

## LYAPUNOV EXPONENTS OF SYSTEMS WITH NOISE AND FLUCTUATING PARAMETERS

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This paper deals with the problem of determination of Lyapunov exponents in dynamical systems with noise or fluctuating parameters. The method for identifying the character of motion in such systems is proposed. This approach is based on the phenomenon of complete synchronization in double-oscillator systems via diagonal, master-slave coupling between them. The idea of effective Lyapunov exponents is introduced for quantifying the local stability in the presence of noise. Examples of the method application and its comparison with bifurcation diagrams representing the system dynamics are demonstrated. Finally, the properties of the method are discussed.

*Key words:* Lyapunov exponents, complete synchronization, systems with noise

### 1. Introduction

The fundamental tasks in studies of dynamical systems are the modelling and the analysis of their dynamics. The crucial problem here is to construct realistic models which are capable to reflect dynamics of a system observed during an experiment. However, we can often observe a quantitative or even qualitative difference between the dynamical behaviour of a real system and its numerical model. It happens, because not all dynamical effects appearing in a real system can be represented by strict mathematical formulas. The factors connected with the influence of environment or unstable conditions of the system work cause indefinite perturbations in its dynamical behaviour. Usually, such an unexpected disturbance of the system dynamics is modelled as a noise with some stochastic process or defined by a parameter mismatch.

One of the most sophisticated tools for identifying the character of motion of dynamical systems are Lyapunov exponents (LEs) (Benettin *et al.*, 1976; Lyapunov, 1947; Oseledec, 1968; Shimada and Nagashima, 1979; Wolf, 1986). These numbers should account for exponential convergence or divergence of trajectories that start close to each other. For practical applications, it is enough to know the largest Lyapunov exponent (LLE). If the LLE is positive, then the system is chaotic. A non-positive maximum number indicates regular system dynamics (periodic or quasi-periodic). The classical algorithms for calculating the spectrum of LEs (Benettin *et al.*, 1976, 1980a,b; Shimada and Nagashima, 1979; Wolf, 1986) have been developed on the basis of Oseledec's theorem (Oseledec, 1968), and they can be applied to the system given by continuous and differentiable ODEs. However, these approaches do not work for systems with discontinuities and in the case when exact equations of motion describing its dynamics are unknown, e.g. due to the presence of noise. Then, the estimation of LEs is not straightforward. In the recent years, several methods for calculation or estimation of LEs of non-smooth dynamical systems have been proposed (De Souza and Caldas, 2004; Hinrichs and Oestreich, 1997; Jin *et al.*, 2006; Müller, 1995; Oestreich *et al.*, 1996; Stefański, 2000, 2004; Stefański and Kapitaniak, 2000, 2003). One of them, elaborated by the author of this paper, is based on the synchronization phenomenon and it allows us to evaluate LLE of an arbitrary dynamical system (smooth or non-smooth). Previously, this approach has been successfully employed for dynamical systems with discontinuities (Stefański, 2000, 2004; Stefański and Kapitaniak, 2000), discrete-time maps (Stefański, 2004; Stefański and Kapitaniak, 2003) and systems with time delay (Stefański, 2004, Stefański *et al.*, 2005).

Here, a proposal how to implement this technique to mechanical systems with noise is presented. During numerical simulations, a classical Duffing oscillator has been considered. In all the above mentioned examples, the phenomenon of synchronization between the identical systems has been used to estimate the LLE. On the other hand, in the case under consideration the process of determining the LLE is based on the synchronization of two slightly different oscillators. Such a parameter mismatch models the influence of the noise. Hence, the term "effective Lyapunov exponents (ELE)" has been introduced in order to define the LLE in such cases. This idea has an innovative character in comparison with previous attempts of the application of the method.

In general, the paper is organized as follows. In Section 2, an issue of the synchronization method of the LLE estimation and the way of its application

in the case under consideration is revealed. The detailed model of the analyzed Duffing oscillator and results of its bifurcation analysis are demonstrated in Section 3. The case with a noise and its undisturbed equivalent are compared. In the last Section, brief conclusions are drawn.

