

## OPTIMAL DESIGN OF FIBER-REINFORCED COMPOSITE DISKS

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The paper is devoted to the modeling, analysis and optimization encountered in the design process of two-dimensional structural components made of fiber-reinforced composite materials subjected to service loading. The problem of optimal layout of reinforcing fibers in the composite in order to obtain the assumed mechanical properties of the disk is considered. The case of creation of linear and curvilinear fibers is discussed. The adequate model, relevant optimality conditions for this type of design problem are derived and the optimization procedure based on the evolutionary algorithm is proposed. The problem considered in the paper is illustrated by some numerical examples.

*Key words:* composite material, fibers layout, optimal design, evolutionary algorithm

### 1. Introduction

Fiber-reinforced composite materials are a group of modern construction materials from which products used in many areas of technical applications are made. These materials are characterized by very good mechanical properties associated with their small weight. The optimal design of these structures is a very complex process. To fulfill the assumed properties of the composite structures, we can modify some their structural parameters, such as mechanical properties of the matrix or reinforcing fibers, percentage participation of fibers in the structure, fiber shape and orientation, etc. Each of these parameters influences the mechanical properties of the composite material and can be treated as the design variable during the optimal design. However, as shown in previous Author's papers (Dems, 1986; Dems and Wiśniewski, 2006, 2007)

the full advantages of the composite materials can be also obtained when the reinforcing fibers are optimally oriented or shaped in the matrix with respect to the assumed objective behavioral measure of the structure under actual loading conditions. The problem of optimal fibers arrangement in the composite material is discussed in the present paper.

The paper constitutes the results of further investigation in the area of designing composite structures. The results can be treated as a starting point for optimal design of real composite structures subjected to service load. Thus, it allows for avoiding expensive experimental testing, which can be reduced to the final phase of structural design.

## 2. Object of analysis

A thin, two-dimensional and linearly elastic disk shown in Fig. 1 is considered. The disk has a uniform thickness and it is supported on the boundary portion  $S_U$  with prescribed displacements  $\mathbf{u}^0$  and loaded by body forces  $\mathbf{f}^0$  within a domain  $A$  and by an external traction  $\mathbf{T}^0$  acting along the boundary portion  $S_T$ .

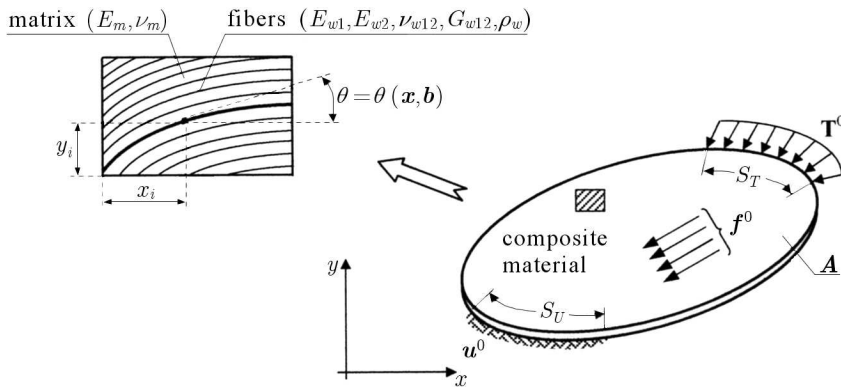


Fig. 1. Two-dimensional composite disk subjected to service loading

The material of the disk is a composite made of a matrix reinforced with a ply of long and unidirectional fibers of an arbitrary shape and assumed cross-section (Fig. 1). Let us assume that:

- The matrix is homogeneous, isotropic and linearly elastic. Young's modulus and Poisson's ratio of the matrix are denoted by  $E_m$  and  $\nu_m$ , respectively.

- The reinforcing fibers are homogeneous, orthotropic and linearly elastic. The mechanical properties of fibers are characterized by Young's moduli  $E_{w1}$  and  $E_{w2}$  with respect to the material axes 1-2, coinciding with the fiber direction and the direction perpendicular to the fiber, Poisson's ratio  $\nu_{w12}$  as well as shear modulus  $G_{w12}$  in the 1-2 plane.
- The fibers are regularly spaced and perfectly aligned in the matrix with global density  $\rho_w$ , and their orientation at any point of the composite material is denoted by the angle  $\theta$  with respect to the  $x$ -axis of global coordinate system  $xy$ .
- The bonds between the matrix and the fibers are perfect.

The angle of fiber orientation  $\theta$  defines the layout of reinforcing fibers at any point of the composite material (Fig. 1). This angle can be constant in the composite, and then the fibers are rectilinearly spaced in the matrix or can vary through the composite domain. In this last case, the fibers are placed curvilinearly in the matrix. In general, the layout of reinforcing fibers can be described by the shape of the so-called *directional fiber* using, for instance, the polynomial, spline or Bezier representation. As the result, the fiber orientation at any point of the composite depends on a set of fiber shape parameters  $\mathbf{b}$  defining that particular representation, i.e.  $\theta = \theta(\mathbf{x}, \mathbf{b})$  and it is treated as an angle between the tangent line to the fiber and  $x$ -axis of the global coordinate system, as it is shown in Fig. 1.

