3.

Tolerance modelling of medium thickness functionally graded plates

Various averaging approaches based on the known Hencky-Bolle-type plate assumptions are proposed in many papers to model medium thickness functionally graded plates with a microstructure. It can be shown that the effect of the microstructure size plays a crucial role in different thermomechanical problems of similar plates, cf. [3.2, 3.5, 3.7, 3.29, 3.68÷3.70]. However, governing equations of most of averaged models neglect the effect of the microstructure size on the overall behaviour of these plates. This lack of these models is supplemented in the tolerance model, which is based on the tolerance modelling approach, cf. [3.64, 3.65, 3.66].

3.1. Introduction

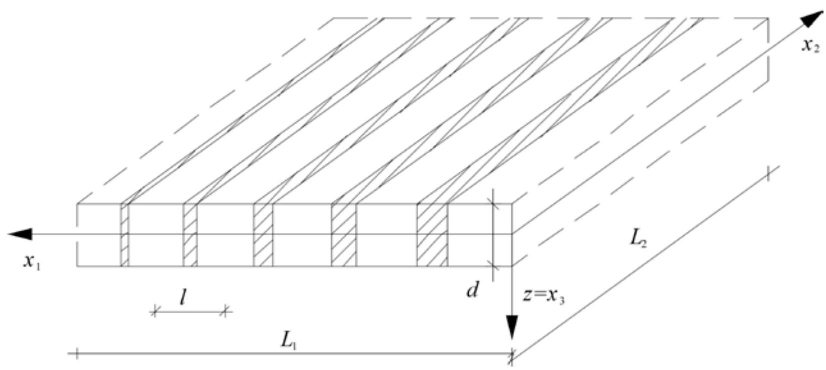

Fig. 3.1. A fragment of a functionally graded plate with a microstructure, cf. [3.23]

In this chapter vibrations of medium thickness functionally graded plates with a microstructure are considered. It is assumed that the plate has tolerance-periodic structure on the microlevel along only one direction parallel to the x_1-axis, but on the macrolevel it has functionally graded properties along this direction, cf. [3.21÷3.24, 3.57]. Material properties of the plate are assumed to be constant along the x_2-axis. In plates of this kind a "basic cell" can be distinguished, with a span l. The length l is assumed to be of an order plate thickness d, $d \sim l$. A fragment of the plate is shown in Fig. 3.1.

Composites and structures with functionally graded properties are usually described using the known methods, which are applied for periodic media. Some of them are shown in [3.57]. Similar approaches can be used for functionally graded plates with microstructure. Models based on the asymptotic homogenization method, cf. [3.4], are very interesting and useful, cf. [3.33]. Other modelling approaches for various periodically microstructured media are also proposed and applied in a series of papers, e.g. a homogenization based on microlocal parameters is used to model periodic plates by Matysiak and Nagórko [3.38] or to analyse temperature distributions in a periodically stratified layer by Matysiak and Perkowski [3.39]; natural frequencies of thick square plates made of orthotropic and hexagonal materials are considered by Batra et al. [3.3]; stability of multi-cell thin-walled columns is analysed by Królak et al. [3.34]; dynamic stability and buckling of beams or plates with metal foam core with variable mechanical properties are considered by Magnucka-Blandzi [3.36], Jasion et al. [3.16], Grygorowicz et al. [3.15].

In a series of papers there are shown many theoretical and numerical results of various problems of functionally graded structures. The modified Donnell type dynamic stability and compatibility equations are used to analyse stability of functionally graded cylindrical shells by Sofiyev and Schnack [3.56], where solutions are obtained by Galerkin's method. Natural frequencies are investigated applying some meshless methods in a few of papers, e.g. for functionally graded plates by Ferreira et al. [3.14], for sandwich beams with functionally graded core by Bui et al. [3.6]. A collocation method with higher-order plate theories is used to analyse vibrations of FG-type plates by Roque et al. [3.53]. A GDQ solution for free vibrations of shells is presented by Tornabene et al. [3.61]. Higher order deformation theories are used to analyse thermomechanical problems for plates, which are functionally graded along their thickness by Akbarzadeha et al. [3.1] and also for functionally graded plates and shells by Oktem et al. [3.47]. Tornabene and Viola [3.60] consider a static behaviour of functionally graded shells. Modal analysis of functionally graded beams with effect of the shear correction function is shown by Murin et al. [3.45]. A new low-order shell element is used to investigate shell-like structures with functionally graded material properties by Kugler et al. [3.35]. In the paper of Jha et al. [3.31] there are analysed free vibrations of functionally graded thick plates with shear and normal deformations effects. Higher-order shear and normal deformable plate theory is applied by Sheikholeslami and Saidi [3.55] to consider vibrations of functionally graded rectangular plates. A numerical analysis of heat transfer in polycrystalline composites, containing metallic or elastic interfaces is shown by Sadowski and Golewski [3.54]. A problem of single-pulse chaos for a functionally graded materials rectangular plate is considered by Yu-Gao Huangfu and Fang-Qi Chen [3.67]. Non-linear analysis of functionally graded plates based on a certain shear deformation theory is presented by Derras et al. [3.10]. Laminated plates are investigated by Fantuzzi et al. [3.13], where a strong formulation finite element method based on GDQ technique is shown.

It is necessary to observe that governing equations of these models neglect usually the effect of the microstructure size, cf. [3.5]. In order to analyse this problem it can be applied the tolerance averaging method, cf. [3.21, 3.64, 3.65, 3.66], which makes it possible to take into account this effect on the overall behaviour of microstructured media. Various problems of dynamics and stability for periodic structures and thermoelastic problems for periodic composites were analysed using this method in a series of papers, e.g. for thin periodic plates by Jędrysiak and Woźniak [3.29], Jędrysiak [3.17÷3.20]; for periodic fluid-saturated grounds by Dell'Isola et al. [3.9]; for plane periodic structures by Wierzbicki and Woźniak [3.62]; for periodic wavy-type plates by Michalak [3.42]; for thin plates reinforced periodically by a system of stiffeners by Nagórko and Woźniak [3.46]; for periodic medium-thickness plates by Baron [3.2]; for periodic thin plates with the microstructure size of an order of the plate thickness by Mazur-Śniady et al. [3.41]; for multiperiodic fibre reinforced composites by Jędrysiak and Woźniak [3.30]; for honeycomb lattice-type plates by Cielecka and Jędrysiak [3.8]; for periodic shells by Tomczyk [3.58, 3.59]; for microperiodic composite rods with uncertain parameters by Mazur-Śniady et al. [3.40]; for medium thickness plates resting on a periodic Winkler's foundation by Jędrysiak and Paś [3.27]; for thin periodic plates with large deflections by Domagalski and Jędrysiak [3.11]; for vibrations of geometrically nonlinear slender periodic beams by Domagalski and Jędrysiak [3.12]; for dynamics of periodic three-layered plates by Marczak and Jędrysiak [3.37].