## 2. The method of LLE estimation

The proposed method of LLE estimation exploits the phenomenon of the so called complete synchronization (CS) between two identical dynamical systems ( $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ ,  $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ , when separated), say, they are given with the same ODEs with identical system parameters. If some kind of linking between them is introduced (direct diffusive or inertial coupling, common external signal, etc), the CS, i.e., full coincidence of phases (frequencies) and amplitudes of their responses becomes possible. Then, for two arbitrarily chosen trajectories  $\mathbf{x}(t)$  and  $\mathbf{y}(t)$ , representing the coupled systems, we have

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = 0 \tag{2.1}$$

Whereas, in the case of slightly different coupled systems (the same ODEs with a small mismatch in the parameters), an imperfect complete synchronization (ICS) can be observed, i.e.

$$\lim_{t \rightarrow \infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| < \varepsilon \tag{2.2}$$

where  $\varepsilon$  is a small scalar quantity determining the ICS threshold. Thus, the ICS manifests in a non-ideal correlation of amplitudes and phases of the systems responses, while the synchronization error remains relatively small during the time evolution.

In order to carry out the estimation procedure, a double-oscillator system with a unidirectional, uniformly diagonal coupling (i.e. realized by all coordinates of the response oscillators with the same coupling strength) has to be constructed. Such a system is described as follows

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad \dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + d\mathbf{I}_n[\mathbf{x} - \mathbf{y}] \tag{2.3}$$

where  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  represent the master (reference trajectory) and slave (disturbed trajectory) system, respectively,  $\mathbf{I}_n$  is an  $n \times n$  unit matrix and  $d \in \mathbb{R}$  is a coefficient of coupling strength.

After introducing a new variable  $\mathbf{z} = \mathbf{y} - \mathbf{x}$ , which represents the synchronization error between both oscillators, and next subtracting Eq. (2.3)<sub>1</sub> from Eq. (2.3)<sub>2</sub>, the ODE describing time evolution of  $\mathbf{z}$  takes the following form

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{x} + \mathbf{z}) - \mathbf{f}(\mathbf{x}) - d\mathbf{1}_n \mathbf{z} \quad (2.4)$$

The variational form of Eq. (2.4) is

$$\dot{\boldsymbol{\zeta}} = (D\mathbf{f}[\mathbf{x}(t), \mathbf{x}_0] - d\mathbf{1}_n)\boldsymbol{\zeta} \quad (2.5)$$

where  $D\mathbf{f}[\mathbf{x}(t), \mathbf{x}_0]$  is the Jacobi matrix of the master system, (2.3)<sub>1</sub>, which is initializing from the generic initial conditions  $\mathbf{x}_0$ . On the basis of this matrix, the LEs of system (2.3)<sub>1</sub> can be calculated according to the classical formula (Benettin *et al.*, 1976; Kapitaniak and Wojewoda, 2000; Shimada and Nagashima, 1979)

$$\lambda_j = \lim_{t \rightarrow \infty} \frac{1}{t} \ln |r_j(t)| \quad (2.6)$$

where  $j = 1, 2, \dots, n$ . From Eq. (2.5) the following relation between the eigenvalues  $s_j(t)$  of the Jacobi matrix of the linearized synchronization error (Eq. (2.5)) defining the transverse stability of the synchronization manifold  $\mathbf{x} = \mathbf{y}$ , and the eigenvalues  $r_j(t)$  of the linearized Jacobi matrix  $D\mathbf{f}[\mathbf{x}(t), \mathbf{x}_0]$  of the reference system (2.3)<sub>1</sub> is given

$$s_j(t) = \exp(-dt)r_j(t) \quad (2.7)$$

On the basis of eigenvalues  $s_j(t)$ , Eq. (2.7), the transversal Lyapunov exponents (TLEs -  $\lambda_j^T$ ), quantifying the synchronizability of diagonally coupled oscillators, Eqs. (2.3), can be calculated in the following way

$$\lambda_j^T = \lim_{t \rightarrow \infty} \frac{1}{t} \ln |s_j(t)| = \lim_{t \rightarrow \infty} \frac{1}{t} \ln |r_j(t)| + \frac{1}{t} \ln \exp(-dt) \quad (2.8)$$

Hence,

$$\lambda_j^T = \lambda_j - d \quad (2.9)$$

The synchronous state is stable if all the TLEs are negative, so the largest one also has to fulfill the condition  $\lambda_1^T < 0$ . Thus, the condition of the complete synchronization of the reference (Eq. (2.3)<sub>1</sub>) and perturbed (Eq. (2.3)<sub>2</sub>) systems is provided by the inequality

$$d > \lambda_1 \quad (2.10)$$

where  $\lambda_1$  is the LLE of the reference system.