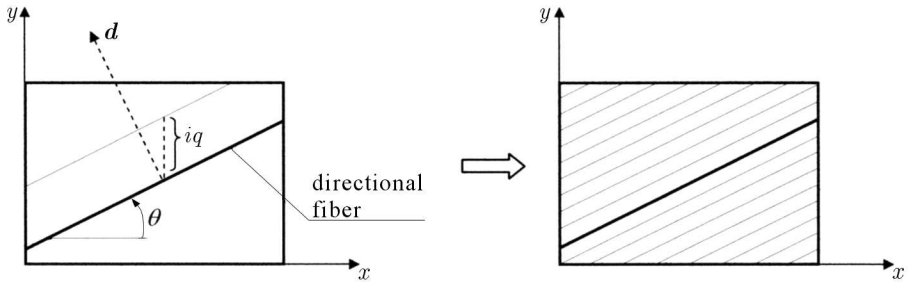


Fig. 2. Family of straight fibers obtained by translation of directional fiber in arbitrary direction

In the first case, the reinforcing fibers constitute a family of straight fibers (Fig. 2), and the shape of directional fiber is described by

$$y(\mathbf{b}) = bx + c \quad (2.1)$$

where  $b$  and  $c$  denote the parameters of a straight line. The fiber orientation angle  $\theta$  for a straight fiber at any point of the composite is related to the parameter  $b$  as follows

$$\tan \theta = b \quad (2.2)$$

or it is directly defined by the angle between the fiber line and the  $x$ -axis of the global coordinate system (Fig. 2). All other fibers in the family are obtained by translation of the directional fiber in the  $\mathbf{d}$ -direction, according to the rule

$$y_i = y(\mathbf{b}) + iq \quad (2.3)$$

where  $i$  is the number of current fiber in the family and  $q$  is the distance between two adjacent fibers measured in the  $y$  direction. As a result, the one-parameter family of reinforcing fibers of the same shape and constant fiber density is created.

Besides the family of straight fibers, also a family of curvilinear fibers can be created. In this case, the fibers layout is described by the shape of the middle line of directional fiber (Fig. 3 and Fig. 4). Now, the shape of this line can be defined using an arbitrary parametric description, namely

$$x = x(\mathbf{b}, t) \quad y = y(\mathbf{b}, t) \quad (2.4)$$

where  $\mathbf{b}$  is a set of fiber shape parameters defining the shape of the curvilinear directional fiber in the composite and  $t$  is a real parameter varying in the range  $\langle \alpha, \beta \rangle$ . The fiber orientation angle  $\theta$  at any point of the composite is defined now by

$$\tan \theta = \frac{y(\mathbf{b}, t)_{,t}}{x(\mathbf{b}, t)_{,t}} = \frac{y_{,t}}{x_{,t}} \quad \text{and} \quad x_{,t} \neq 0 \quad (2.5)$$

where  $y_{,t}$  and  $x_{,t}$  denote derivatives of the function  $y(\mathbf{b}, t)$  and  $x(\mathbf{b}, t)$  with respect to the parameter  $t$ , respectively.

For the case of curvilinear fibers, the layout of all other fibers in the family can be obtained by translation of the directional fiber in an arbitrary direction (Fig. 3), or by its shifting in the direction normal to its middle line (Fig. 4).

In the case of creation of the fiber family by translation of directional fiber (Fig. 3), the directional fiber is translated in the  $\mathbf{d}$ -direction according to the rule

$$x_i = x(\mathbf{b}_k, t) + id_x \quad y_i = y(\mathbf{b}_k, t) + id_y \quad (2.6)$$

where  $i$  is the number of current fiber in the family, while  $d_x$  and  $d_y$  denote components of the distance vector between two adjacent fibers in the direction of translation. Thus, all fibers constitute a one-parameter family of fibers of

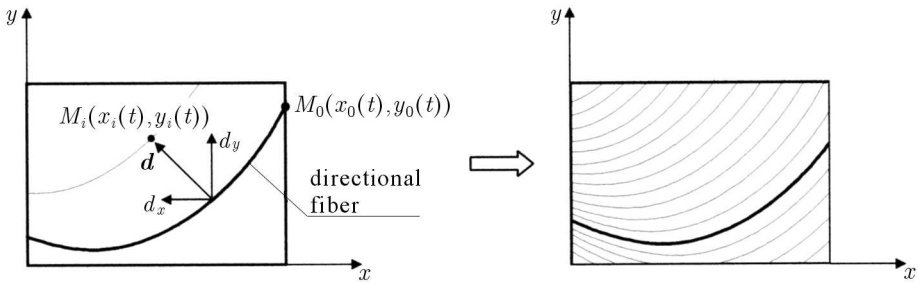


Fig. 3. Family of curvilinear fibers obtained by translation of directional fiber in arbitrary direction

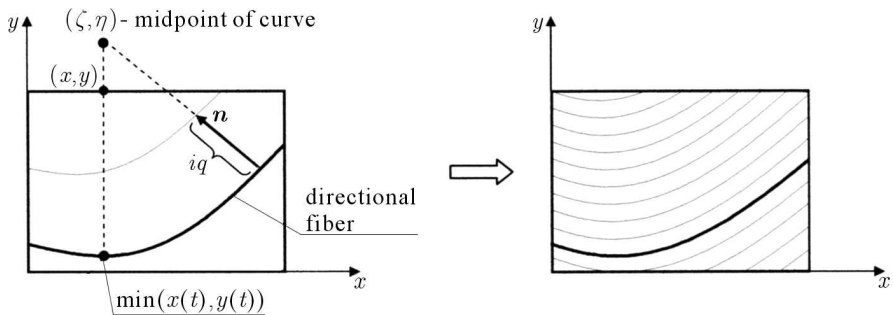


Fig. 4. Family of curvilinear fibers obtained by shifting of directional fiber in normal direction

the same shape, but the local fiber density is varying in the composite domain. The fiber density at any point  $M_i(x_i(t), y_i(t))$  of the composite is specified as follows

$$\rho_{w(i)} = \rho_{w(0)} \frac{\mu x_{0,t} - y_{0,t}}{\mu x_{i,t} - y_{i,t}} \sqrt{\frac{(x_{i,t})^2 + (y_{i,t})^2}{(x_{0,t})^2 + (y_{0,t})^2}} \quad \mu = \frac{dx}{dy} \quad (2.7)$$

where  $\rho_w(0)$  denotes the fiber density in the assumed point  $M_0(x_0(t), y_0(t))$ .

