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# Level Set Method in Inverse Problem Solution

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***Abstract.*** *The optimal shape design of the capacitor with Laplace equation of state and inverse problem solution in Electrical Impedance Tomography using level set method are presented in the paper. The inverse problem solution determines the positions of capacitor plates, which were optimized to achieve required potential distribution, while the inverse problem solution in EIT enables the identification of the size and the position of internal areas with different conductivity.*

## 1. Introduction

The electromagnetic structures have been widely investigated in recent years. The most frequent approach to this problem is to come it down to the problem of optimization and solve it by deterministic or stochastic methods. The optimization problem solution deals with minimizing the objective function of the problem [1, 2]. There are many different algorithms for this problem solution: deterministic methods (e.g. back-projection, perturbation, and Newton-Raphson, Conjugate Gradient method) [1,2,3], stochastic methods (e.g. Genetic Algorithms, Monte-Carlo method, Simulated Annealing)[1,2,3], and also Artificial Neural Network (ANN). In the first case, although the number of function evaluations needed to be reached is generally small, the risk of being stopped in local minima is very high. Stochastic algorithms deal with finding the global minimum. In this case the continuity and differentiability of the objective function are not required. Main disadvantages of these methods are weak convergence and high time complexity. In order to solve the inverse problem using iterative optimization method it is necessary to repeat forward problem solution many times – to determine the distribution of potential [4].

In Electrical Impedance Tomography (EIT) the reconstruction algorithm uses the knowledge of applied current patterns and measured electrode voltages to solve the inverse problem, computing electrical conductivity distribution in the object. The inverse problem in EIT is nonlinear, because the current flow strongly depends on the unknown conductivity within the object. The solution of the inverse problem in EIT is significantly more difficult than in case of e.g. X-ray computed tomography, where the photon paths are essentially straight lines. Furthermore, the problem is ill-posed due to its instability – small errors in the measurements can produce large errors in reconstruction of conductivity.

In our case the level set method approach to the inverse problem solution was used. The solution determines the positions of the capacitor plates, which were optimized to achieve required potential distribution. The inverse problem solution in EIT provides the identification of the size and the position of internal areas with different conductivity.

The level set method is chosen to describe moving shapes, since this method is able to simplify model topological changes of the boundaries. In this technique, the shapes are given as the zero level set of a higher dimensional level set function.

The level set methods approach in optimal shape design involving partial differential equation has received little attention up till now.

The method, that uses both level set for determining the shape of the domains and essentially nonoscilatory schemes to solve the Hamilton-Jacobi equation is known as an efficient method for wide class of shape optimization problems involving partial differential equation [5,6,7].

## 2. Level Set Method

Level set methods were proposed as a versatile tool for representing moving fronts in a variety of physical processes, involving flow phenomena, crystal growth and phase changes among others. The use of level set methods for shape optimization involving partial differential equations apparently received little attention so far [10].

Given an interface $\Gamma$ and region $\Omega$ its subsequent motion under a velocity field was analyzed and computed. The level set function $\varphi$ has the following properties (see Fig. 1) [6,7]:

$$
\begin{aligned}
&\varphi(x,t) > 0 \quad x \in \Omega_1 \\
&\varphi(x,t) < 0 \quad x \in \Omega_2 \\
&\varphi(x,t) = 0 \quad x \in \partial\Omega
\end{aligned}
\tag{1}
$$