Thus, the diagonal coupling of identical systems (2.3) leads to the appearance (for a sufficiently small distance  $z$ ) of two mutually counteracting effects: exponential divergence or convergence (it depends on the sign and magnitude of  $\lambda_1$  and  $d$ ) of the reference  $\mathbf{x}(t)$  and disturbed  $\mathbf{y}(t)$  trajectories. This property of diagonal coupling causes a linear dependence between the LLE and the value of the coupling coefficient for which synchronization appears (see inequality (2.10)). Consequently, it can be used to determine the LLE via numerical investigations of the synchronization process. Such a direct approach works equally well in an arbitrary kind of a dynamical system (not only given by continuous ODEs), because it allows us to avoid the problem of defining the Jacobi matrices for some singular points on the system trajectory, e.g. during transition via discontinuity.

In fact, inequality (2.10) states that the smallest value of the coupling coefficient  $d$ , for which the synchronization occurs, is approximately equal to  $\lambda_1$ . Thus, in order to apply this method to numerical simulations, it is necessary to build an augmented system according to Eqs. (2.3). The next step is numerical search for the synchronous value of  $d$  approaching the largest LLE of the investigated system. The simplest way to evaluate the smallest synchronous value of the coupling  $d$  is to construct a bifurcation diagram of the synchronization error  $z$  versus  $d$ . The magnitude of the synchronization error can be computed according to the following formula

$$z = \|z\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \quad (2.11)$$

Then, the largest LLE can be estimated as a value of  $d$  where  $z$  approaches zero, i.e. when the CS takes place. However, such a way of LLE estimation is usually time-consuming.

Therefore, for the calculations presented in this paper, a method of fast search for the synchronous value of  $d$  has been applied. This method uses the following numerical procedures improving its effectiveness and speeding up the time of estimation:

- The application of the ICS (inequality (2.2)) also allows us to reduce the estimation time, because in practice it is enough to confirm that the synchronization state is asymptotically stable for a currently investigated value of  $d$ . For this purpose, for each iteration of  $d$ , a constant estimation period and transient period has to be assumed. If during the testing period the ICS condition (inequality (2.2)) is fulfilled for all the time, then the synchronization is recognized to be stable. In the opposite case, it is unstable.

- Elastic coupling, i.e. a dependence of the parameter  $d$  on the synchronization error  $z$ . If this error crosses the boundary value  $\varepsilon$  of the ICS threshold (inequality (2.2)), then a strong coupling ( $D \gg d$ ) is introduced between systems (2.3) in order to avoid a disadvantageous effect of long time transient motion before the appearance of synchronization. Consequently, it accelerates the CS process because the slave system (2.3)<sub>2</sub> is forced to evolve in the neighborhood of the master one (Eq. (2.3)<sub>1</sub>).

The application of the above described procedures in the numerical simulation allows us to estimate the LLE relatively fast with a satisfactory precision. Their more detailed description can be found in Stefański (2004), Stefański and Kapitaniak (2000, 2003).

Up to now, the presented method, and its variant for maps, has been applied to various cases of non-smooth oscillators (with impacts and friction), systems with time-delay and discrete maps (as mentioned in Introduction). Here, a small modification of the method in order to apply it effectively for the systems with noise is proposed. This is connected with a bit different dynamical nature of such systems in comparison with typical, unperturbed oscillators. Generally, noise can be treated as a relatively small but an unexpected disturbance of the input/output signal or the system parameters, which is of an unknown source and character. Thus, the classical algorithmic methods for calculating the LEs cannot be used here because such a system is non-differentiable. Normally, in a periodic system, the convergence of nearby trajectories leading to the CS should be observed. However, after introducing noise, there appears divergence of close orbits due to permanent disturbance caused by the noise, while the dynamics of the systems remains still regular. Obviously, the proposed method of the LLE estimation can be used for systems with noise exactly in the form described above. We can add the same stochastic component modelling the noise to Eqs. (2.3) and next simulate the estimation procedure. However, in such a case, the noise plays in fact the role of external drive only and, then, the CS of systems (2.3)<sub>1</sub> and (2.3)<sub>2</sub> is possible. This is, the author's opinion, in contradiction with the real nature of systems with noise, where the ideal convergence (say, the CS) of trajectories should not take place. Therefore, a mismatch between systems (2.3) is introduced by adding the noise component to the slave system (Eq. (2.3)<sub>2</sub>) only. Then, Eq. (2.3)<sub>2</sub> assumes the form