The family of curvilinear reinforcing fibers can also be obtained by shifting the directional fiber in the direction normal to its middle line, as it is shown in Fig. 4. In this case, the local fiber density is constant in the composite domain,

but each particular fiber in this family has a slightly different shape described by the following expressions

$$\begin{aligned} x_i &= x(\mathbf{b}_k, t) + iq \left( -\frac{y_{,t}}{\sqrt{(x_{,t})^2 + (y_{,t})^2}} \right) \\ y_i &= y(\mathbf{b}_k, t) + iq \left( \frac{x_{,t}}{\sqrt{(x_{,t})^2 + (y_{,t})^2}} \right) \end{aligned} \quad (2.8)$$

where  $i$  is the number of current fiber in the family and  $q$  is the distance between two adjacent fibers. Moreover, the creation procedure of the family of curvilinear fibers by shifting the directional fiber has to be subjected to the following condition

$$\sqrt{[\zeta - x(t)_{min}]^2 + [\eta - y(t)_{min}]^2} \geq \sqrt{[x - x(t)_{min}]^2 + [y - y(t)_{min}]^2} \quad (2.9)$$

which prevents the fibers from intersection of their middle lines.

### 3. Model of fiber-reinforced composite material

The microscopically non-homogeneous composite material is treated on a macro-scale level as a plane, homogeneous, orthotropic and linearly elastic material. In conformity with this assumption, the relevant model of the composite is built (Fig. 5). The purpose of the modeling process is to determine the extensional stiffness matrix  $\mathbf{D}$  for that model in the global coordinate system and to express its components in terms of mechanical and geometrical properties of the matrix and the reinforcing fibers.

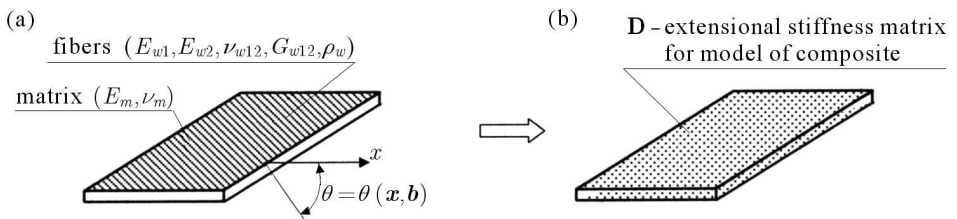


Fig. 5. Fiber-reinforced composite material: (a) real composite, (b) model of composite

The extensional stiffness matrix  $\mathbf{D}$  for the assumed model of the composite material in the global coordinate system  $xy$  can be expressed by (German, 1996)

$$\mathbf{D} = \mathbf{T}^{-1} \mathbf{C} \mathbf{T}^{-\top} \quad (3.1)$$

The matrix  $\mathbf{C}$  denotes here the stiffness matrix for the composite with respect to the material axes 1-2, coinciding with the fiber direction and the direction perpendicular to the fiber, and has the form

$$\mathbf{C} = \begin{bmatrix} \frac{E_1}{1 - \nu_{12}\nu_{21}} & \frac{E_1\nu_{21}}{1 - \nu_{12}\nu_{21}} & 0 \\ \frac{E_2\nu_{12}}{1 - \nu_{12}\nu_{21}} & \frac{E_2}{1 - \nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \quad (3.2)$$

whereas the matrix  $\mathbf{T}$  denotes the transformation matrix from the global coordinate system  $xy$  to the material axes 1-2, and it is expressed as follows

$$\mathbf{T} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \quad (3.3)$$

The components of matrix  $\mathbf{C}$  depend on the so-called engineering constants for orthotropic lamina, where  $E_1$  and  $E_2$  are apparent Young's moduli in the fiber direction and in the direction perpendicular to the fibers, respectively, while  $\nu_{12}$  and  $\nu_{21}$  are major and minor Poisson's ratios, and  $G_{12}$  denotes the in-plane shear modulus. Using the model of lamina presented by Halpin and Tsai (1969) these engineering constants have the following form

$$\begin{aligned} E_1 &= E_{w1}\rho_w + E_m(1 - \rho_w) \\ E_2 &= E_m \left( \frac{1 + \xi\chi\rho_w}{1 - \chi\rho_w} \right) \quad \text{where} \quad \chi = \frac{\frac{E_{w2}}{E_m} - 1}{\frac{E_{w2}}{E_m} + \xi} \\ \nu_{12} &= \nu_{w12}\rho_w + \nu_m(1 - \rho_w) \\ \nu_{21} &= \frac{E_2}{E_1}\nu_{12} \\ G_{12} &= G_m \left( \frac{1 + \xi\chi\rho_w}{1 - \chi\rho_w} \right) \quad \text{where} \quad \chi = \frac{\frac{G_{w12}}{G_m} - 1}{\frac{G_{w12}}{G_m} + \xi} \end{aligned} \quad (3.4)$$

$G_m$  denotes here the shear modulus for the isotropic matrix, while the parameter  $\xi$  is a measure of fiber reinforcement of the composite that depends on the cross-section of fibers and packing geometry. The values of  $\xi$  for typical

**Table 1.** Values of  $\xi$  for typical fibers

	value of $\xi$ for $E_2$	value of $\xi$ for $G_{12}$
circular fibers in square array	$\xi = 2$	$\xi = \begin{cases} 1 & \text{for } \rho_w \leq 0.5 \\ 1 + 40\rho_w^{10} & \text{for } \rho_w > 0.5 \end{cases}$
quadratic cross-section fibers in diamond array	$\xi = 2$	$\xi = 1$
rectangular cross-section fibers in diamond array	$\xi = 2\frac{a}{b}$	$\xi = \sqrt{3} \ln \frac{a}{b}$

fibers obtained by Adams and Doner (1967a,b) and Foye (1966) are given in Table 1.