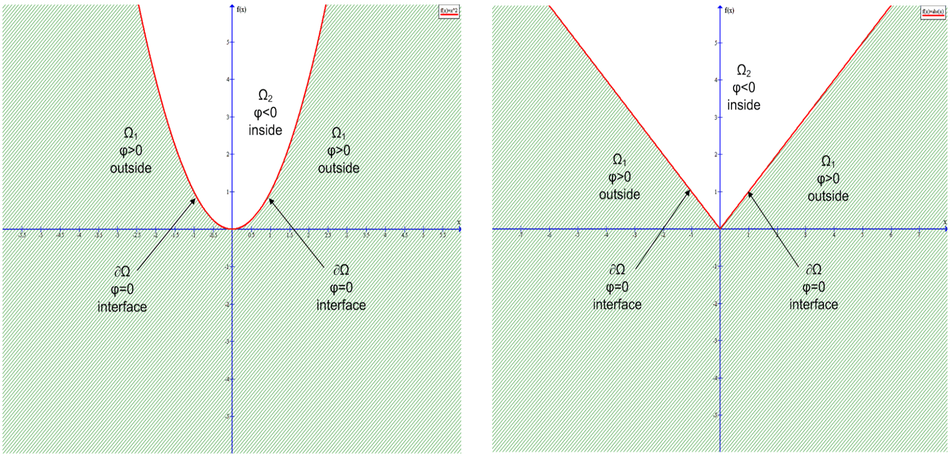


Fig.1. Considered regions – convex functions

The level set method allows to move the interface $\Gamma$ along the direction $\vec{V}$ within a neighborhood of $\Gamma$. To derive the equation for level set function, let $\varphi(t,\cdot)$ denote a family of functions from $R^2$ to $R$. If $x \to \varphi(t,x)$ is perturbed to $x \to \varphi(t, x+th(x))$ then differentiating level contours $\{x : \varphi(t, x+th(x)) = const\}$ with respect to *t* we obtain [8]:

$$\varphi_t + h\nabla\varphi = 0 \tag{2}$$

This leads to the Hamilton-Jacobi equation for level set function.

$$\frac{\partial\varphi}{\partial t} + \vec{V}\nabla\varphi = 0\text{, where } \vec{V}_N = \vec{V}\frac{\nabla\varphi}{|\nabla\varphi|} \tag{3}$$

The updating of level set is possible using Hamilton-Jacobi equation (3) in which velocity field is the function of the direction of the unit normal [7,8]:

The first step in the solution of a problem using set level method is determination of zero-level set (Fig. 2) $\Gamma_0 = \varphi(x,0)$ for *t*=0 [6].

Do the following steps in the numerical algorithm until its convergence:

– solve the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0 \text{ with boundary conditions } \begin{matrix} u(a) = 1 \\ u(b) = 0 \end{matrix} \tag{4}$$

– determine the potential distribution $u_0$ [6];

- compute the difference of the observed value and expected value $u - u_0$;
- solve the Poisson equation:

$$-\Delta p = u - u_0 \tag{5}$$

- evaluate the normal velocity:

$$v_k = \nabla p_k \cdot \nabla u_k \tag{6}$$

- update the level set function $\varphi(x,t)$ by solving the Hamilton-Jacobi equation:

$$\varphi_t^k + v_k \left|\nabla \varphi^k\right| = 0 \tag{7}$$

$$\varphi_t^{k+1} = \varphi_t^k - \Delta t\, v_k \left|\nabla \varphi^k\right| \tag{8}$$

- check the convergence; options include repeating the process, stopping if convergence criteria are satisfied [6].