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}) + \mathbf{\Delta}(\mathbf{y}) + d\mathbf{I}_n[\mathbf{x} - \mathbf{y}] \quad (2.12)$$

where  $\Delta(\mathbf{y}) \in \mathbb{R}^n$  represents the *mismatch vector*. Applying the proposed method to slightly different systems (2.3)<sub>1</sub> and (2.12), we can estimate the magnitude of the trajectories divergence in the presence of noise (detecting their ICS – Eq.(2.2)), which is quantified with the coupling parameter approaching the practical value of the LLE. Such an *effective Lyapunov exponent* (ELE) can be treated as a measure of ICS robustness between the reference system and its replica disturbed by the noise. The most decisive factor for ELE evaluation is the ICS threshold  $\varepsilon$  assumed in numerical investigations. Hence, the ELE can be defined as follows:

The number called *the practical Lyapunov exponent*  $\lambda^E$  is equal to the minimum strength of the diagonal coupling coefficient  $d$  linking the reference and the disturbed (with noise) systems Eqs. (2.3)<sub>1</sub> and (2.12), which is required to maintain the synchronization error  $z$ , Eq. (2.11), between them in the specified  $\varepsilon$ -range.

The ELE has an analogical practical sense like the LLE, because it is also based on the definition of stability by Lyapunov (1947). Therefore, it has been named similarly, although the above informal definition of the ELE is not directly related to the basic definition of Lyapunov exponents (Oseledec, 1968).

### 3. Numerical example

In this Section, examples of determination of the LLE and the ELE by means of the proposed method are presented. The numerical experiments have been carried out by means of *dynamics* (Nusse and Yorke, 1997) and *delphi* (www.borland.com) software with the use of a mechanical oscillator of Duffing's type with the non-linear spring characteristic  $kx^2$ , linear viscous friction  $c$  and a harmonic excitation by a force of the amplitude  $F_0$  and frequency  $\Omega$  (see Fig. 1), in which the influence of noise is also considered. The dynamics of this oscillator is governed by the following non-autonomous, dimensionless equations of motion

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -\alpha x_1^3 - hx_2 + q \sin(\eta\tau) \quad (3.1)$$

where the parameters  $\alpha$ ,  $h$ ,  $q$ ,  $\eta$  and  $\tau$  are dimensionless representations of the real parameters  $k$ ,  $c$ ,  $F_0$ ,  $\Omega$  and time  $t$  (Fig. 1), respectively. In all numerical experiments presented here:  $\alpha = 10.0$ ,  $h = 0.3$  and  $q = 10.0$ . In the bifurcation analysis, the frequency of excitation  $\eta$  has been used as the control parameter.

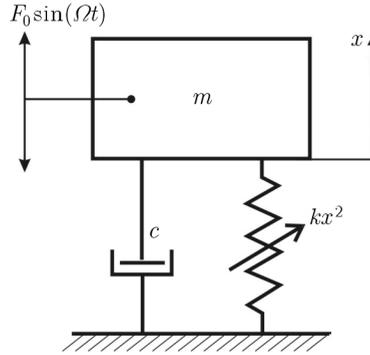


Fig. 1. Non-linear mechanical oscillator of Duffing's type

In order to compare the outcomes of LEs estimation with and without the noise, an auxiliary double system has been constructed in two variants:

(a) by substituting Eq. (3.1) to Eqs. (2.3) for estimation of the LLE of the oscillator without noise, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2 & \dot{x}_2 &= -\alpha x_1^3 - hx_2 + q \sin(\eta\tau) \\ \dot{y}_1 &= y_2 + d(x_1 - y_1) & \dot{y}_2 &= -\alpha y_1^3 - hy_2 + q \sin(\eta\tau) + d(x_2 - y_2) \end{aligned} \quad (3.2)$$