The matrix  $\mathbf{T}$  is considered as the matrix function of fiber orientation angle  $\theta$ . When the fiber line is described by (2.1), its components are simply defined by

$$\begin{aligned}\sin \theta &= \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{b}{\sqrt{1 + b^2}} \\ \cos \theta &= \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{1}{\sqrt{1 + b^2}}\end{aligned}\tag{3.5}$$

or they can be explicitly defined by the angle between the fiber line and the  $x$ -axis of the global coordinate system. For the case of curvilinear fibers,  $\sin \theta$  and  $\cos \theta$  follow from (2.5), and are expressed in the form

$$\begin{aligned}\sin \theta &= \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{y_{,t}}{\sqrt{(x_{,t})^2 + (y_{,t})^2}} \\ \cos \theta &= \frac{1}{\sqrt{1 + \tan^2 \theta}} = \frac{x_{,t}}{\sqrt{(x_{,t})^2 + (y_{,t})^2}}\end{aligned}\tag{3.6}$$

#### 4. Analysis of structural behavior

Under the applied load, the structure shown in Fig. 1 undergoes some deformations described by the displacement field  $\mathbf{u}$ , the strain field  $\mathbf{e}$  and the



stress field  $\boldsymbol{\sigma}$ . Thus, the behavior of the structure can be described by the equilibrium equation given in the form (Zienkiewicz, 1971)

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f}^0 = \mathbf{0} \quad (4.1)$$

as well as the kinematical relation between strain and displacement fields

$$\mathbf{e} = \mathbf{B}\mathbf{u} \quad (4.2)$$

where  $\mathbf{B}$  is a linear differential operator relating the displacement field with the strain field. A linear stress-strain relation is assumed in the form of generalized Hooke's law

$$\boldsymbol{\sigma} = \mathbf{D}\mathbf{e} \quad (4.3)$$

where  $\mathbf{D}$  denotes the extensional stiffness matrix for the orthotropic model of the composite material, and it is given by (3.1)-(3.6). Besides, the structure is subjected to the boundary conditions expressed as follows

$$\begin{cases} \boldsymbol{\sigma}\mathbf{n} = \mathbf{T}^0 & \text{on } S_T \\ \mathbf{u} = \mathbf{u}^0 & \text{on } S_U \end{cases} \quad (4.4)$$

where  $\mathbf{n}$  denotes the normal unit vector on the external boundary  $S$  of the disk.

To solve the set of equations (4.1)-(4.4), the finite element method (FEM) is proposed. The structure domain is divided into a set of finite elements connected at nodes each to an other. In the present paper, the two-dimensional four-node quadrilateral elements are used. The detailed description of the finite element methods is presented, for instance, in Zienkiewicz (1971).

## 5. Optimization problem

The mechanical properties of composite disks depend on mechanical properties of the matrix and reinforcing fibers, percentage participation of fibers in this material, fiber shape and orientation as well as on the cross-section of fibers. Thus, each of these parameters can be treated as a design variable during the optimization procedure. However, full advantages of the composite disk can mainly be obtained when the reinforcing fibers are optimally arranged in the matrix with respect to the assumed objective behavioral measure under actual loading conditions of the structure.

The problem of optimal design of fibers arrangement in the composite, so that the structure should satisfy assumed requirements in the range of mechanical properties can be formulated in general as the minimization or maximization of the proper quality index of structural behavior, expressed in the global form as follows

$$\min (\text{or } \max) \quad F_c = \int_A \Gamma(\boldsymbol{\sigma}, \mathbf{e}, \mathbf{u}, \mathbf{b}) dA + \int_{S_T} \Psi(\mathbf{T}^0, \mathbf{u}) dS_T \quad (5.1)$$

subjected to the global or local behavioral constraints

$$\left( \int_A \Gamma_1(\boldsymbol{\sigma}, \mathbf{e}, \mathbf{u}, \mathbf{b}) dA + \int_{S_T} \Psi_1(\mathbf{T}^0, \mathbf{u}) dS_T \right) - G_0 \leq 0 \quad (5.2)$$

and/or the constraint imposed on the total cost of the composite structure

$$[c_w \rho_w + c_m (1 - \rho_w)]V - C_0 \leq 0 \quad (5.3)$$

where  $\Gamma$ ,  $\Gamma_1$  and  $\Psi$ ,  $\Psi_1$  are continuous functions depending on the state fields induced in the deformed structure, while  $c_w$  and  $c_m$  are the costs per unit volume of the fiber and matrix materials, respectively. The fiber orientation of a straight fiber or parameters defining the shape of a curvilinear fiber are selected as components of the vector of design variables  $\mathbf{b}$  during this optimization process.

The optimization task, defined by (5.1)-(5.3), will be performed with the aid of the evolutionary algorithm shown in Fig. 6. This method based on the imitation of the evolution processes occurring in the nature (Michalewicz, 1996) still finds the growing interest in engineering design problems. The evolutionary algorithm is a simple, powerful and effective tool used for finding the best solution in a complicated space of design parameters and it is not limited by a restrictive assumption about the search space. This method only needs the information based on the objective functional, which is its main advantage in comparison to the methods based on gradient information of the objective functional and constraints. Besides, in contrast to deterministic methods, which often fall into a local optimum, the evolutionary algorithm always finds the global optimum or the solution close to this optimum.