The tolerance modelling can be successfully used to consider various thermomechanical problems of functionally graded structures, e.g. for stability of transversally and longitudinally graded plates by Jędrysiak and Michalak [3.26]; for heat transfer in transversally graded laminates by Jędrysiak and Radzikowska [3.28]; for dynamics of plates with longitudinally graded structure by Michalak and Wirowski [3.44], Wirowski [3.63], Perliński et al. [3.51]; for vibrations of transversally graded thin plates with the plate thickness small in compare to the microstructure size by Jędrysiak [3.21], Kaźmierczak and Jędrysiak [3.32], Jędrysiak and Kaźmierczak-Sobińska [3.25]; for dynamics of thin plates having the microstructure size of an order of the plate thickness by Jędrysiak [3.22-3.24]; for dynamic problems of a thin-walled structure with dense system of ribs by Michalak [3.43]; for non-stationary heat transfer in a hollow cylinder with functionally graded material properties by Rabenda [3.52]; for heat conduction in cylindrical composite conductors with non-uniform microstructure by Ostrowski and Michalak [3.48, 3.49]; for thermoelastic problems in transversally graded laminates by Pazera and Jędrysiak [3.50]. A lot of examples of applications of the tolerance method to analyse these composites and structures can be found in the books [3.21, 3.64, 3.65].

In this chapter there are derived the tolerance model equations of the medium thickness microstructured functionally graded plates with the microstructure size of an order of the plate thickness, which describe the effect of the microstructure size. Moreover, these equations and equations of the asymptotic model are applied to analyse vibrations for a simply supported microstructured plate band. Formulas of vibration amplitudes and resonance frequencies are obtained by using the Ritz method.

3.2. Modelling foundations

3.2.1. Preliminaries

A plate is considered in the orthogonal Cartesian coordinate system $Ox_1x_2x_3$. Let t be the time coordinate and subscripts i,k,l run over 1,2,3, but α,β,γ run over 1,2. Introduce $\mathbf{x}\equiv(x_1,x_2)$, $x\equiv x_1$, $z\equiv x_3$ and denote the region of the undeformed plate by $\Omega\equiv\{(\mathbf{x},z):-d/2\le z\le d/2, \mathbf{x}\in\Pi\}$, where Π is the plate midplane and $d(\cdot)$ is the plate thickness, which can be a tolerance-periodic function in x. Derivatives of x_α are denoted by ∂_α and also $\partial_{\alpha\ldots\delta}\equiv\partial_\alpha\ldots\partial_\delta$. Let $\Delta\equiv[-l/2,l/2]\times\{0\}$ be the "basic cell" on Ox_1x_2, where l is its length dimension along the x_1-axis, which satisfies conditions $d\sim l$ and $l<<L_1$. Hence, it is called *the microstructure parameter*. Introduce also an interval $\Lambda\equiv[0,L_1]$. All material and inertial properties of the plate, as mass density $\rho=\rho(\cdot,x_2,z)$ and elastic moduli $a_{ijkl}=a_{ijkl}(\cdot,x_2,z)$, are assumed to be also tolerance-periodic functions in x, even functions in z and independent (constant) of x_2. Denote $c_{\alpha\beta\gamma\delta}\equiv a_{\alpha\beta\gamma\delta}-a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$, $c_{\alpha3\gamma3}\equiv a_{\alpha3\gamma3}-a_{\alpha333}a_{33\gamma3}(a_{3333})^{-1}$, where $a_{\alpha\beta\gamma\delta}$, $a_{\alpha\beta33}$, $a_{\alpha3\gamma3}$, a_{3333} are the non-zero components of the elastic moduli tensor. Denote also plate displacements by u_i (i=1,2,3) and total loadings in the z-axis direction by p.

3.2.2. Governing equations

The medium thickness plates under consideration have properties described by tolerance-periodic functions of x - a mass density per unit area μ, a rotational inertia ϑ and stiffnesses $b_{\alpha\beta\gamma\delta}$, $d_{\alpha\beta}$, defined by the following formulas

$$
\begin{aligned}
&\mu(x) \equiv \int_{-d/2}^{d/2} \rho(x,z)dz, && \vartheta(x) \equiv \int_{-d/2}^{d/2} \rho(x,z)z^2 dz, \\
&b_{\alpha\beta\gamma\delta}(x) \equiv \int_{-d/2}^{d/2} c_{\alpha\beta\gamma\delta}(x,z)z^2 dz, && d_{\alpha\beta}(x) \equiv \tfrac{5}{6}\int_{-d/2}^{d/2} c_{\alpha 3\beta 3}(x,z)dz.
\end{aligned} \tag{3.1}
$$

Using the kinematic assumptions of the Hencky-Bolle-type plate theory, the following action functional can be written

$$
\mathsf{A}(u_i(\cdot), p(\cdot)) = \int_{\Pi}\int_{t_0}^{t_1} \mathsf{L}(\boldsymbol{y}, \nabla u_i(\boldsymbol{y},t), \dot{u}_i(\boldsymbol{y},t), u_i(\boldsymbol{y},t), p(\boldsymbol{y},t))dtd\boldsymbol{y}, \tag{3.2}
$$

with Lagrangean $\mathcal{L}$ defined by the formula

$$
\begin{aligned}
\mathcal{L} = {} & \tfrac{1}{2}(\mu\dot{w}\dot{w} + \vartheta\dot{\phi}_\alpha\dot{\phi}_\beta\delta_{\alpha\beta}) - \\
& -\tfrac{1}{2}(b_{\alpha\beta\gamma\delta}\partial_\alpha\phi_\beta\partial_\gamma\phi_\delta + d_{\alpha\beta}\partial_\alpha w\partial_\beta w + 2d_{\alpha\beta}\partial_\alpha w\phi_\beta + d_{\alpha\beta}\phi_\alpha\phi_\beta) + pw.
\end{aligned} \tag{3.3}
$$

where: $w=u_3(\mathbf{x},t)$ is a plate deflection; $\phi_\alpha(\mathbf{x},t)$, $\alpha=1,2$, are plate rotations. It is assumed that $\mathcal{L}$ is tolerance-periodic, highly oscillating function of x. Using the principle of stationary action to functional $\mathcal{A}$, (3.2), and Lagrangean $\mathcal{L}$, (3.3), we arrive at the known system of partial differential equations for deflection $w(\mathbf{x},t)$ and rotations $\phi_\alpha(\mathbf{x},t)$ of the medium thickness plate:

$$
\begin{aligned}
&\partial_\beta(b_{\alpha\beta\gamma\delta}\partial_\delta\phi_\gamma) - d_{\alpha\beta}(\partial_\beta w + \phi_\beta) - \vartheta\ddot{\phi}_\alpha = 0, \\
&\partial_\alpha[d_{\alpha\beta}(\partial_\beta w + \phi_\beta)] - \mu\ddot{w} = -p\,.
\end{aligned} \tag{3.4}
$$

The above equations describe vibrations of medium thickness functionally graded plates with microstructure. Equations (3.4) have highly oscillating, non-continuous functional coefficients. Hence, an application of these equations to special problems is rather difficult and it is necessary to propose an averaged approach of them or Lagrangean $\mathcal{L}$, (3.3).