(b) by substituting Eq. (3.1) to Eqs. (2.3) in order to determine the ELE of the system disturbed with noise, i.e.

$$\begin{aligned} \dot{x}_1 &= x_2 & \dot{x}_2 &= -\alpha x_1^3 - hx_2 + q \sin(\eta\tau) \\ \dot{y}_1 &= y_2 + d(x_1 - y_1) & & \\ \dot{y}_2 &= -[\alpha + \Delta\alpha(t)]y_1^3 - hy_2 + q \sin(\eta\tau) + d(x_2 - y_2) \end{aligned} \quad (3.3)$$

where  $\Delta\alpha(t)$  represents the time-varying mismatch of the parameter  $\alpha$ . Hence, the related *mismatch vector* from Eq. (2.12) is  $\mathbf{\Delta}(\mathbf{y}) = [0, \Delta\alpha(t)y_1^3]^\top$ . As we can see, in the example under consideration, the influence of noise is manifested with the randomly fluctuating parameter  $\alpha$ . Such a fluctuation can be generated using numerical techniques modelling any stochastic process, e.g. Gaussian process by the spectral representation method (Shinozuka and Deodatis, 1992; Wiercigroch and Cheng, 1997). In the simulations carried out, a random number generator embedded in the *delphi* environment ([www.borland.com](http://www.borland.com)) has been applied. Time variations of the mismatch parameter are determined with the formula

$$\Delta\alpha(t) = 0.03\text{rand}[-1, 1] \quad (3.4)$$

where  $\text{rand}[-1, 1]$  is a stochastic function returning a random number uniformly distributed over the interval  $[-1, 1]$  in each step of numerical computations. These computations have been carried out employing the RK4 method with a fixed time step  $dt = 0.01$ . The assumed form of noise (Eq. (3.4)) means that the fluctuations of the parameter  $\alpha$  are restricted to the range  $\pm 3\%$  of its magnitude.

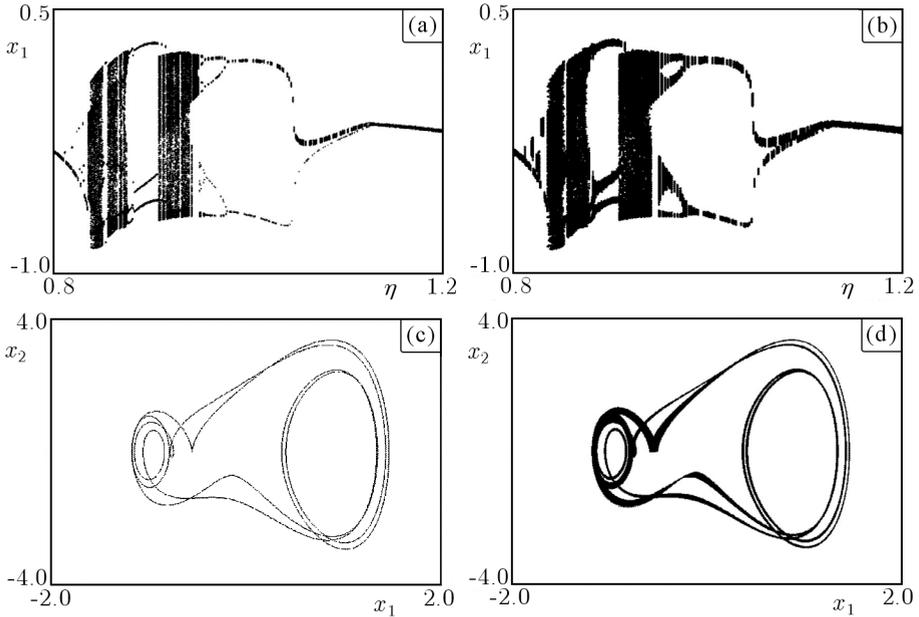


Fig. 2. Bifurcation diagrams of the Duffing oscillator (Eq. (3.1)) versus excitation frequency  $\eta$ : (a) undisturbed, (b) disturbed with Eq. (3.4), and exemplary phase portraits for  $\eta = 0.95$ : (c) without noise, (d) with noise