It should be added that in the case of the optimization problem with constraints, the penalty function approach (Findeisen *et al.*, 1980) is applied in the proposed algorithm. Using this approach, the constrained problem is transformed to an unconstrained one as follows

$$\min Z(\mathbf{b}, \boldsymbol{\alpha}) = \min \left[ F_c(\mathbf{b}) + \frac{1}{2} \sum_{i=1}^{n_g} \alpha_i [\max(0; G_i(\mathbf{b}))]^2 \right] \quad (5.4)$$

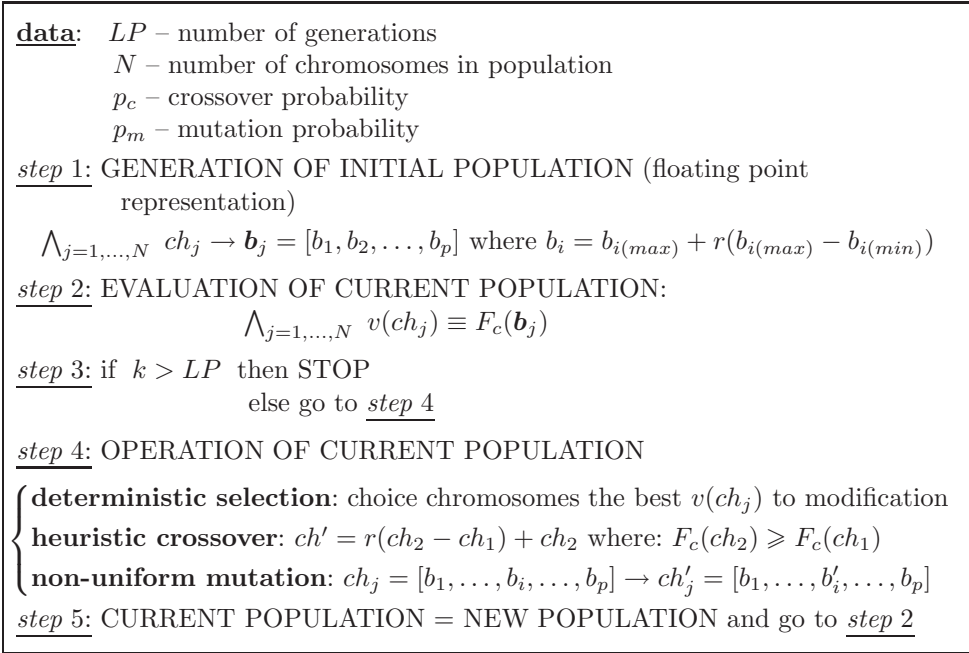


Fig. 6. Flow chart of evolutionary algorithm

where  $\alpha$  is a vector of positive coefficients of penalty functions, and  $n_g$  is a number of inequality constraints in the constrained problem.

The evolutionary algorithm starts from random selection of the initial population of  $N$  chromosomes. Each chromosome is a coded vector of design parameters  $\mathbf{b}$  defining the shape or orientation of reinforcing fibers and describes one possible solution to the given problem. This population is processed by three main operators of the evolutionary algorithm. They are deterministic selection, heuristic crossover and non-uniform mutation. Applying these three operators, a new population of solutions is created and the single cycle of the evolutionary algorithm, which is known as a generation, comes to the end. Each successive generation contains better „partial solutions than in the previous generations, and converges towards the global optimum. This procedure is continued until the best solution is found according to the assumed stop criterion or the specified number of generations is attained.

As it is shown in Fig. 6, all chromosomes in the current population are evaluated using the objective functional  $F_c$  in the second step of the evolutionary algorithm. The values of state fields appearing in the objective functional are calculated using the finite element method in the analysis of the structural behavior (Section 4).

## 6. Optimal design of composite disks

To illustrate the applicability of the proposed approach to the optimal design of fiber-reinforced composite disks, some simple examples are presented in this Section. The particular forms of optimization problem (5.1)-(5.3) will be introduced in the successive examples, in order to obtain the optimal layout of reinforcing fibers in the composite structure.

As the **first example**, let us consider the problem of optimal strength design of a composite disk for which the optimal fiber shape and orientation should be derived. Let us assume that the allowable stress level for the composite structure after homogenization is defined by the longitudinal tensile strength  $R_{r1}$ , longitudinal compressive strength  $R_{c1}$ , transverse tensile strength  $R_{r2}$ , transverse compressive strength  $R_{c2}$  and shear strength  $R_s$ .

For an orthotropic material, the strength function  $\Phi$  corresponding to the stress field  $\sigma$  induced in the deformed structure is assigned using the bi-axial theory (Tsai and Wu, 1971). In conformity with this theory, the value of the strength function  $\Phi$  at any point of the composite structure should satisfy the following relationship

$$\Phi(\sigma) = F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{33}\tau_{12}^2 \leq 1 \quad (6.1)$$

where the strength coefficients  $F_i$  and  $F_{ij}$  appearing in (6.1) have the form

$$\begin{aligned} F_1 &= \frac{1}{R_{r1}} - \frac{1}{R_{c1}} & F_2 &= \frac{1}{R_{r2}} - \frac{1}{R_{c2}} & F_{11} &= \frac{1}{R_{r1}R_{c1}} \\ F_{22} &= \frac{1}{R_{r2}R_{c2}} & F_{33} &= \frac{1}{R_s^2} & F_{12} &= -\frac{\sqrt{F_{11}F_{22}}}{2} \end{aligned} \quad (6.2)$$

whereas  $\sigma_1$ ,  $\sigma_2$  and  $\tau_{12}$  are components of the stress field with respect to material axes of the composite, coinciding with the fiber direction and the direction perpendicular to the fiber. They are obtained using the transformation rule in the form

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad (6.3)$$

The main idea of the strength design results in reduction of values of the strength function, which violate criterion (6.1), and then in redistribution of the stress field in the domain of the structure. Thus, the global measure of

local strength criterion can be introduced as a behavioral functional and the optimization problem is formulated in the following form

$$\min F_c(\mathbf{b}) = \sqrt[n]{\int_A [\Phi(\mathbf{b})]^n dA} \quad (6.4)$$

where  $A$  is the area of structure domain of unitary thickness and  $\mathbf{b}$  denotes a vector of design parameters defining the layout of fibers in the composite, and  $n$  denotes an even integer number. When  $n \rightarrow \infty$ , the functional is a strict measure of the maximum local values of strength criterion. In practice, the value of  $n$  does not exceed the upper bound limit following from numerical constraints. In all presented examples, the value of  $n$  was set to 20.