3.3. Tolerance modelling

3.3.1. Basic concepts

Basic concepts of the tolerance modelling method, which is used here, were defined in the books [3.21, 3.64÷3.66] and also in a series of papers, e.g. for transversally graded plates in [3.23÷3.24]. Here, these concepts can be only mentioned: the tolerance system, the tolerance-periodic function f, $f \in TP_\delta^r(\Lambda,\Delta)$, the slowly-varying function F, $F \in SV_\delta^r(\Lambda,\Delta)$, the highly oscillating function ϕ, $\phi \in HO_\delta^r(\Lambda,\Delta)$, the fluctuation shape function g,

$g \in FS^r_\delta(\Lambda,\Delta)$, where δ is a tolerance parameter, 0<δ<<1, *r* is a kind of the function, *r*>0.

Introducing a cell $\Delta(x) \equiv x + \Delta$ at $x \in \Lambda_\Delta$, $\Lambda_\Delta = \{x \in \Lambda : \ \Delta(x) \subset \Lambda\}$, the *averaging operator* for an integrable function *f* can be defined by

$$< f > (x,x_2) = \tfrac{1}{l}\int_{\Delta(x)} f(y,x_2)dy, \quad x \in \Lambda_\Delta. \tag{3.5}$$

The averaged value of a tolerance-periodic function *f*, calculated from (3.5), is a slowly-varying function in *x*.

3.3.2. Fundamental modelling assumptions

Using the introductory concepts two fundamental assumptions of the tolerance modelling can be formulated, cf. [3.64÷3.66] and for thin functionally graded plates in [3.23÷3.24].

The first assumption is *the micro-macro decomposition*, which lets to decompose medium thickness plate displacements in the form

$$\begin{aligned} &u_3(\mathbf{x},z,t) = w(\mathbf{x},t), \\ &u_\alpha(\mathbf{x},z,t) = z[\varphi_\alpha(\mathbf{x},t) + g(x)\theta_\alpha(\mathbf{x},t)], \end{aligned} \tag{3.6}$$

where: new unknowns - *macrodeflection w*, *macrorotations* φ_α (α=1,2), and *fluctuation variables* θ_α (α=1,2), are slowly-varying functions in *x* ($w(\cdot,x_2,t), \varphi_\alpha(\cdot,x_2,t), \theta_\alpha(\cdot,x_2,t) \in SV^1_\delta(\Lambda,\Delta)$); the known fluctuation shape function *g*, $g(\cdot) \in FS^1_\delta(\Lambda,\Delta), g \in O(l)$, has the form of a saw-type function of *x*. Similar assumptions were introduced for periodic plates - thin, cf. [3.41], and medium thickness, cf. [3.2].

The next fundamental assumption is *the tolerance averaging approximation*, such that terms of an order of tolerance parameter δ are negligibly small in the modelling procedure, e.g. for functions $f \in TP^1_\delta(\Lambda,\Delta)$, $h \in FS^1_\delta(\Lambda,\Delta)$, $F \in SV^1_\delta(\Lambda,\Delta)$, in formulas: $< f > (x) =< \bar{f} > (x) + O(\delta)$, $< fF > (x) =< f > (x)F(x) + O(\delta)$, $< f\partial(hF) > (x) =< f\partial h > (x)F(x) + O(\delta)$, and they can be neglected.

3.3.3. Modelling procedure

The tolerance modelling procedure of thin functionally graded plates, having thickness *d*, which is small in comparing to the span of cell *l*, cf. [3.21, 3.32], can be easily adopted to consider plates with the span l being of an order of the plate thickness, cf. [3.22÷3.24].

In the first step Lagrangean $\mathcal{L}$ in the form (3.3) is formulated. The second step is the substitution of the micro-macro decomposition (3.6) into formula (3.3). In the next step the averaging operator (3.5) is used to the resulting equation. Applying in the fourth step the tolerance averaging approximation the tolerance averaged lagrangean $<\mathcal{L}_g>$ is derived in the following form

$$
\begin{aligned}
<\mathcal{L}_g> =& \tfrac{1}{2}(<\mu>\dot{w}\dot{w} + <\vartheta>\dot{\varphi}_\alpha\dot{\varphi}_\beta\delta_{\alpha\beta} + \underline{<\vartheta gg>\dot{\theta}_\alpha\dot{\theta}_\beta\delta_{\alpha\beta}}) - \\
&-\tfrac{1}{2}(<b_{\alpha\beta\gamma\delta}>\partial_\alpha\varphi_\beta\partial_\gamma\varphi_\delta + 2<b_{\alpha\beta1\delta}\partial_1 g>\partial_\alpha\varphi_\beta\theta_\delta + \\
&+<b_{1\beta1\delta}\partial_1 g\partial_1 g>\theta_\beta\theta_\delta + \underline{<b_{2\beta2\delta}gg>\partial_2\theta_\beta\partial_2\theta_\delta}) - \\
&-\tfrac{1}{2}(<d_{\alpha\beta}>\partial_\alpha w\partial_\beta w + 2<d_{\alpha\beta}>\partial_\alpha w\varphi_\beta + \\
&+<d_{\alpha\beta}>\varphi_\alpha\varphi_\beta + \underline{<d_{\alpha\beta}gg>\theta_\alpha\theta_\beta}) + <p>w.
\end{aligned}
\tag{3.7}
$$

From the principle of stationary action used to formula (3.7) the Euler-Lagrange equations for unknown functions $w(\cdot,x_2,t), \varphi_\alpha(\cdot,x_2,t), \theta_\alpha(\cdot,x_2,t)$ can be derived

$$
\begin{aligned}
&-\frac{\partial}{\partial t}\frac{\partial<\mathcal{L}_g>}{\partial\dot{w}} - \partial_\alpha\frac{\partial<\mathcal{L}_g>}{\partial\,\partial_\alpha w} + \frac{\partial<\mathcal{L}_g>}{\partial w} = 0, \\
&-\frac{\partial}{\partial t}\frac{\partial<\mathcal{L}_g>}{\partial\dot{\varphi}_\alpha} - \partial_\alpha\frac{\partial<\mathcal{L}_g>}{\partial\,\partial_\alpha\varphi_\beta} + \frac{\partial<\mathcal{L}_g>}{\partial\varphi_\alpha} = 0, \\
&-\frac{\partial}{\partial t}\frac{\partial<\mathcal{L}_g>}{\partial\dot{\theta}_\alpha} - \partial_2\frac{\partial<\mathcal{L}_g>}{\partial\,\partial_2\theta_\alpha} + \frac{\partial<\mathcal{L}_g>}{\partial\theta_\alpha} = 0.
\end{aligned}
\tag{3.8}
$$