The results of typical bifurcation analysis (variable  $x_1$  versus control parameter  $\eta$ ) of the considered Duffing oscillator are presented in Fig. 2. We can see (Fig. 2a) that for smaller values of  $\eta$  chaotic motion appears as a result of period-doubling bifurcations,  $\eta \approx (0.83, 0.94)$ . In the middle of this region, a "periodic window" is observed  $\eta \approx (0.87, 0.91)$ . Next, with the increase of the frequency  $\eta$ , chaos disappears in a few sequences of the inverted period doubling bifurcations. Comparing the bifurcation diagrams depicted in Figs. 2a and 2b, i.e. computed for the perturbed and unperturbed versions of the oscillator (Eq. (3.1)), respectively, we can evaluate the influence of the fluctuating parameter  $\alpha$  on its dynamics. This influence is clearly visible especially in the intervals of regular motion (see Fig. 2b), where it is manifested with significantly thicker branches of the plot in parallel with the undisturbed case

(Fig. 2a). However, the disturbance does not introduce qualitative changes to the global view of the system dynamics (Fig. 2b), because the overall structure of the bifurcation plot is kept in spite of noise. The same effect can be observed in the accompanying phase portraits illustrating the dynamics of both cases of the system under consideration (Figs. 2c and 2d). Here, the noise-induced perturbation of the system trajectory is also evident (Fig. 2d).

In Fig. 3, bifurcation graphs of the synchronization error  $z$  which correspond to the diagrams in Figs. 2a and 2b are demonstrated. The intervals of desynchronous motion (large  $z$ ) reflect the chaotic ranges from Figs. 2a and 2b. In the case of identical master and slave systems (Eq. (3.2)), the distance  $z$  is equal to zero (Fig. 3a) in the range of regular motion due to asymptotical convergence of the periodic trajectories, i.e. the CS takes place there. Obviously, the CS is impossible for non-identical systems (Eq. (3.3)). However, we can see that the synchronization error remains relatively small when motion is regular (Fig. 3b), i.e., the ICS occurs. As it was mentioned above, the crucial parameter for quantifying the ICS and estimating the ELE is the ICS threshold  $\varepsilon$ . The bifurcation graph shown in Fig. 3b helps us to evaluate the boundary value of  $\varepsilon$ , for which the disturbed motion can be still recognized as the regular one. From simulations it results that the value of  $\varepsilon$  should be related to the magnitude of the mismatch, i.e. the ratio of  $\varepsilon$  and the maximum synchronization error  $\sup(z)$  should approximate the maximum variations of the mismatch. Thus, in the case under consideration, we have

$$\frac{\varepsilon}{\sup(z)} \approx \frac{\sup(\Delta\alpha(t))}{\alpha} \quad (3.5)$$

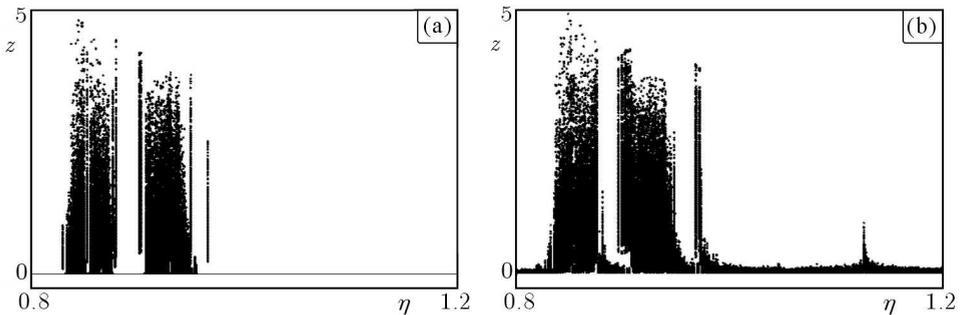


Fig. 3. Bifurcation diagrams of the synchronization error  $z$  corresponding to Figs. 2a and 2b, respectively, computed for (a) identical systems (Eq. (3.2)) and (b) systems with mismatch (Eq. (3.3))

In the given example, the maximum variation of the parameter  $\alpha$  amounts 6% percent of its value (between 97% and 103% – see Eq. (3.4)). Therefore, it

has been taken  $\varepsilon = 0.25$  in the numerical experiment, i.e. about 5-6% of the maximum synchronization error approaching value 5 (see Fig. 3).