The above formulated problem can be illustrated by a simple example. Let us consider a disk supported along its lower boundary and loaded by linearly varying traction on the upper boundary (Fig. 7). The disk has a unitary thickness and it is made of the composite material with mechanical properties given in Tables 2 and 3.

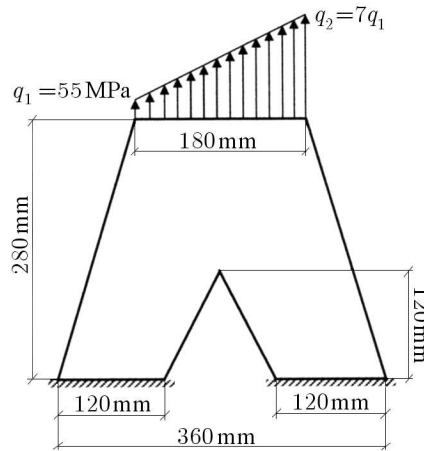


Fig. 7. Composite disk subjected to load and boundary conditions

Optimization problem (6.4) was solved for two classes of the fiber shape. In the first case, the composite material was reinforced with one family of the straight fibers, and the angle of fiber orientation  $\theta$  was selected as the design parameter, i.e.  $\mathbf{b} = \{\theta\}$ . Next, the design problem was considered for the case of one family of curvilinear fibers for which the shape of the middle line of directional fiber was described by Bezier's curve (Kiciak, 2000). In this case, the coordinates of four nodes of the Bezier polygon were chosen as the design

**Table 2.** Material data of components of composite

	$E$ [GPa]	$\nu$	$G$ [GPa]	$\rho$ [%]
fibers (graphite HS)	220.0 13.8	0.22	8.90	45
matrix (epoxy)	3.5	0.38	–	55
fibers array	quadratic cross-section fibers in diamond array			

source: Adopted from German (1996)

**Table 3.** Allowable stress level for composite (graphite HS/epoxy)

$R_{r1}$ [MPa]	$R_{c1}$ [MPa]	$R_{r2}$ [MPa]	$R_{c2}$ [MPa]	$R_s$ [MPa]
1531	1390	41	145	98

source: Adopted from German (1996)

parameters, i.e.  $\mathbf{b} = \{x_i, y_i\}$ ,  $i = 0, 1, 2, 3$ . The family of curvilinear reinforcing fibers was obtained by shifting the directional fiber in the direction normal to its middle line.

The optimization problem discussed here was solved using the proposed evolutionary algorithm, discussed already in Section 5. To analyze the behavior of the disk, its domain was discretized into 196 two-dimensional four-node quadrilateral elements. The results of the optimization process are given in Table 4, and the optimal layouts of reinforcing fibers in the disk domain are shown in Fig. 8a and 8b, respectively.

**Table 4.** Results of optimization process

	design parameters	value of local strength function
optimal disk with one family of straight fibers	$\theta = 85.17^\circ$	$\Phi_{max} = 0.92$ $\Phi_{min} = -0.13$
optimal disk with one family of curvilinear fibers described by Bezier function	$P_0$ (0.317, 0.000) $P_1$ (0.131, 0.075) $P_2$ (0.142, 0.205) $P_3$ (0.148, 0.280)	$\Phi_{max} = 0.84$ $\Phi_{min} = -0.12$
reference disk with one family of straight fibers	$\theta = 90.00^\circ$	$\Phi_{max} = \underline{\mathbf{1.15}}$ $\Phi_{min} = -0.11$

These optimal solutions were also compared with the reference solution for the disk reinforced with one family of straight fibers parallel to the  $y$ -axis (Fig. 8c) in order to qualify the results of optimization. One can observe that

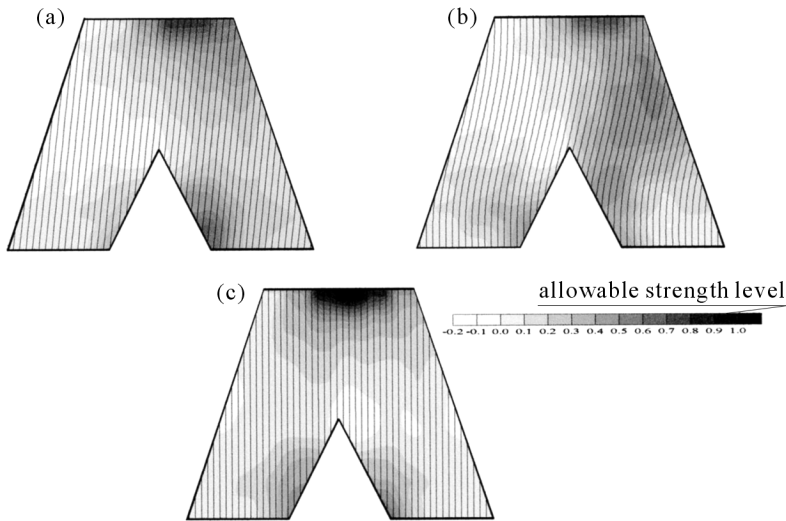


Fig. 8. Plot of strength function in: (a) optimal disk with one family of straight fibers, (b) optimal disk with one family of curvilinear fibers described by Bezier representation, (c) reference disk with one family of straight fibers

the distribution of strength function  $\Phi$  in the reference disk is considerably reduced after the optimization process, and the values of local strength function violating criterion (6.1) are eliminated for both optimal solutions (cf. the plots shown in Fig. 8 and maximal values of  $\Phi$  presented in Table 4). It must be also noted that the distribution of the stress field in the optimal structures is practically uniform which allows for better usage of the construction material or allows increasing the allowable upper level of service load.