3.4. Governing equations

Substituting Lagrangean (3.7) into equations (3.8), after some manipulations governing equations for functions $w(\cdot,x_2,t), \varphi_\alpha(\cdot,x_2,t), \theta_\alpha(\cdot,x_2,t)$, α=1,2, are obtained

$$
\begin{aligned}
&\partial_\beta(<b_{\alpha\beta\gamma\delta}>\partial_\delta\varphi_\gamma) + \partial_\beta(<b_{\alpha\beta\gamma1}\partial_1 g>\theta_\gamma) - <d_{\alpha\beta}>(\partial_\beta w + \varphi_\beta) - <\vartheta>\ddot{\varphi}_\alpha = 0, \\
&\partial_\alpha(<d_{\alpha\beta}>(\partial_\beta w + \varphi_\beta)) - <\mu>\ddot{w} = -p, \\
&-<b_{\alpha1\gamma\delta}\partial_1 g>\partial_\delta\varphi_\gamma - \\
&-(<b_{\alpha1\beta1}\partial_1 g\partial_1 g> + \underline{\underline{<d_{\alpha\beta}gg>}})\theta_\beta + \underline{\underline{<b_{\alpha2\gamma2}gg>}}\partial_{22}\theta_\gamma - \underline{<\vartheta gg>}\ddot{\theta}_\alpha = 0,
\end{aligned}
\tag{3.9}
$$

which are a system of partial differential equations. Equations (3.9) with micro-macro decomposition (3.6) stand *the tolerance model of medium thickness functionally graded plates with a microstructure*. Underlined terms of equations (3.9) depend on the microstructure parameter l. Hence, the tolerance model takes into account the effect of the microstructure size. Coefficients of (3.9) are slowly-varying functions in x. The basic unknowns - w, φ_α, θ_α, are slowly-varying functions in x. Boundary conditions should be formulated for macrodeflection w and macrorotations φ_α on all edges, but for fluctuation variables θ_α only for x_2=const.

In order to compare and evaluate obtained results an approximate model, which governing equations neglect the effect of the microstructure size, is introduced. The equations of this model can be derived from equations (3.9) after vanishing underlined terms and can be written as

$$\begin{aligned}
&\partial_\beta(<b_{\alpha\beta\gamma\delta}>\partial_\delta\varphi_\gamma)+\partial_\beta(<b_{\alpha\beta\gamma1}\partial_1 g>\theta_\gamma)-<d_{\alpha\beta}>(\partial_\beta w+\varphi_\beta)-<\vartheta>\ddot{\varphi}_\alpha=0,\\
&\partial_\alpha(<d_{\alpha\beta}>(\partial_\beta w+\varphi_\beta))-<\mu>\ddot{w}=-p,\\
&-<b_{\alpha1\gamma\delta}\partial_1 g>\partial_\delta\varphi_\gamma-<b_{\alpha1\beta1}\partial_1 g\partial_1 g>\theta_\beta=0.
\end{aligned} \tag{3.10}$$

The above equations stand *the asymptotic model of medium thickness functionally graded plates with a microstructure*. On the contrary to equations (3.9) they do not describe the effect of the microstructure size on vibrations. Equations (3.10) have also slowly-varying coefficients.

3.5. Example - vibrations of medium thickness functionally graded plate band

3.5.1. Preliminaries

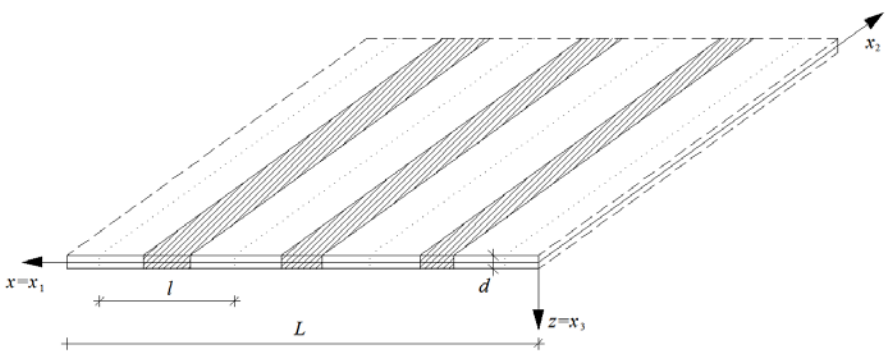

Fig. 3.2. A fragment of a functionally graded plate band

Let us consider vibrations of a simply supported plate band, with a span $L \equiv L_1$, cf. Figure 3.2. It is assumed that the plate band is made of two elastic isotropic materials, with Young's moduli E', E'', Poisson's ratios ν', ν'' and mass densities ρ', ρ''. Both materials are perfectly bonded across interfaces. It is assumed that $E(x), \rho(x), x \in \Lambda$, are tolerance periodic, highly oscillating functions in x, $E(\cdot), \rho(\cdot) \in TP^0_\delta(\Lambda, \Delta) \subset H^0(\Lambda)$, but Poisson's ratio $\nu \equiv \nu' = \nu''$ is constant. Under condition $E' \neq E''$ and/or $\rho' \neq \rho''$ the material structure of the plate can be treated as functionally graded in the x-axis direction. Hence, these plate properties can be assumed in the form

$$
\begin{aligned}
&\rho(\cdot, z) = \begin{cases} \rho', & \text{for} \quad z \in ((1-\gamma(x))\lambda/2, (1+\gamma(x))\lambda/2), \\ \rho'', & \text{for} \quad z \in [0, (1-\gamma(x))\lambda/2] \cup [(1+\gamma(x))\lambda/2, \lambda], \end{cases} \\
&E(\cdot, z) = \begin{cases} E', & \text{for} \quad z \in ((1-\gamma(x))\lambda/2, (1+\gamma(x))\lambda/2), \\ E'', & \text{for} \quad z \in [0, (1-\gamma(x))\lambda/2] \cup [(1+\gamma(x))\lambda/2, \lambda], \end{cases}
\end{aligned}
\tag{3.11}
$$

where $\gamma(x)$ is a distribution function of material properties, cf. Figure 3.3.

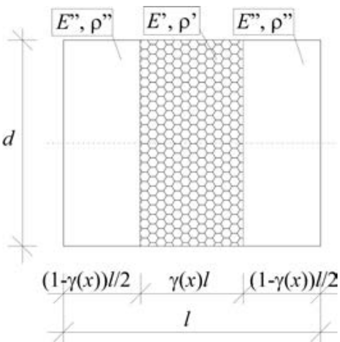

Fig. 3.3. A basic cell of a functionally graded plate under consideration, cf. [3.23]

Because the cell $\Delta(x)$, $x \in \Lambda$, of the plate band, has the form shown in Fig. 3.3 the periodic approximation of the fluctuation shape function can be assumed in as

$$
\tilde{h}(x, y) = \begin{cases} -2y[1-\tilde{\gamma}(x)]^{-1}, & y \in [0, (1-\tilde{\gamma}(x))\lambda/2], \\ (2y-l)\tilde{\gamma}(x)^{-1}, & y \in [(1-\tilde{\gamma}(x))l/2, (1+\tilde{\gamma}(x))l/2], \\ -2(y+l)[1-\tilde{\gamma}(x)]^{-1}, & y \in [(1+\tilde{\gamma}(x))l/2, l], \end{cases}
\tag{3.12}
$$

where $x \in \overline{\Lambda}$, $y \in \Delta(x)$; $\tilde{\gamma}(x)$ is a periodic approximation of the distribution function of material properties $\gamma(x)$, cf. Fig. 3.4.

