beta} &= < \partial_\alpha h^a D^{\alpha\beta\gamma\delta} > \partial_\delta u_\gamma^0 + < \partial_\alpha h^a D^{\alpha\beta\gamma\delta} \partial_\delta h^b > U_\gamma^b + \\
&\quad + r^{-1}(< \partial_\alpha h^a D^{\alpha\beta11} > w^0 + \underline{< \partial_\alpha h^a D^{\alpha\beta11} g^B >} W^B), \\
G^A &= r^{-1} \underline{< g^A D^{11\gamma\delta} >} \partial_\delta u_\gamma^0 + r^{-2} \underline{< g^A D^{1111} >} w^0 + \\
&\quad + < \partial_{\gamma\delta} g^A B^{\alpha\beta\gamma\delta} > \partial_{\alpha\beta} w^0 + r^{-1} \underline{< g^A D^{11\beta\gamma} >} \partial_\beta h^b > U_\gamma^b + \\
&\quad + [< \partial_{\alpha\beta} g^A B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g^B > + r^{-2} \underline{< g^A D^{1111} g^B >}] W^B,
\end{aligned} \qquad (15)$$

and the dynamic equilibrium equations

$$\partial_\alpha N^{\alpha\beta} - <\mu> a^{\alpha\beta}\ddot{u}_\alpha^0 + <f^\beta> = 0,$$

$$\partial_{\alpha\beta}M^{\alpha\beta} + r^{-1}N^{11} + \overline{N}^{\alpha\beta}\partial_{\alpha\beta}w^0 + <\mu>\ddot{w}^0 - <f> = 0,$$

$$<\mu\, h^a h^b> a^{\alpha\beta}\ddot{U}_\alpha^b + h^{\alpha\beta} - \underline{<f^\beta h^a>} = 0, \quad a,b = 1,2,...,n, \quad (16)$$

$$<\mu\, g^A\, g^B> \ddot{W}^B + G^A - \overline{N}^{\alpha\beta}\underline{<\partial_\alpha g^A \partial_\beta g^B>}W^B +$$

$$-\underline{<fg^A>} = 0, \quad A,B = 1,2,...,N.$$

Equations (15) and (16) together with *micro-macro decomposition* (11) and *physical reliability conditions* (12) constitute *the tolerance model for analysis of selected dynamic and stability problems for biperiodically stiffened shells under consideration.* In contrast to starting equations (4) with discontinuous, highly oscillating and periodic coefficients, the tolerance model equations presented here *have constant coefficients.* Moreover, *some of them depend on microstructure length parameter* λ (underlined terms). Hence, the tolerance model makes it possible to *describe the effect of length scale on the shell behaviour.*

It has to be emphasized that solutions to selected initial/boundary value problems formulated in the framework of the tolerance model have a physical sense only if conditions (12) hold for the pertinent tolerance parameter δ. These conditions can be also used for the *a posteriori* evaluation of tolerance parameter δ and hence, for the verification of the physical reliability of the obtained solutions.

It is easy to prove, that for a homogeneous shell and homogeneous initial conditions for fluctuation amplitudes the resulting equations reduce to the starting equations (4).

3.4. Asymptotic model equations

The asymptotic model equations can be derived directly from the tolerance model equations (15), (16) by neglecting the underlined terms which depend on the microstructure length parameter λ. Hence, after calculating the fluctuation amplitudes by means of

$$U_\gamma^b = -(G^{-1})_{\gamma\eta}^{bc}\left[<\partial_\alpha h^c D^{\alpha\eta\mu\vartheta}>\partial_\vartheta u_\mu^0 + \rho^{-1}<\partial_\alpha h^c D^{\alpha\eta 11}>w^0\right],$$

$$w^A = -(S^{-1})^{AB}<\partial_{\alpha\beta}g^B B^{\alpha\beta\gamma\delta}>\partial_{\gamma\delta}w^0, \quad (17)$$