As the **second example**, the problem of optimal layout of reinforcing fibers in the composite for the case of mean stiffness design of the disk is discussed. Thus, the minimization of the work done by external forces is selected as the objective functional, and the optimization problem can be written in the following form

$$\min F_c(\mathbf{b}) = \int_{S_T} \mathbf{u}^\top \mathbf{T}^0 dS_T \quad (6.5)$$

where  $\mathbf{b}$  is a vector of design parameters defining the shape of the middle line of the directional fiber.

To illustrate this kind of design problem, the composite disk shown in Fig. 9 was considered. Similarly as in the first example, the disk has a unitary thickness and it is composed from polyester matrix reinforced with graphite

fibers symmetrically placed with respect to the  $x$  axis. The material data of the components of the composite material are given in Table 5.

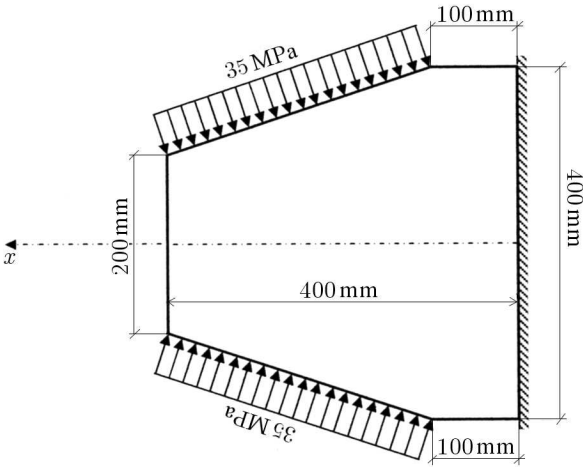


Fig. 9. Composite disk subjected to load and boundary conditions

**Table 5.** Material data of composite components

	$E$ [GPa]	$\nu$	$G$ [GPa]	$\rho$ [%]
fibers (graphite HS)	220.0 13.8	0.22	8.90	50
matrix (epoxy)	3.2	0.41	—	50
fibers array	quadratic cross-section fibers in diamond array			

source: Adopted from German (1996)

The problem of mean stiffness design was discussed, as previously, for two classes of fibers shape, and due to symmetry of the problem, only one half of the disk was considered. To solve the design problem, the evolutionary algorithm presented in Section 5 was used. The values of state fields and objective functional  $F_c$  were calculated using the finite element method with 100 two-dimensional four-node quadrilateral elements applied for discretization of the disk domain.

In the first case, the composite material was reinforced with one family of straight fibers for which the angle of orientation with respect to the symmetry axis of the disk was selected as the design parameter, i.e.  $\mathbf{b} = \{\theta\}$ . The results of optimization process for this case are given in Table 6 and the optimal layout of reinforcing fibers in disk is shown in Fig. 10a.



In the second case, the material of the disk was reinforced with one family of parabolic fibers. Thus, the shape of directional fibers was assumed in the form  $x = ay^2$ , where the coefficient  $a$  was treated as design parameters, i.e.  $\mathbf{b} = \{a\}$ . All other fibers in the family were obtained by translation of the directional fiber in the  $x$ -direction. The results obtained after the optimization process for this case are also given in Table 6, and the optimal layout of fibers is depicted in Fig. 10b.

**Table 6.** Result of optimization process

	design parameters	objective functional
optimal disk with straight fibers	$\theta = 76.98^\circ$	$F_c = 15.40 \text{ J}$
optimal disk with curvilinear fibers	$a = 0.72$	$F_c = 14.15 \text{ J}$
reference disk with straight fibers	$\theta = 158.0^\circ$	$F_c = 38.76 \text{ J}$

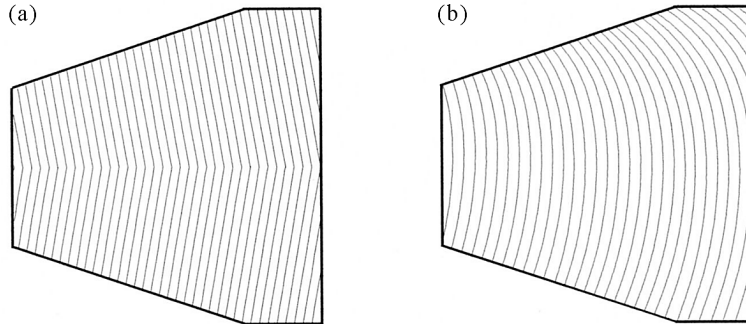


Fig. 10. Optimal disk with family of straight fibers (a), optimal disk with one family of parabolic fibers (b)

To evaluate the results obtained for optimal solutions, the mean stiffness of the reference disk reinforced with the family of straight fibers of orientation  $\theta = 158^\circ$  with respect to the symmetry axis of the disk (the most flexible disk) was also calculated. The optimal layout in the case of family of straight fibers increases the mean stiffness of the disk by 60%, while in the case of family of parabolic fibers, this stiffness increases by 68%, when compared to the reference disk.