The plots of LLE (in grey) and ELE (in black) corresponding to Figs. 2a and 2b, respectively, are depicted in Fig. 4. They were determined using the proposed method on the basis of Eqs (3.2) (the LLE) and (3.3) (the ELE). It is observable that in the chaotic ranges of bifurcation diagrams (Figs. 2a and 2b), the positive LLE and ELE are detected and vice versa. The intervals of the positive ELE are wider than the analogous range of the LLE and additional fields, where the ELE is larger than zero, appear in the neighborhood of the control parameter values corresponding to period-doubling bifurcation. The comparison of LLE and ELE plots allows us to evaluate the qualitative and quantitative influence of noise on the system dynamics.

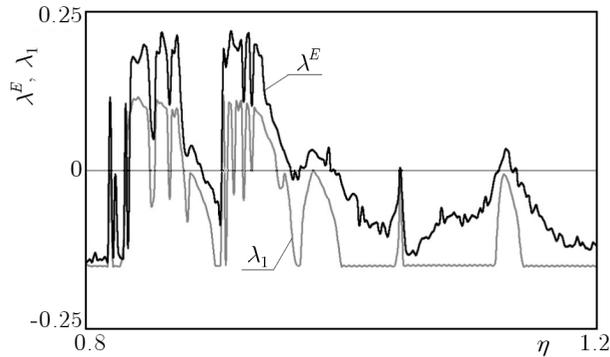


Fig. 4. Bifurcation plots of LLE (in grey) found from Eq. (3.2) and ELE (in black) found from Eq. (3.3) for the ICS threshold  $\varepsilon = 0.25$  which correspond to Figs. 2a and 2b, respectively

#### 4. Conclusions

In this paper, the synchronization based approach for determination of the dominant Lyapunov exponent of the systems with noise and fluctuating parameters has been discussed. This technique can be treated as a slight development of its existing version, which is based on the properties of the CS via a diagonal coupling of two identical systems (Eqs. (2.3)). The method has been modified by introducing a small mismatch between them, which models the influence of noise on the system dynamics. This change can be considered as an innovative consequence of this paper. After such a modification, the esti-

mation procedure is based on numerical investigations of the ICS process of slightly different oscillators.

Therefore, a new instrument called the *effective Lyapunov exponent* (ELE), quantifying the local stability of the system responses in the presence of noise, has been introduced. For the precision of the ELE estimation, the proper choice of the ICS threshold  $\varepsilon$  determining the limit of stability of the disturbed solution plays the decisive role. Its connection to the magnitude of the mismatch in practical applications is proposed in Sec. 3 (see Eq. (3.5)). The presented examples of LLE and ELE estimation (Sec. 3) show that this method allows us to quantify the dynamical character of the disturbed oscillator response. This fact is confirmed by the coincidence of the estimated ELE (Fig. 4) with the corresponding bifurcation diagram (Fig. 2b).

However, at the end of this paper it should be pointed out that particular conclusions refer to the case when the amplitude of noise oscillations is uniformly distributed over the range  $[-1, 1]$ , i.e. it is restricted with a probability equal to 1. Thus, there is no possibility to observe a sudden jump of the noise amplitude.

To summarize, an important practical advantage of this approach, in comparison to other known algorithmic methods, is its universality, i.e. it works equally well both for time-continuous, undisturbed, disturbed and non-smooth systems. The applications of the proposed method in various kinds of dynamical systems will be reported soon.

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## References

1. BENETTIN G., GALGANI L., STRELCYN J.M., 1976, Kolmogorov entropy and numerical experiment, *Physical Review A*, **14**, 2338-2345
2. BENETTIN G., GALGANI L., GIORGILLI A., STRELCYN J.M., 1980a, Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, Part I: Theory, *Meccanica*, **15**, 9-20
3. BENETTIN G., GALGANI L., GIORGILLI A., STRELCYN J.M., 1980b, Lyapunov exponents for smooth dynamical systems and Hamiltonian systems; a method for computing all of them, Part II: Numerical application, *Meccanica*, **15**, 21-30