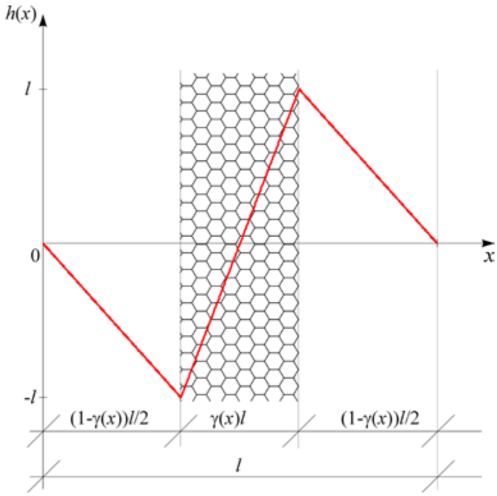

Fig. 3.4. A fluctuation shape function for the cell of the plate, cf. [3.23]

3.5.2. Governing equations of vibrations

Because vibrations of a medium thickness plate band are considered it is assumed that all basic unknowns are independent of argument x_2. Hence, the governing equations of *the tolerance model* (3.9) take the form

$$
\begin{aligned}
&\partial_1(<d_{11}>(\partial_1 w+\varphi_1)) - <\mu>\ddot{w} = -<p>,\\
&\partial_1(<b_{1111}>\partial_1\varphi_1) + \partial_1(<b_{1111}\partial_1 g>\theta_1) - <d_{11}>(\partial_1 w+\varphi_1) - <\vartheta>\ddot{\varphi}_1 = 0,\\
&-<b_{1111}\partial_1 g>\partial_1\varphi_1 - (<b_{1111}\partial_1 g\partial_1 g> + \underline{\underline{<d_{11}gg>}})\theta_1 - \underline{<\vartheta gg>}\ddot{\theta}_1 = 0,\\
&\partial_1(<b_{2121}>\partial_1\varphi_2) + \partial_1(<b_{2121}\partial_1 g>\theta_2) - <\vartheta>\ddot{\varphi}_2 = 0,\\
&-<b_{2121}\partial_1 g>\partial_1\varphi_2 - (<b_{2121}\partial_1 g\partial_1 g> + \underline{\underline{<d_{22}gg>}})\theta_2 - \underline{<\vartheta gg>}\ddot{\theta}_2 = 0.
\end{aligned}
\tag{3.13}
$$

Equations (3.13) are decoupled on two systems of equations: the first of differential equations for unknown functions - macrodeflection w, macrorotation φ_1, fluctuation variable θ_1, and the second - for macrorotation φ_2 and fluctuation variable θ_2.

Obtained results can be evaluated using the governing equations of *the asymptotic model* (3.10), which have the form

$$
\begin{aligned}
&\partial_1(<d_{11}>(\partial_1 w+\varphi_1))-<\mu>\ddot{w}=-<p>,\\
&\partial_1(<b_{1111}>\partial_1\varphi_1)+\partial_1(<b_{1111}\partial_1 g>\theta_1)-<d_{11}>(\partial_1 w+\varphi_1)-<\vartheta>\ddot{\varphi}_1=0,\\
&-<b_{1111}\partial_1 g>\partial_1\varphi_1-<b_{1111}\partial_1 g\partial_1 g>\theta_1=0,\\
&\partial_1(<b_{2121}>\partial_1\varphi_2)+\partial_1(<b_{2121}\partial_1 g>\theta_2)-<\vartheta>\ddot{\varphi}_2=0,\\
&-<b_{2121}\partial_1 g>\partial_1\varphi_2-<b_{2121}\partial_1 g\partial_1 g>\theta_2=0.
\end{aligned}
\tag{3.14}
$$

They are also decoupled on two systems of equations. It can be observed that for fluctuation variables θ_α, α=1,2, there are only algebraic equations $(3.14)_{3,5}$.

3.5.3. Approximate solutions to the governing equations

Equations (3.13), (3.14) have slowly-varying functional coefficients of x_1 argument. Hence, they are not a good tool to solve special problems of these plates. But some known approximate methods can be used, for instance the Ritz method, such for thin functionally graded plates in [3.21, 3.22÷3.25, 3.32]. For the plate band under consideration and using the following denotations

$$
\begin{array}{ll}
\widetilde{B}_{1111}=<b_{1111}>, & \widetilde{B}_{1212}=<b_{1212}>,\\
\widetilde{B}_{111}=<b_{1111}\partial_1 g>, & \widetilde{B}_{122}=<b_{1212}\partial_1 g>,\\
\widetilde{B}_{11}=<b_{1111}\partial_1 g\partial_1 g>, & \widetilde{B}_{22}=<b_{1212}\partial_1 g\partial_1 g>,\\
\widetilde{D}_{11}=<d_{11}>, & \widetilde{D}_{22}=<d_{22}>,\\
\overline{D}_{11}=l^{-2}<d_{11}gg>, & \overline{D}_{22}=l^{-2}<d_{22}gg>,\\
\widetilde{\mu}=<\mu>, & \widetilde{p}=<p>\\
\widetilde{\vartheta}=<\vartheta>, & \overline{\vartheta}=l^{-2}<\vartheta gg>,
\end{array}
\tag{3.15}
$$

Lagrangean $<\mathcal{L}_g>$, (3.7), takes the form

$$
\begin{aligned}
<\mathcal{L}_g>=&\tfrac{1}{2}[\widetilde{\mu}\dot{w}\dot{w}+\widetilde{\vartheta}(\dot{\varphi}_1\dot{\varphi}_1+\dot{\varphi}_2\dot{\varphi}_2)+l^2\overline{\vartheta}(\dot{\theta}_1\dot{\theta}_1+\dot{\theta}_2\dot{\theta}_2)]-\\
&-\tfrac{1}{2}(\widetilde{B}_{1111}\partial_1\varphi_1\partial_1\varphi_1+\widetilde{B}_{1212}\partial_1\varphi_2\partial_1\varphi_2+\\
&\quad+2\widetilde{B}_{111}\partial_1\varphi_1\theta_1+2\widetilde{B}_{122}\partial_1\varphi_2\theta_2+\widetilde{B}_{11}\theta_1\theta_1+\widetilde{B}_{22}\theta_2\theta_2)-\\
&-\tfrac{1}{2}[\widetilde{D}_{11}(\partial_1 w\partial_1 w+\varphi_1\varphi_1+2\partial_1 w\varphi_1)+\widetilde{D}_{22}\varphi_2\varphi_2+\\
&\quad+\underline{l^2\overline{D}_{11}\theta_1\theta_1}+\underline{l^2\overline{D}_{22}\theta_2\theta_2})]+\widetilde{p}w.
\end{aligned}
\tag{3.16}
$$