where

$$G_{\beta\gamma}^{ab} = <\partial_\alpha h^a D^{\alpha\beta\gamma\delta}\partial_\delta h^b>, \quad S^{AB} = <\partial_{\alpha\beta}g^A B^{\alpha\beta\gamma\delta}\partial_{\gamma\delta}g^B>$$

and denoting

$$D_h^{\alpha\beta\gamma\delta} \equiv\, <D^{\alpha\beta\gamma\delta}> - <D^{\alpha\beta\eta\chi}\,\partial_\chi h^a > (G^{-1})_{\eta\zeta}^{ab} <\partial_\chi h^b D^{\chi\zeta\gamma\delta}>,$$

$$B_h^{\alpha\beta\gamma\delta} \equiv\, <B^{\alpha\beta\gamma\delta}> - <B^{\alpha\beta\mu\zeta}\partial_{\mu\zeta} g^A > (S^{-1})^{AB} <\partial_{\mu\zeta} g^B B^{\mu\zeta\gamma\delta}>,$$

we arrive at *the asymptotic model equations* for unknowns $u_\alpha^0(\mathbf{x},t), w^0(\mathbf{x},t)$

$$D_h^{\alpha\beta\gamma\delta}\partial_{\beta\delta} u_\gamma^0 + r^{-1} D_h^{\alpha\beta11}\partial_\beta w^0 - <\mu> a^{\alpha\beta}\ddot{u}_\beta^0 + <f^\alpha> = 0,$$

$$B_h^{\alpha\beta\gamma\delta}\partial_{\alpha\beta\gamma\delta} w^0 + r^{-1} D_h^{11\gamma\delta}\partial_\delta u_\gamma^0 + r^{-2} D_h^{1111} w^0 + \tag{18}$$

$$+ \overline{N}^{\alpha\beta}\partial_{\alpha\beta} w^0 + <\mu> \ddot{w}^0 - <f> = 0.$$

Equations (18) have to be considered together with decomposition of $u_\alpha(\mathbf{x},t), w(\mathbf{x},t)$ in $\Omega \times (t_0, t_1)$ given by (11) with U_α^a, W^A calculated by means of (17). Coefficients in equations (18) are constant in contrast to coefficients in equations (4) which are discontinuous, highly oscillating and periodic. It has to be emphasized that equations (18) are not able to describe the length-scale effect on the overall shell dynamics and stability being independent of the microstructure size.

It has to be emphasized that equations (18) coincide with the *consistent asymptotic model equations* derived in [20] from Euler-Lagrange equations (3) by applying a new approach to the asymptotic modelling of problems for micro-heterogeneous media. This new approach is proposed in [24, 25]

The subsequent analysis dealing with a certain stationary stability problem will be based on tolerance model equations (15), (16) and asymptotic model equations (18). Moreover, the results obtained for biperiodic shells under consideration will be compared with the corresponding results derived in [21] for uniperiodic shells.

4. Applications

Now, the tolerance model equations (15) and (16) will be applied to determine and investigate critical forces of the biperiodically stiffened shells under consideration. In order to evaluate the effect of a cell size in this stability problem, the critical forces obtained from the tolerance model will be compared with those derived from asymptotic model (18). Moreover, comparison of results obtained here with corresponding those derived in [21] for uniperiodic shells will be discussed.

4.1. Formulation of the problem

Let the shell, considered here, be closed and circular. It means that $L_1 = 2\pi r$, where r is the midsurface curvature radius. The shell is reinforced by n families of stiffeners, which are periodically and densely distributed in circumferential and axial directions; an example of such a shell is shown in (Fig. 1). The stiffeners have constant cross-sections. Moreover, the gravity centres of the stiffener cross-sections are situated on the shell midsurface. It is assumed that both the shell and ribs are made of homogeneous isotropic materials.