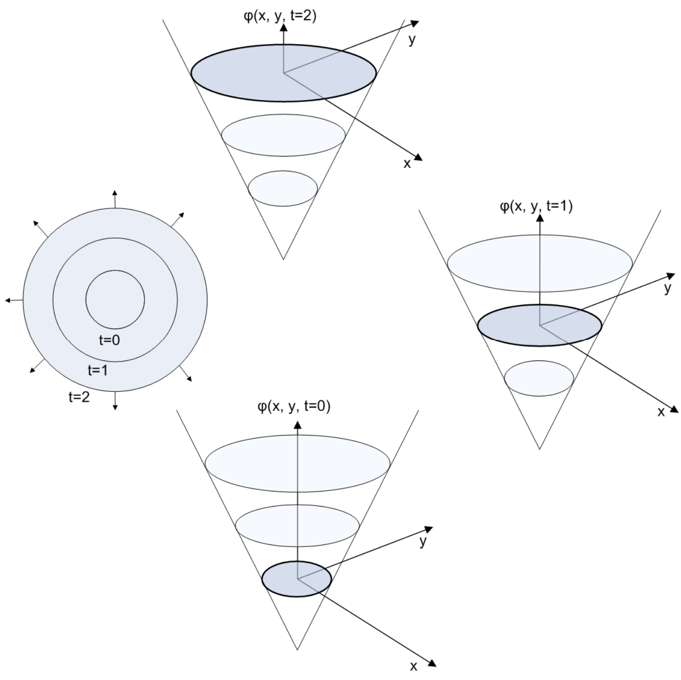


Fig.2. Zero-level set

The stability of received solution is guaranteed by Courant-Friedreichs-Levy condition, which asserts, that the numerical waves should propagate at least as fast as the physical waves. The CFL time step restriction is described as:

$$\Delta t < \frac{\Delta x}{\max\{|u|\}} \tag{9}$$

Equation is usually enforced by choosing a CFL number $\alpha$ with [6]:

$$\Delta t\left(\frac{\max\{|u|\}}{\Delta x}\right) = \alpha \text{ and } 0 < \alpha < 1 \tag{10}$$

An optimum value of $\alpha$ that provides best stability and convergence is 0.9, and a common conservative choice is $\alpha = 0.5$.

## 2.1 Hamilton-Jacobi ENO

The scheme ENO – essentially nonoscillatory polynomial interpolation was used to determine the function of level set gradient $\left|\nabla\varphi^k\right|$ in the following iterations.,

The zeroth divided differences of $\varphi$, as a standard with Newton polynomial interpolation are defined at the grid nodes and defined by [6]:

$$D_i^0\varphi = \varphi_i \tag{11}$$

at each grid node *i*. The first divided differences of defined midway between grid nodes as:

$$D_{i+1/2}^1\varphi = \frac{D_{i+1}^0\varphi - D_i^0\varphi}{\Delta x} \tag{12}$$

The divided differences are used to reconstruct a polynomial of the form:

$$\varphi(x) = Q_0(x) + Q_1(x) \tag{13}$$

that can be differentiated and evaluated at $x_i$ to find $\left(\varphi_x^+\right)_i$ and $\left(\varphi_x^-\right)_i$. That is, we use

$$\varphi_x(x_i) = Q_0^{'}(x) + Q_1^{'}(x) \tag{14}$$

to define $\left(\varphi_x^+\right)_i$ and $\left(\varphi_x^-\right)_i$.

# 3. Boundary Element Method

Integral equations are used to the solution to the forward problems (analysis) and the inverse problems (synthesis and identification) [3, 4]. The function and Green's formulas are used to the transformation of the integral equations. The Green's function satisfies the Laplace's equation:

$$\nabla^2 G = 0 \tag{15}$$

and for 2D domain is:

$$G(M,P) = -\frac{1}{2\pi}\ln(r) \tag{16}$$

where: r – the distance between M and P

$$r(M,P)=\sqrt{(x_M - x_P)^2 + (y_M - y_P)^2} \tag{17}$$

The second Green's identity (symmetrical), which was used in transformation of the integral equation, is described as follows:

$$\int_\Omega (u\nabla^2 G - G\nabla^2 u)d\Omega = \oint_S \left( u\frac{\partial \mathrm{G}}{\partial \mathrm{n}} - G\frac{\partial u}{\partial \mathrm{n}} \right) dS$$

where: $u$ – the potential in 2D domain Ω.
To solve this problem numerically, the surface has to be discretized into elements. The element, which was modeled as a constant value were used in this case.

# 4. Numerical experiments and results

## 4.1 Cylindrical capacitor

The cylindrical capacitor (Fig. 3) with the potential distribution described by Laplace equation (4) was considered [8, 9] with boundary conditions.