Solutions to equations (3.13) can be assumed in the form satisfying proper boundary conditions for a simply supported plate band

$$
\begin{aligned}
&w(x,t)=A_w\sin(\alpha x)\cos(\omega t),\\
&\varphi_1(x,t)=A_{\varphi_1}\cos(\alpha x)\cos(\omega t),\qquad \theta_1(x,t)=A_{\theta_1}\cos(\alpha x)\cos(\omega t),\\
&\varphi_2(x,t)=A_{\varphi_2}\cos(\alpha x)\cos(\omega t),\qquad \theta_2(x,t)=A_{\theta_2}\cos(\alpha x)\cos(\omega t),
\end{aligned}
\tag{3.17}
$$

where: α is a wave number; ω is a free vibration frequency; $A_w, A_{\varphi_1}, A_{\theta_1}, A_{\varphi_2}, A_{\theta_2}$ are amplitudes.

Using these solutions (3.17) and introducing the following denotations

$$
\begin{aligned}
&\breve{B}_1 = \widetilde{B}_{1111} = \tfrac{d^3}{12(1-\nu^2)}\int_0^L \{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}[\sin(\alpha x)]^2 dx,\\
&\breve{B}_2 = \widetilde{B}_{1212} = \tfrac{d^3}{12(1+\nu)}\int_0^L \{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}[\sin(\alpha x)]^2 dx,\\
&\widetilde{B}_1 = \widetilde{B}_{111} = \tfrac{-d^3}{6(1-\nu^2)}\int_0^L (E'-E'')\sin(\alpha x)\cos(\alpha x) dx,\\
&\widetilde{B}_2 = \widetilde{B}_{122} = \tfrac{-d^3}{6(1+\nu)}\int_0^L (E'-E'')\sin(\alpha x)\cos(\alpha x) dx,\\
&\widehat{B}_1 = \widetilde{B}_{11} = \tfrac{d^3}{3(1-\nu^2)}\int_0^L \tfrac{E''[1-\tilde{\gamma}(x)]+\tilde{\gamma}(x)E'}{\tilde{\gamma}(x)(1-\tilde{\gamma}(x))}[\cos(\alpha x)]^2 dx,\\
&\widehat{B}_2 = \widetilde{B}_{22} = \tfrac{d^3}{3(1+\nu)}\int_0^L \tfrac{E''[1-\tilde{\gamma}(x)]+\tilde{\gamma}(x)E'}{\tilde{\gamma}(x)(1-\tilde{\gamma}(x))}[\cos(\alpha x)]^2 dx,\\
&\breve{D}_1 = \breve{D}_{11} = \tfrac{5}{6}\tfrac{d}{1-\nu^2}\int_0^L \{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}[\cos(\alpha x)]^2 dx,\\
&\breve{D}_2 = \breve{D}_{22} = \tfrac{5}{6}\tfrac{d}{1+\nu}\int_0^L \{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}[\cos(\alpha x)]^2 dx,\\
&\overline{D}_1 = \overline{D}_{11} = \tfrac{5}{6}\tfrac{d}{3(1-\nu^2)}\int_0^L \{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}[\cos(\alpha x)]^2 dx,\\
&\overline{D}_2 = \overline{D}_{22} = \tfrac{5}{6}\tfrac{d}{3(1+\nu)}\int_0^L \{E''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)E'\}[\cos(\alpha x)]^2 dx,\\
&\breve{\mu} = d\int_0^L \{\rho''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)\rho'\}[\sin(\alpha x)]^2 dx,\\
&\widehat{\vartheta} = \tfrac{d^3}{12}\int_0^L \{\rho''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)\rho'\}[\cos(\alpha x)]^2 dx,\\
&\breve{\vartheta} = \tfrac{d^3}{36}\int_0^L \{\rho''[1-\tilde{\gamma}(x)] + \tilde{\gamma}(x)\rho'\}[\cos(\alpha x)]^2 dx,\\
&P = \int_0^L p(x)\sin(\alpha x) dx,
\end{aligned}
\tag{3.18}
$$

the maximal kinetic energy $\mathrm{K}_{\max}^{TM}$ and the maximal potential energy $\mathcal{U}_{\max}^{TM}$ by the tolerance model can be written as:

– the maximal kinetic energy $\mathcal{K}_{\max}^{TM}$

$$
\mathcal{K}_{\max}^{TM} = \tfrac{1}{2}\omega^2[\breve{\mu}{A_w}^2 + \widehat{\vartheta}({A_{\varphi_1}}^2 + {A_{\varphi_2}}^2) + l^2\breve{\vartheta}({A_{\theta_1}}^2 + {A_{\theta_2}}^2)],
\tag{3.19a}
$$

– the maximal potential energy $\mathcal{U}_{\max}^{TM}$

$$\begin{aligned}\mathcal{U}_{\max}^{TM} = \tfrac{1}{2}[\breve{D}_1\alpha^2 {A_w}^2 + 2\breve{D}_1\alpha A_w A_{\varphi_1} + (\breve{B}_1\alpha^2 + \breve{D}_1){A_{\varphi_1}}^2 + \\ +2\widetilde{B}_1\alpha A_{\varphi_1} A_{\theta_1} + (\widehat{B}_1 + l^2\overline{D}_1){A_{\theta_1}}^2 + \\ +(\breve{B}_2\alpha^2 + \breve{D}_2){A_{\varphi_2}}^2 + 2\widetilde{B}_2\alpha A_{\varphi_2} A_{\theta_2} + (\widehat{B}_2 + l^2\overline{D}_2){A_{\theta_2}}^2] + PA_w.\end{aligned} \tag{3.19b}$$

Similarly, these energies in the framework of the asymptotic model take the form:

– the maximal kinetic energy $\mathcal{K}_{\max}^{AM}$

$$\mathcal{K}_{\max}^{AM} = \tfrac{1}{2}\omega^2[\breve{\mu}{A_w}^2 + \widehat{\vartheta}({A_{\varphi_1}}^2 + {A_{\varphi_2}}^2)], \tag{3.20a}$$

– the maximal potential energy $\mathcal{U}_{\max}^{AM}$

$$\begin{aligned}\mathcal{U}_{\max}^{AM} = \tfrac{1}{2}[\breve{D}_1\alpha^2 {A_w}^2 + 2\breve{D}_1\alpha A_w A_{\varphi_1} + (\breve{B}_1\alpha^2 + \breve{D}_1){A_{\varphi_1}}^2 + \\ +2\widetilde{B}_1\alpha A_{\varphi_1} A_{\theta_1} + \widehat{B}_1{A_{\theta_1}}^2 + \\ +(\breve{B}_2\alpha^2 + \breve{D}_2){A_{\varphi_2}}^2 + 2\widetilde{B}_2\alpha A_{\varphi_2} A_{\theta_2} + \widehat{B}_2{A_{\theta_2}}^2] + PA_w.\end{aligned} \tag{3.20b}$$