We define $\lambda \equiv \lambda_1 = \lambda_2$ as the period length of the stiffened shell structure. The period λ represents the distance between axes of two neighbouring ribs belonging to the same family, cf. (Fig 1). It assumed that the following conditions hold: $\lambda/d_{max} \gg 1$, $\lambda/r \ll 1$ and $\lambda/L_1 \ll 1$, where d_{max} is the maximum height of stiffeners. We also assume that $L_2 > L_1$ and hence $\lambda/L_2 \ll 1$. We recall that inside the cell $\Delta \equiv [-\lambda/2, \lambda/2] \times [-\lambda/2, \lambda/2]$, the geometrical, elastic and inertial properties of the stiffened shell are described by symmetric (i.e. even) functions of arguments $\mathbf{z} \equiv (z^1, z^2) \in [-\lambda_1/2, \lambda_1/2] \times [-\lambda_2/2, \lambda_2/2]$.

The stiffened shell under consideration will be treated as a shell having a λ-periodic thickness $\tilde{d}(\mathbf{x})$ and λ-periodic elastic properties described by stiffness tensors $D^{\alpha\beta\gamma\delta}(\mathbf{x}), B^{\alpha\beta\gamma\delta}(\mathbf{x})$, $\mathbf{x} \in [0, L_1] \times [0, L_2]$.

It assumed that the shell is simply supported on edges $x^2 = 0$, $x^2 = L_2$, cf. [31].

In order to analyse the problem of stationary stability, argument t and the terms involving time derivatives (i.e. inertial forces) as well as external forces will be neglected in the tolerance and asymptotic model equations. Now, forces $\overline{N}^{\alpha\beta}$ in the stability terms of the model equations are time-independent constant compressive membrane forces.

We assume that the shell is uniformly compressed in axial direction by constant forces $\overline{N}^{22}$; hence $\overline{N}^{12} = \overline{N}^{21} = \overline{N}^{11} = 0$.

Let the investigated problem be rotationally symmetric with a period λ/r; hence $u_1^0 = U_1^a = 0$ and the remaining basic unknowns are only the functions of x^2-midsurface parameter. It has to be emphasized that the total displacement u_2, w in the micro-macro decomposition (11) are functions of both arguments, because the fluctuation shape functions depend on x^1 and x^2.

For the sake of simplicity, we shall confine ourselves to the simplest form of the tolerance model in which $a = n = A = N = 1$. Hence, we introduce only two λ-periodic fluctuation shape functions $h(\mathbf{z}) \equiv h^1(\mathbf{z}) \in HO^1(\Omega, \Delta)$ and $g(\mathbf{z}) \equiv g^1(\mathbf{z}) \in HO^2(\Omega, \Delta)$, $\mathbf{z} \equiv (z^1, z^2) \in \Delta(\mathbf{x})$, which have to satisfy condition $<h> = <g> = 0$. Bearing in mind the symmetry of the cell geometry and symmetric distribution of the material properties inside the cell we assume that

$h(\mathbf{z})$ and $g(\mathbf{z})$ are respectively *odd* and *even* functions of $\mathbf{z}$; i.e. $h(\mathbf{z})$ and $g(\mathbf{z})$ are respectively *antisymmetric* and *symmetric* functions on the cell. It assumed that these functions are known in the problem under consideration.

In the sequel denotations $U_2(x^2,t) \equiv U_2^1(x^2,t)$, $W(x^2,t) \equiv W^1(x^2,t)$ will be used.

Bearing in mind the conditions and denotations given above we will derive below the formulae for critical forces of the considered biperiodic shell by using both the tolerance model governed by equations (15), (16) and the asymptotic model represented by equations (18).

4.2. Analysis in the framework of tolerance model for biperiodic shells

Under assumptions given in the previous Subsection and under extra approximation $1 + \lambda / r \approx 1$, governing equations (16) of the tolerance model take the form

$$< D^{2222} > \partial_{22} u_2^0 + r^{-1} < D^{2211} > \partial_2 w^0 + < D^{2222} \partial_2 h > \partial_2 U_2 = 0,$$

$$r^{-1} < D^{1122} > \partial_2 u_2^0 + < B^{2222} > \partial_{2222} w^0 + r^{-2} < D^{1111} > w^0 +$$
$$+ \overline{N}^{22} \partial_{22} w^0 + r^{-1} < D^{1122} \partial_2 h > U_2 + < B^{22\alpha\alpha} \partial_{\alpha\alpha} g > \partial_{22} W = 0,$$