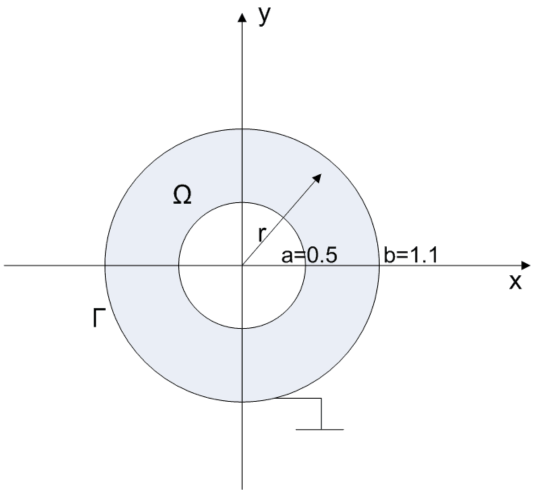


Fig.3. Cross-section of the capacitor

The potential distribution is known in the original region Ω and defined as $u_0(r) = -0.721348 \ln(0.714286r)$.

The solution of considered problem is finding the region $\Omega_0$, where the function $u_0$ is the solution of Laplace equation with boundary conditions (4).

Potential distribution was evaluated using Boundary Element Method. Both covers of the capacitor (two circles) were discretized into 32 segments (boundary elements). In the next step the inverse problem was calculated on the basis of the potential values obtained by using BEM. Level set method, shortly described in section 2, was used to the solution of inverse problem. These procedures were repeated until the values of potential distribution were equal to $u_0$.

The conjugate equation for Laplace equation is defined as:

$$\Delta q = u \ln r \left( \frac{1}{\ln \frac{a_0}{b_0}} - \frac{1}{\ln \frac{a}{b}} \right) + u \left( \frac{\ln b}{\ln \frac{a}{b}} - \frac{\ln b_0}{\ln \frac{a_0}{b_0}} \right) \tag{18}$$

Figures 4 – 7 present the iteration process for two different values of $\alpha$. For $\alpha$=0.8 the function $u_0$ is the solution of Laplace equation for the radii values $a$=0.349 and $b$=1.401 (Figs. 4 and 5) in 48 iterations. For $\alpha$=0.9 we got the radii values of capacitor plates $a$ =0.35 and $b$=1.40 in 29 iterations (Figs. 6 and 7). In comparison, the material derivative (one of the most popular method in shape optimization) the approach to the solution gave results just after 672 iterations [10].

Our experiments confirm that in this case, the selection of $\alpha$ value is very important.