4. DE SOUZA S.L.T., CALDAS I.L., 2004, Calculation of Lyapunov exponents in systems with impacts, *Chaos, Solitons and Fractals*, **19**, 3, 569-579
5. HINRICHS N., OESTREICH M., POPP K., 1997, Dynamics of oscillators with impact and friction, *Chaos, Solitons and Fractals*, **4**, 8, 535-558
6. JIN L., LU Q.-S., TWIZELL E.H., 2006, A method for calculating the spectrum of Lyapunov exponents by local maps in non-smooth impact-vibrating systems, *Journal of Sound and Vibration*, **298**, 4/5, 1019-1033
7. KAPITANIAK T., WOJEWODA J., 2000, *Bifurkacje i chaos*, Politechnika Łódzka, Wydawnictwo Naukowe PWN, Warszawa-Łódź
8. LYAPUNOV A.M., 1947, Probleme general de la stabilite du mouvement, *Annales Mathematical Study*, **17**
9. MÜLLER P., 1995, Calculation of Lyapunov exponents for dynamical systems with discontinuities, *Chaos, Solitons and Fractals*, **5**, 9, 1671-1681
10. NUSSE H.E., YORKE J.A., 1997, *Dynamics: Numerical Explorations*, Springer-Verlag, New York
11. OESTREICH M., HINRICHS N., POPP K., 1996, Bifurcation and stability analysis for a non-smooth friction oscillator, *Archive of Applied Mechanics*, **66**, 301-314
12. OSELEDEC V.I., 1968, A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems, *Trans. Moscow Math. Soc.*, **19**, 197-231
13. SHIMADA I., NAGASHIMA T., 1979, A numerical approach to ergodic problem of dissipative dynamical systems, *Progress of Theoretical Physics*, **61**, 6, 1605-1616
14. SHINOZUKA M., DEODATIS G., 1992, Simulation of stochastic processes by spectral representation, *Applied Mechanics Review*, **44**, 191-203
15. STEFAŃSKI A., KAPITANIAK T., 2000, Using chaos synchronization to estimate the largest Lyapunov exponent of non-smooth systems, *Discrete Dynamics in Nature and Society*, **4**, 207-215
16. STEFAŃSKI A., 2000, Estimation of the largest Lyapunov exponent in systems with impacts, *Chaos, Solitons and Fractals*, **11**, 15, 2443-2451
17. STEFAŃSKI A., KAPITANIAK T., 2003, Estimation of the dominant Lyapunov exponent of non-smooth systems on the basis of maps synchronization, *Chaos, Solitons and Fractals*, **15**, 233-244
18. STEFAŃSKI A., 2004, Estymacja maksymalnego wykładnika Lapunowa układów dynamicznych w oparciu o zjawisko synchronizacji, *Zeszyty Naukowe Politechniki Łódzkiej*, **941**, *Rozprawy Naukowe*, **333**, str. 114, Łódź

19. STEFAŃSKI A., DĄBROWSKI A., KAPITANIAK T., 2005, Evaluation of the largest Lyapunov exponent in dynamical systems with time delay, *Chaos, Solitons and Fractals*, **23**, 1651-1659
20. WIERCIGROCH M., CHENG A.D.-H., 1997, Chaotic and stochastic dynamics of metal cutting process, In: *Multi-Body Dynamics: Monitoring and Simulation Techniques*, Rahnejat H. and Whalley R. (Edit.), MEP, London, 351-366
21. WOLF A., 1986, Quantifying chaos with Lyapunov exponents, In: *Chaos*, V. Holden (Edit.), Manchester University Press, Manchester, 273-290
22. [www.borland.com](http://www.borland.com)

## Wykładniki Lapunowa układów dynamicznych z szumem i oscylującymi parametrami

### Streszczenie

Niniejszy artykuł dotyczy problemu wyznaczania wykładników Lapunowa układów dynamicznych z szumem lub zmiennymi w czasie parametrami. Zawiera on propozycję nowej metody identyfikacji charakteru ruchu tych układów, która wykorzystuje zjawisko synchronizacji kompletnej dwóch oscylatorów połączonych jednokierunkowym diagonalnym sprzężeniem. Istotą proponowanej metody są tzw. efektywne wykładniki Lapunowa, które są miarą lokalnej stateczności układu dynamicznego w obecności szumu lub zaburzenia. W artykule przedstawiono zastosowanie efektywnych wykładników Lapunowa na przykładzie zaburzonego oscylatora typu Duffinga w zestawieniu z jego analizą bifurkacyjną. We wnioskach zawarto dyskusję własności proponowanej metody.

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