$$< \partial_2 h D^{2222} > \partial_2 u_2^0 + r^{-1} < \partial_2 h D^{2211} > w^0 + \qquad\qquad (19)$$
$$+ < \partial_\alpha h D^{\alpha22\alpha} \partial_\alpha h > U_2 = 0,$$

$$< \partial_{\alpha\alpha} g B^{22\alpha\alpha} > \partial_{22} w^0 +$$
$$+ (< \partial_{\alpha\beta} g B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g > - \lambda^2 \overline{N}^{22} < (\partial_2 \widetilde{g})^2 >) W = 0,$$

with the basic unknowns $u_2^0(x^2)$, $w^0(x^2)$, $U_2(x^2,t) \equiv U_2^1(x^2,t)$, $W(x^2) \equiv W^1(x^2)$ and where $\widetilde{g} = \lambda^{-1} g$. All coefficients of (19) are constant and some of them depend explicitly on microstructure length parameter λ.

Solutions to equations (19) satisfying the boundary conditions for a simply supported shell can be assumed in the form (see [31])

$$u_2^0(x^2) = \sum_{n=1}^{\infty} A_n \cos(\beta_n x^2), \qquad w^0(x^2) = \sum_{n=1}^{\infty} B_n \sin(\beta_n x^2),$$
$$U_2(x^2) = \sum_{n=1}^{\infty} C_n \sin(\beta_n x^2), \qquad W(x^2) = \sum_{n=1}^{\infty} D_n \sin(\beta_n x^2), \qquad (20)$$

where $\beta_n = n\pi / L_2$, $n = 1,2,...$; n represents the number of buckling half-waves in axial direction. Substituting these solutions into (19) we obtain the system of

four linear homogeneous algebraic equations for A_n, B_n, C_n, D_n, which has nontrivial solutions provided that its determinant is equal to zero. In this manner we arrive at the characteristic equation for critical forces $(\overline{N}^{22})_{cr}^{tm} \equiv \overline{N}^{22}$, from which for every n we derive the formulae for *fundamental lower* $(\overline{N}_{-}^{22})_{cr}^{tm}$ and *new additional higher* $(\overline{N}_{+}^{22})_{cr}^{tm}$ critical values of compressive force $(\overline{N}^{22})_{cr}^{tm}$. Under denotations

$$
\begin{aligned}
\tilde{a} &\equiv\; < \partial_{\alpha\beta} g \, B^{\alpha\beta\gamma\delta} \partial_{\gamma\delta} g >, \\
\tilde{b} &\equiv\; < \partial_2 h \, D^{2211} > - < D^{2211} >, \\
\tilde{c} &\equiv\; < D^{2222} >< \partial_\alpha h \, D^{\alpha 22\alpha} \partial_\alpha h >< D^{2222} \partial_2 h >^{-1} - < \partial_2 h \, D^{2222} >, \\
\tilde{d} &\equiv\; < D^{1122} \partial_2 h >< D^{2222} >< D^{2222} \partial_2 h >^{-1} - < D^{1122} >, \\
\tilde{e} &\equiv r^{-r} \tilde{d}\, \tilde{b} (\tilde{c})^{-1}, \qquad \tilde{k}_n \equiv < B^{2222} > (\beta_n)^4 + r^{-2} < D^{1111} >, \\
\tilde{p}_n &\equiv\; < B^{22\alpha\alpha} \partial_{\alpha\alpha} g > (\beta_n)^2, \qquad \tilde{s}_n \equiv (\tilde{p}_n)^2 + \tilde{a}(\tilde{e} - \tilde{k}_n), \\
E &\equiv\; < (\partial_2 \tilde{g}^2) >,
\end{aligned}
\tag{21}
$$