Using the conditions of the Ritz method

$$\begin{aligned}&\frac{\partial(\mathcal{U}_{\max} - \mathcal{K}_{\max})}{\partial A_w} = 0, \\ &\frac{\partial(\mathcal{U}_{\max} - \mathcal{K}_{\max})}{\partial A_{\varphi_1}} = 0, \qquad \frac{\partial(\mathcal{U}_{\max} - \mathcal{K}_{\max})}{\partial A_{\varphi_2}} = 0, \\ &\frac{\partial(\mathcal{U}_{\max} - \mathcal{K}_{\max})}{\partial A_{\theta_1}} = 0, \qquad \frac{\partial(\mathcal{U}_{\max} - \mathcal{K}_{\max})}{\partial A_{\theta_2}} = 0,\end{aligned} \tag{3.21}$$

to formulas (3.19) and (3.20) systems of linear algebraic equations for amplitudes $A_w, A_{\varphi_1}, A_{\theta_1}, A_{\varphi_2}, A_{\theta_2}$ can be obtained.

For the tolerance model these algebraic equations take the form of two decoupled systems:

– the first for amplitudes $A_w, A_{\varphi_1}, A_{\theta_1}$

$$\begin{aligned}&(\breve{D}_1\alpha^2 - \omega^2\breve{\mu})A_w + \breve{D}_1\alpha A_{\varphi_1} = -P, \\ &\breve{D}_1\alpha A_w + (\breve{B}_1\alpha^2 + \breve{D}_1 - \omega^2\widehat{\vartheta})A_{\varphi_1} + \widetilde{B}_1\alpha A_{\theta_1} = 0, \\ &\widetilde{B}_1\alpha A_{\varphi_1} + [\widehat{B}_1 + l^2(\overline{D}_1 - \omega^2\breve{\vartheta})]A_{\theta_1} = 0,\end{aligned} \tag{3.22a}$$

– the second for amplitudes $A_{\varphi_2}, A_{\theta_2}$

$$
\begin{aligned}
&(\breve{B}_2\alpha^2 + \breve{D}_2 - \omega^2\hat{\vartheta})A_{\varphi_2} + \tilde{B}_2\alpha A_{\theta_2} = 0, \\
&\tilde{B}_2\alpha A_{\varphi_2} + [\hat{B}_2 + l^2(\overline{D}_2 - \omega^2\breve{\vartheta})]A_{\theta_2} = 0.
\end{aligned}
\tag{3.22b}
$$

Below, our considerations are restricted only to equations (3.22a). Solving this system formulas of amplitudes $A_w, A_{\varphi_1}, A_{\theta_1}$ take the form

$$
\begin{aligned}
A_w &= \frac{\{\omega^4 l^2\hat{\vartheta}\breve{\vartheta} - \omega^2[l^2(\breve{B}_1\alpha^2 + \breve{D}_1)\breve{\vartheta} + (\hat{B}_1 + l^2\overline{D}_1)\hat{\vartheta}]\}P}{l^2\breve{\mu}\hat{\vartheta}\breve{\vartheta}[(\varpi_{-1})^2 - \omega^2][(\varpi_{-2})^2 - \omega^2][(\varpi_{+1})^2 - \omega^2]} + \\
&\quad + \frac{\{(\breve{B}_1\alpha^2 + \breve{D}_1)(\hat{B}_1 + l^2\overline{D}_1) - (\tilde{B}_1)^2\alpha^2\}P}{l^2\breve{\mu}\hat{\vartheta}\breve{\vartheta}[(\varpi_{-1})^2 - \omega^2][(\varpi_{-2})^2 - \omega^2][(\varpi_{+1})^2 - \omega^2]}, \\
A_{\varphi_1} &= \frac{[\omega^2 l^2\alpha\breve{D}_1\breve{\vartheta} - \alpha\breve{D}_1(\hat{B}_1 + l^2\overline{D}_1)]P}{l^2\breve{\mu}\hat{\vartheta}\breve{\vartheta}[(\varpi_{-1})^2 - \omega^2][(\varpi_{-2})^2 - \omega^2][(\varpi_{+1})^2 - \omega^2]}, \\
A_{\theta_1} &= \frac{\alpha^2\breve{D}_1\tilde{B}_1 P}{l^2\breve{\mu}\hat{\vartheta}\breve{\vartheta}[(\varpi_{-1})^2 - \omega^2][(\varpi_{-2})^2 - \omega^2][(\varpi_{+1})^2 - \omega^2]},
\end{aligned}
\tag{3.23}
$$

where ϖ_{-1}, ϖ_{-2} are *two lower* and ϖ_{+1} *the higher resonance frequencies*, respectively. Introducing the following denotations

$$
\begin{aligned}
&\overline{a} \equiv \breve{\mu}\hat{\vartheta}\breve{\vartheta}l^2, \\
&\overline{b} \equiv l^2\breve{\vartheta}[\breve{\mu}\breve{D}_1 + \alpha^2(\breve{\mu}\breve{B}_1 + \breve{D}_1\hat{\vartheta})] + \breve{\mu}\hat{\vartheta}(l^2\overline{D}_1 + \hat{B}_1), \\
&\overline{c} \equiv [\breve{\mu}\breve{D}_1 + \alpha^2(\breve{\mu}\breve{B}_1 + \breve{D}_1\hat{\vartheta})](l^2\overline{D}_1 + \hat{B}_1) - \alpha^2(\breve{\mu}\tilde{B}_1^{\,2} - l^2\breve{B}_1\breve{D}_1\breve{\vartheta}\alpha^2), \\
&\overline{d} \equiv \breve{D}_1[(l^2\overline{D}_1 + \hat{B}_1)\breve{B}_1 - \tilde{B}_1^{\,2}]\alpha^4,
\end{aligned}
\tag{3.24}
$$

and also

$$
\begin{aligned}
&\overline{\alpha} \equiv 27\overline{d}\,\overline{a}^2 + 2\overline{b}^3 - 9\overline{a}\,\overline{b}\,\overline{c}, \\
&\overline{\beta} \equiv 3\overline{a}\,\overline{c} - \overline{b}^2, \\
&\overline{\delta} \equiv \sqrt[3]{\overline{\alpha} + i\sqrt{-\overline{\alpha}^2 - 4\overline{\beta}^3}},
\end{aligned}
\tag{3.25}
$$

formulas of the abovementioned resonance frequencies take the following form

$$
\begin{aligned}
&\varpi_{-1} = \sqrt{(3\overline{a})^{-1}[\overline{b} - (\sqrt[3]{2})^{-1}(\mathrm{Re}\,\overline{\delta} + \sqrt{3}\,\mathrm{Im}\,\overline{\delta})]}, \\
&\varpi_{-2} = \sqrt{(3\overline{a})^{-1}[\overline{b} - (\sqrt[3]{2})^{-1}(\mathrm{Re}\,\overline{\delta} - \sqrt{3}\,\mathrm{Im}\,\overline{\delta})]}, \\
&\varpi_{+1} = \sqrt{(3\overline{a})^{-1}(\overline{b} + \sqrt[3]{4}\,\mathrm{Re}\,\overline{\delta})}.
\end{aligned}
\tag{3.26}
$$