As the **last example**, let us consider the problem of optimal design of the composite disk with respect to the cost of its components. The disk is

additionally subjected to the global and/or local behavioral constraints. Thus, the optimization problem can be formulated as follows

$$\min C(\mathbf{b}) = [c_w \rho_w + c_m (1 - \rho_w)] V \tag{6.6}$$

subjected to the global behavioral constraint

$$\left( \int_A \Gamma(\boldsymbol{\sigma}, \mathbf{e}, \mathbf{u}, \mathbf{b}) \, dA + \int_{S_T} \Psi(\mathbf{T}^0, \mathbf{u}) \, dS_T \right) - G_0 \leq 0 \tag{6.7}$$

where  $\mathbf{b}$  denotes a vector of design parameters,  $c_w$  and  $c_m$  are the costs per unit volume of the fiber and matrix materials, respectively, while  $\Gamma$  and  $\Psi$  are continuous functions depending on the state fields induced in the deformed structure.

To illustrate this kind of optimization problem, let us consider the composite disk shown in Fig. 11. The disk has a unitary thickness and the mechanical properties as well as unit costs of the components are given in Table 7.

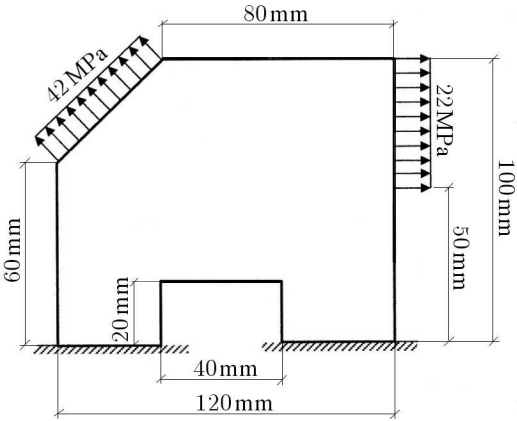


Fig. 11. Composite disk subjected to load and boundary conditions

**Table 7.** Material data of composite components

	$E$ [GPa]	$\nu$	cost [PLN]
fibers (glass E)	75	0.25	30
matrix (epoxy)	3.5	0.38	12
fibers array	circular fibers in square array		

source: Adopted from German (1996)

The minimization of the total cost of the disk reinforced with one family of straight fibers was considered. It was also assumed that the stiffness of

reinforced disk should be equal to 170 J. The angle of fiber orientation  $\theta$  and fiber density  $\rho_w$  were selected as the design parameters, i.e.  $\mathbf{b} = \{\theta, \rho_w\}$ , and they were determined during the optimization process.

**Table 8.** Results of optimization process

	design parameters	stiffness [J]	cost [PLN]
optimal disk	$\theta = 134^\circ$ , $\rho_w = 0.52$	170.0	2.48
reference disk	$\theta = 90^\circ$ , $\rho_w = 0.70$	170.0	2.96

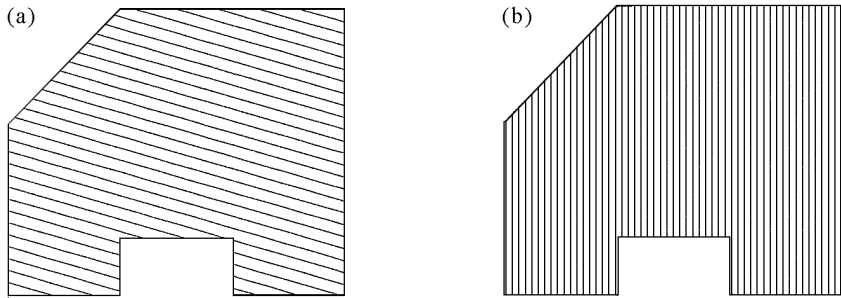


Fig. 12. Optimal disk (a) and reference disk (b) reinforced with one family of straight fibers

The design problem was solved using the proposed evolutionary algorithm and the results, obtained after of the optimization process, are given in Table 8. In this example, the disk was divided into 120 two-dimensional four-node quadrilateral elements. The optimal solution, shown in Fig. 12a, was next compared with the solution for the same disk reinforced with one family of straight fibers parallel to the  $y$ -axis (Fig. 12b). The optimal layout of fibers decreases the total cost of disk by 17% when compared to the reference design shown in Fig. 12b.

## 7. Conclusion

The full advantages of a composite disk subjected to a particular load can be obtained when fibers are optimally distributed and oriented in the structure with respect to the assumed measure of structural behavior. This optimal layout can be obtained using the proposed evolutionary optimization algorithm, which can be a very effective tool for finding a reasonable solution to a given problem. Thus, this algorithm can constitute an alternative technique for

the classical gradient-oriented methods applied in optimization of structural elements, or can supplement them.

The presented analysis can be treated as a starting point for computer-oriented optimal design procedures of real structures made of composite materials subjected to actual loading conditions. Such a procedure can allow for avoiding expensive experimental testing, which can be reduced to the final phase of structural design.

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## Optymalne projektowanie tarcz kompozytowych wzmocnionych włóknami

### Streszczenie

Praca dotyczy zagadnień modelowania, analizy i optymalizacji zachodzących w procesie projektowania płaskich, dwuwymiarowych elementów konstrukcyjnych wykonanych z materiałów kompozytowych i obciążonych statycznie siłami działającymi w ich płaszczyźnie. Rozpatrzono problem projektowania optymalnego ułożenia włókien wzmacniających w kompozycie pod kątem uzyskania przez element konstrukcyjny wykonany z tego materiału wymaganych własności mechanicznych. Rozważania dotyczyły sposobu projektowania oraz generowania rodziny prostoliniowych, jak również krzywoliniowych włókien wzmacniających. Do poszukiwania optymalnych rozwiązań zastosowano opracowaną do tego celu metodę optymalizacyjną opartą na algorytmie ewolucyjnym. Rozpatrywany w pracy problem zilustrowano przykładami numerycznymi.

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