these formulae are written as

$$
\begin{aligned}
(\overline{N}_{-}^{22})_{cr}^{tm} &= \frac{1}{2}\left(\frac{\tilde{a}}{\lambda^2 E} + \frac{\tilde{k}_n - \tilde{e}}{(\beta_n)^2} \right) + \\
&\quad - \frac{1}{2}\frac{\tilde{a}}{\lambda^2 E}\sqrt{1 + \left(\frac{4\tilde{s}_n}{(\beta_n)^2(\tilde{a})^2} - \frac{2(\tilde{e}-\tilde{k}_n)}{(\beta_n)^2 \tilde{a}} \right)\lambda^2 E + \frac{(\tilde{e}-\tilde{k}_n)^2}{(\beta_n)^4(\tilde{a})^2}\lambda^4 E^2}, \\
(\overline{N}_{+}^{22})_{cr}^{tm} &= \frac{1}{2}\left(\frac{\tilde{a}}{\lambda^2 E} + \frac{\tilde{k}_n - \tilde{e}}{(\beta_n)^2} \right) + \\
&\quad + \frac{1}{2}\frac{\tilde{a}}{\lambda^2 E}\sqrt{1 + \left(\frac{4\tilde{s}_n}{(\beta_n)^2(\tilde{a})^2} - \frac{2(\tilde{e}-\tilde{k}_n)}{(\beta_n)^2 \tilde{a}} \right)\lambda^2 E + \frac{(\tilde{e}-\tilde{k}_n)^2}{(\beta_n)^4(\tilde{a})^2}\lambda^4 E^2}.
\end{aligned}
\tag{22}
$$

Some terms in (22) depend explicitly on period length λ.

4.3. Analysis in the framework of asymptotic biperiodic shell model

In order to evaluate the obtained results, let us consider the above problem within the asymptotic model, which can be derived from governing equations (19) by neglecting the terms involving microstructure length parameter λ.

Setting $(\overline{N}^{22})_{cr}^{asym} \equiv \overline{N}^{22}$ and assuming solutions to the resulting asymptotic model equations in the form (20), we derive from this asymptotic model the following formula for critical values of compressive forces $\overline{N}^{22}$

$$(\overline{N}^{22})_{cr}^{asym} = (\beta_n)^{-2}(\tilde{k}_n - \tilde{e}), \tag{23}$$

where $\tilde{e}, \tilde{k}_n$ are defined by $(21)_{5,6}$.

It is easy to see that in the above formula the cell size is neglected and that in the framework of asymptotic model it is not possible to determine *the additional higher critical force*, caused by the periodic structure of the stiffened shell.

4.4. Analysis in the framework of tolerance uniperiodic shell model

This same stability problems has been analysed in [21] for shells with a periodic micro-heterogeneous structure only in circumferential direction. Example of such a shell is shown in (Fig. 2). Below, following [21] we recall the results obtained from the tolerance model for uniperiodic shells.

Under denotations

$$\begin{aligned}
\tilde{a}_n &\equiv <B^{2222}>(\beta_n)^4 + r^{-2}<D^{1122}>(1-<D^{1122}><D^{2222}>^{-1}), \\
\tilde{b}_n &\equiv (-<\partial_{11}g\,B^{1122}>+\lambda^2<\overline{g}B^{2222}>(\beta_n)^2)^2(\beta_n)^4\lambda^{-4}<\overline{g}^2>^{-1}, \\
\tilde{c}_n &\equiv (<(\partial_{11}g)^2 B^{1111}>-2(\beta_n)^2\lambda^2(<\partial_{11}g\,B^{1122}\overline{g}>+ \\
&\quad +2<(\partial_1\tilde{g})^2 B^{1212}>))\lambda^{-4}<\overline{g}^2>^{-1},
\end{aligned} \tag{24}$$

where $\tilde{g} = \lambda^{-1}g$, $\overline{g} = \lambda^{-2}g$ and setting $(\overline{N}^{22})_{cr}^{utm} \equiv \overline{N}^{22}$, the formulae for *fundamental lower* $(\overline{N}_-^{22})_{cr}^{utm}$ and *new additional higher* $(\overline{N}_+^{22})_{cr}^{utm}$ critical values of compressive force $(\overline{N}^{22})_{cr}^{tm}$ obtained in the framework of tolerance uniperiodic shell model are written as