It can be observed that formulas (3.26) are identical to these, which describe free vibration frequencies of medium thickness functionally graded plate band with microstructure in the framework of *the tolerance model*. There are two *fundamental lower frequencies* ϖ_{-1}, ϖ_{-2} *of free macro-vibrations* and one *higher frequency* ϖ_{+1} *of free micro-vibrations*.

On the other side, using the conditions of the Ritz method (3.21) to the asymptotic model formulas of the maximal energies (3.20) the systems of algebraic equations are obtained:

– the first for amplitudes $A_w, A_{\varphi_1}, A_{\theta_1}$

$$
\begin{aligned}
&(\breve{D}_1\alpha^2 - \omega^2\breve{\mu})A_w + \breve{D}_1\alpha A_{\varphi_1} = -P,\\
&\breve{D}_1\alpha A_w + (\breve{B}_1\alpha^2 + \breve{D}_1 - \omega^2\hat{\vartheta})A_{\varphi_1} + \tilde{B}_1\alpha A_{\theta_1} = 0,\\
&\tilde{B}_1\alpha A_{\varphi_1} + \hat{B}_1 A_{\theta_1} = 0,
\end{aligned}
\tag{3.27a}
$$

– the second for amplitudes $A_{\varphi_2}, A_{\theta_2}$

$$
\begin{aligned}
&(\breve{B}_2\alpha^2 + \breve{D}_2 - \omega^2\hat{\vartheta})A_{\varphi_2} + \tilde{B}_2\alpha A_{\theta_2} = 0,\\
&\tilde{B}_2\alpha A_{\varphi_2} + \hat{B}_2 A_{\theta_2} = 0.
\end{aligned}
\tag{3.27b}
$$

Restricting our considerations to equations (3.27a) and solving this system formulas of amplitudes $A_w, A_{\varphi_1}, A_{\theta_1}$ have the form

$$
\begin{aligned}
&\tilde{A}_w = \frac{\{\omega^2\hat{B}_1\hat{\vartheta} - (\breve{B}_1\alpha^2 + \breve{D}_1)\hat{B}_1 + (\tilde{B}_1)^2\alpha^2\}P}{\breve{\mu}\hat{\vartheta}\hat{B}_1[(\tilde{\varpi}_{-1})^2 - \omega^2][(\tilde{\varpi}_{-2})^2 - \omega^2]},\\
&\tilde{A}_{\varphi_1} = \frac{\alpha\breve{D}_1\hat{B}_1 P}{\breve{\mu}\hat{\vartheta}\hat{B}_1[(\tilde{\varpi}_{-1})^2 - \omega^2][(\tilde{\varpi}_{-2})^2 - \omega^2]},\\
&\tilde{A}_{\theta_1} = \frac{\alpha^2\breve{D}_1\tilde{B}_1 P}{\breve{\mu}\hat{\vartheta}\hat{B}_1[(\tilde{\varpi}_{-1})^2 - \omega^2][(\tilde{\varpi}_{-2})^2 - \omega^2]},
\end{aligned}
\tag{3.28}
$$

where $\tilde{\varpi}_{-1}, \tilde{\varpi}_{-2}$ are *two lower resonance frequencies*. Introducing denotations similar to (3.24)

$$
\begin{aligned}
&\tilde{b} \equiv \breve{\mu}\hat{\vartheta}\hat{B}_1,\\
&\tilde{c} \equiv [\breve{\mu}\breve{D}_1 + \alpha^2(\breve{\mu}\breve{B}_1 + \breve{D}_1\hat{\vartheta})]\hat{B}_1 - \alpha^2\breve{\mu}\tilde{B}_1^{\,2},\\
&\tilde{d} \equiv \breve{D}_1(\hat{B}_1\breve{B}_1 - \tilde{B}_1^{\,2})\alpha^4,
\end{aligned}
\tag{3.29}
$$

formulas of the abovementioned resonance frequencies take the following form

$$\tilde{\omega}_{-1} = \sqrt{(2\tilde{b})^{-1}[\tilde{c} - \sqrt{\tilde{c}^2 - 4\tilde{b}\tilde{d}}\,]},$$
$$\tilde{\omega}_{-2} = \sqrt{(2\tilde{b})^{-1}[\tilde{c} + \sqrt{\tilde{c}^2 - 4\tilde{b}\tilde{d}}\,]}. \tag{3.30}$$

Formulas (3.30) are identical to these of free vibration frequencies of medium thickness functionally graded plate band with microstructure in the framework of *the asymptotic model*. They are only two *fundamental lower frequencies* $\tilde{\omega}_{-1}, \tilde{\omega}_{-2}$ *of free macro-vibrations*.

It can be observed that only in the framework of the tolerance model the effect of the microstructure size of the plate strip can be analysed in the form of higher vibration frequencies, $(3.26)_3$. However, in the asymptotic model this effect is neglected and the fundamental lower frequencies can be only investigated, (3.30).

3.6. Final remarks

The main problem considered in this chapter is modelling of vibrations of medium thickness functionally graded plates having a microstructure. Unfortunately, most averaging approaches applied to analyse these problems neglects phenomena related to the microstructure size of the plate. In order to take into account the effect of the microstructure size the tolerance method is used. Applying this method the known differential equations, based on the Hencky-Bolle-type plate assumptions, with tolerance-periodic, non-continuous, functional coefficients is replaced by governing equations with smooth, slowly-varying coefficients. The derived tolerance model equations describe the effect of the microstructure size on the overall behaviour of microstructured medium thickness functionally graded plates under consideration. However, the asymptotic model equations neglect this effect and describe these plates on the macrolevel only.

Following the obtained analytical results some general remarks can be formulated.

1 *The tolerance model* take into account *the effect of the microstructure size* in dynamic problems of microstructured medium thickness functionally graded plates, e.g. the “higher order” vibrations related to the plate microstructure;

2 *The asymptotic model* lets to investigate only lower order vibrations of these microstructured plates;

3 Solutions obtained in the framework of the tolerance model have to satisfy the condition to be slowly-varying functions in x_1. This condition stands *a posteriori* verification of results of this model.

Some other thermoelasticity problems of the medium thickness functionally graded plates will be considered in forthcoming papers, where certain evaluations and comparisons with other averaged models could be presented.

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