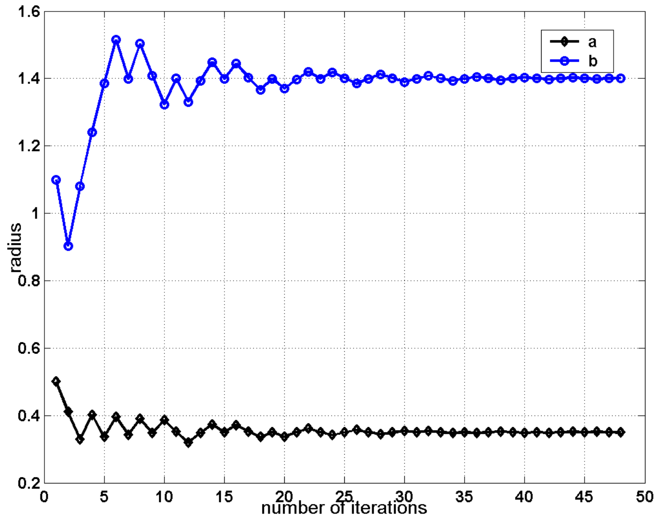


Fig.4. Iteration process for $\alpha$ =0.8

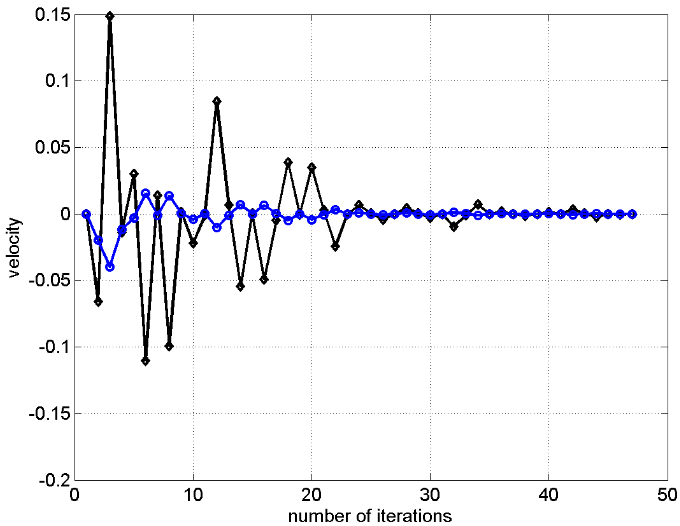


Fig.5. The normal velocity in following iterations for α=0.8

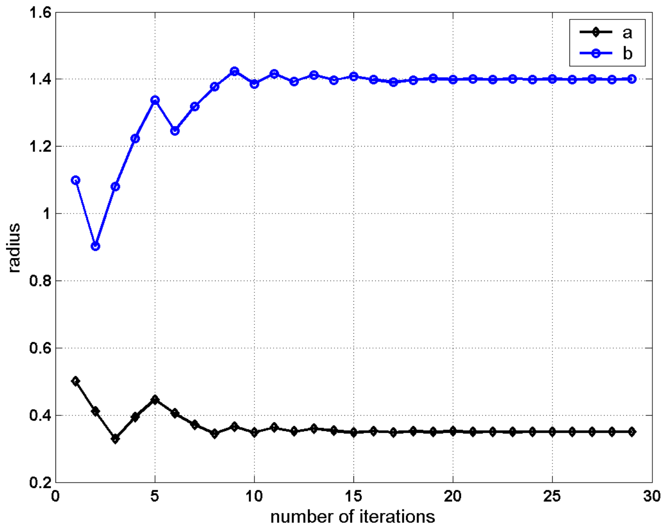


Fig.6. Iteration process for $\alpha$ =0.9

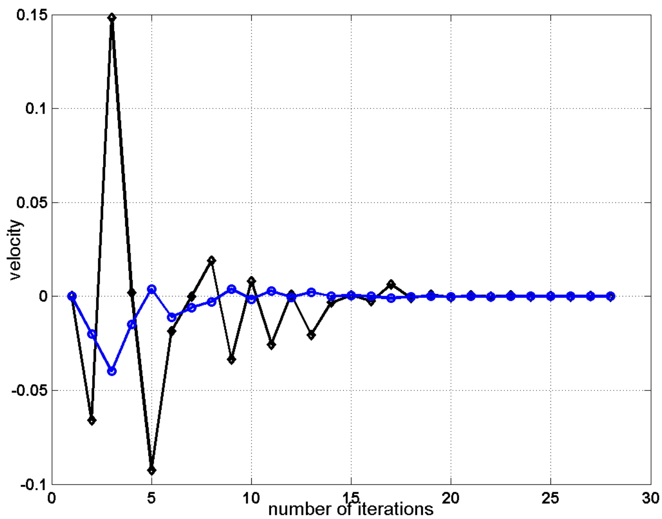


Fig.7. The normal velocity in following iterations for $\alpha$ =0.9

## 4.2 Impedance Tomography

The solution of the forward problem in EIT is to determine the distribution of potential for a given conductivity geometry and for a given set of current injection electrodes (Fig. 8).

Algebraic solution of the forward problem in 2D EIT can be formulated as follows:

$$\nabla \cdot [\gamma(x,y)\nabla\varphi(x,y)] = 0 \text{ in } \Omega \tag{19}$$

where:
$\vec{E} = -\nabla\varphi(x,y)$ – the vector of the electric field intensity;

$\gamma(x,y)$ – the conductivity.