$$\begin{aligned}
(\overline{N}_-^{22})_{cr}^{utm} &= 0{,}5(\beta_n)^{-2}\left((\tilde{a}_n + \tilde{c}_n) - \sqrt{(\tilde{a}_n + \tilde{c}_n)^2 - 4(\tilde{a}_n\tilde{c}_n - \tilde{b}_n)}\right), \\
(\overline{N}_+^{22})_{cr}^{utm} &= 0{,}5(\beta_n)^{-2}\left((\tilde{a}_n + \tilde{c}_n) + \sqrt{(\tilde{a}_n + \tilde{c}_n)^2 - 4(\tilde{a}_n\tilde{c}_n - \tilde{b}_n)}\right).
\end{aligned} \tag{25}$$

In (25) the period length λ is contained in terms $\tilde{b}_n, \tilde{c}_n$.

4.5. Comparison of results, discussion and conclusions

- In the framework of the tolerance models, not only the fundamental lower, but also *the new additional higher critical forces* can be derived; cf. (22) and (25). The higher-order critical forces, caused by a periodic structure of the stiffened shell, cannot be determined using the asymptotic models.
- In order to evaluate the length-scale effect in the stability problem considered here, let us compare the lower critical force $(\overline{N}_{-}^{22})_{cr}^{tm}$ derived from the tolerance biperiodic shell model, cf. $(22)_1$, with critical force $(\overline{N}^{22})_{cr}^{asym}$ obtained from the asymptotic model, cf. (23). It can be shown that introducing a small parameter $\varepsilon \equiv (\lambda / L_2)^2$ $(\lambda / L_2 \ll 1)$ and then representing the square root in $(22)_1$ in the form of the power series with respect to ε, we arrive at the result:

$$(\overline{N}_{-}^{22})_{cr}^{tm} = (\overline{N}^{22})_{cr}^{asym} - O(\lambda^2) \tag{26}$$

It means that the differences between the fundamental lower critical force derived from the tolerance biperiodic shell model and critical force obtained from the asymptotic one are *negligibly small*. Thus, *the effect of microstructure length parameter λ on the fundamental critical forces of the shells under consideration can be neglected*. It means *that the asymptotic models* (for example asymptotic model given by (18)) *are sufficient to determine and investigate the critical forces of biperiodically densely stiffened cylindrical shells under consideration*. This is a very important conclusion from an engineering point of view, because of the asymptotic models are more simple then the non-asymptotic models. Similar results have been obtained for uniperiodically stiffened shells, cf. [21].

- Comparing critical forces (22) derived from the tolerance biperiodic shell model with the corresponding critical forces (25) obtained from the tolerance uniperiodic shell model, we conclude that there are more terms depending on λ in formulae (25) then in (22). It means that the length-scale effect is stronger in the stability problems for uniperiodic shells. It follows from the fact that reliability conditions (12) for biperiodic shells are more restrictive then the corresponding reliability conditions for uniperiodic shells. In the tolerance model equations for uniperiodic shells we deal with functions which are slowly-varying or highly-oscillating in only one direction, while for biperiodic shells these functions are slowly-varying or highly-oscillating in two directions.

5. Final remarks

Thin linear-elastic Kirchhoff-Love-type circular cylindrical shells with a micro-periodically inhomogeneous structure along the axial and circumferential directions are objects under consideration. Shells of this kind are termed *biperiodic*. As an example we can mention cylindrical shells with periodically and densely spaced families of longitudinal and circular stiffeners as shown in (Fig. 1). Dynamic and stability behaviour of such shells are described by Euler-Lagrange equations (3) generated by the well known Lagrange function (2). The explicit form of (3), given by (4), coincides with the governing equations of the simplified Kirchhoff-Love second-order theory for elastic shells. For periodic shells coefficients of these equations are highly oscillating non-continuous periodic functions. That is why the direct application of equations (4) to investigations of specific problems is non-effective even using computational methods.

The new mathematical non-asymptotic model for analysis of selected dynamic and stability problems for periodic shells under consideration was formulated in [16] by applying *the tolerance modelling procedure* given in [23]. Contrary to the "exact" shell equations (4) with highly oscillating non-continuous periodic coefficients, *the tolerance model equations* have coefficients which are constant or depend only on the time coordinate. Moreover, in contrast to the known asymptotic models commonly used to analysis of dynamics and stability of densely stiffened shells, the non-asymptotic tolerance model takes into account the effect of a cell size on the overall shell behaviour (*the length-scale effect*). Here, this model is applied to investigate the influence of a microstructure length on the stationary stability of biperiodically stiffened shell subjected to constant compressive axial forces. The tolerance model derived in [16] is recalled here by means of constitutive relations (15) and of dynamic balance equations (16) for *averaged shell displacements* and *fluctuation amplitudes* as the basic unknowns as well as by means of *micro-macro decomposition* (11) of the total shell displacements and *the physical reliability conditions* (12) making it possible to determine *a posteriori* an accuracy of the obtained solutions to special problems. Decomposition (11) and hence also resulting tolerance equations (15) and (16) are uniquely determined by the periodic linear independent *fluctuation shape functions*, which have to be known in every problem under consideration. These functions can be obtained as solutions to certain periodic eigenvalue problems describing free vibrations of the cell, cf. [17] or can be treated as the shape functions resulting from the finite element periodic discretization of the cell.

In the framework of the non-asymptotic models of periodic shells *the fundamental lower and the new additional higher critical forces* can be calculated and analysed, cf. formulae (21), (25). *The higher-order critical forces* depend on the microstructure length λ and cannot be derived from the asymptotic models; they can be analysed only in the framework of the tolerance (non-asymptotic) models. The differences between the fundamental lower critical forces derived from the tolerance models and critical forces obtained from the asymptotic models are *negligibly small*. Thus, *the effect of microstructure length parameter λ on the fundamental critical forces of the shells under consideration can be neglected.* Hence, the asymptotic models being more simple then the non-asymptotic models are sufficient from the point of view of calculations made for this problem.

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EFEKT SKALI W ZAGADNIENIACH STATECZNOŚCI BIPERIODYCZNIE UŻEBROWANTCH POWŁOK WALCOWYCH

Streszczenie

Przedmiotem rozważań są cienkie, liniowo-sprężyste powłoki walcowe mające periodycznie mikro-niejednorodną strukturę w dwóch kierunkach stycznych do powierzchni środkowej powłoki (*powłoki biperiodyczne*). Celem pracy jest zbadanie wpływu wielkości komórki periodyczności na stacjonarną stateczność takich powłok. Aby uwzględnić *efekt skali* w zagadnieniach stateczności, zastosowano nowy, uśredniony, nieasymptotyczny model służący do analizy dynamiki i stateczności biperiodycznie użebrowanych powłok. Model ten zaproponowano w pracy [Tomczyk B.: Thin cylindrical shells, in: Thermomechanics of micro-heterogeneous solids and structures. Tolerance averaging approach, Part II: Model equations. Ed. by C. Woźniak, B. Michalak, J. Jędrysiak, Lodz Technical University Press, Lodz 2008, pp. 165-175]. Równania modelu wyprowadzone z wykorzystaniem *techniki tolerancyjnego modelowania* mają stałe współczynniki i wiele z nich zależy od długości okresu periodyczności struktury. Wzięcie pod uwagę efektu skali pozwala wyznaczać i analizować *nowe, dodatkowe, wyższego rzędu siły krytyczne, zależne od wielkości mikrostruktury*. Siły te nie mogą być wyprowadzone w ramach modeli asymptotycznych, powszechnie stosowanych do badania stateczności powłok. Różnice i podobieństwa między wynikami otrzymanymi z modelu tolerancyjnego dla powłok biperiodycznych oraz wynikami uzyskanymi z modelu asymptotycznego a także z modelu tolerancyjnego dla powłok z periodyczną strukturą w jednym kierunku stycznym do powierzchni środkowej są dyskutowane.