With $n$ being the unit outward normal vector to the boundary surface, $\varphi$ is subjected to the following boundary conditions:

$\varphi_D$ – is known potential at the electrodes connected to the voltage source (20)

$\frac{\partial\varphi}{\partial n}_N = 0$ – at the rest of the boundary including the bottom of the hemisphere (21)

The model of the computer simulation consists of layer with 16 electrodes. In all experiments, the protocol files so-called the polar voltage excitation has been used (Fig. 8).

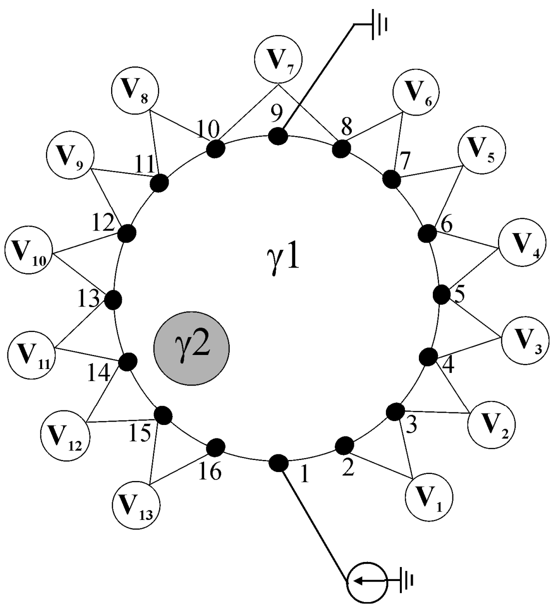


Fig. 8. Configuration of electrode-to-electrode voltages – protocol file

The solution to the inverse problem deals with finding the position and radius of the internal object with different conductivity. The procedure of this kind problem solution based on electrode-to-electrode voltages obtained on the surface of tested object.

In order to solve the EIT problem for the identification of the size and the position of the anomalies, the LSM was used.

Figures 9 and 10 present the iteration process for two different locations of the region with different conductivity. In Fig. 9 the solution after 25 iterations and in Fig 10 – after 50 iterations are shown.

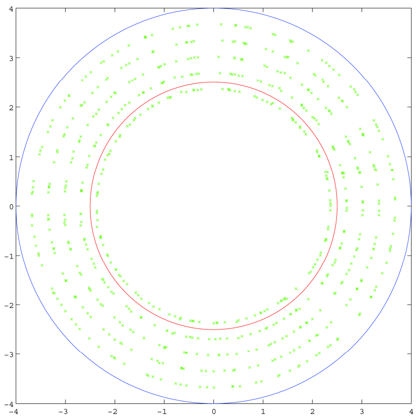


Fig. 9. Iteration process for central region with different conductivity

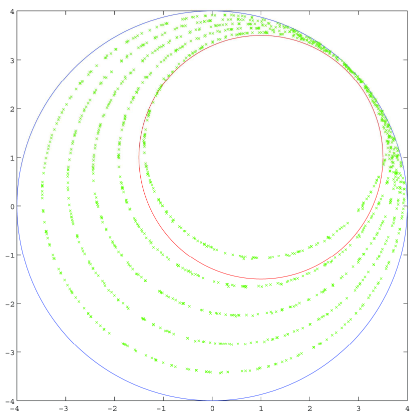


Fig. 10. Iteration process for region with different conductivity located in the near of outer boundary

## 5. Conclusions

The presented experiments indicated the efficiency of combination level set method with BEM to the solution of the forward and inverse problems. Level set formulation to describe the shapes of the domains combined with essentially nonoscilatory schemes to solve the Hamilton-Jacobi equation is efficient method for optimal shape design involving partial differential equation.

The proposed method can be used in the inverse problem solution of electromagnetic fields (e.g. image reconstruction in impedance or optical tomography).

In the further experiments we are going to use WENO (weighted essentially nonoscillatory polynomial interpolation) instead of ENO and solve Poisson equation by BEM.

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