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FREE AND FORCED OSCILLATIONS OF TIMOSHENKO BEAM MADE OF VISCOELASTIC MATERIAL

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Dynamics of Timoshenko's beam made of a viscoelastic material is studied. Dimensionless equations of motion are obtained, depending only on two parameters, one of which relates to the shear flexibility and the second – to the viscous internal friction. Advantages of the proposed equations are illustrated by solutions to the free and forced oscillation problems for the simplest case of hinged-hinged beams. The influence of the beam shear flexibility and viscous internal friction on the natural frequencies and the dynamic amplification factor is studied.

Key words: Timoshenko beam, viscoelasticity, oscilation, internal friction

1. Introduction

It is well known that oscillation of structures, in particular, beams, in the vicinity of resonances may be correctly described only with account of the internal friction. This means that the elastic model of the material is insufficient, and its viscoelastic properties should be taken into account.

For the classical beam model such solutions for beams made of viscoelastic materials have been obtained in many known works, see e.g., Panovko (1960), Filippov (1956).

For the Timoshenko model of a beam, which is necessary for shear-deformable beams (short beams, composite beams), the known solutions mostly relate to an elastic material (see References and others).

The aim of this paper is to provide the analysis of transverse oscillations of Timoshenko beam made of a viscoelastic material that obeys the Voigt law.

We would like to show that the use of the proposed generalized dimensionless equation, which depends only on two parameters, considerably facilitates the general analysis of the beam dynamics.

2. Governing equations

2.1. Equations of motion

Equations of motion are derived using known hypotheses. Deformations of the beam are described by two independent functions – the angle of the cross section rotation ψ and the shear angle (at the neutral axis) γ, see Fig. 1.

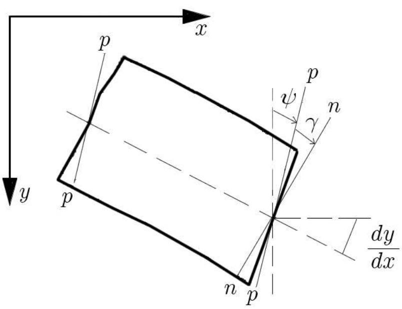

Fig. 1. Element of the beam and scheme of deformation of the beam flat cross section; n-n is the normal plane (to the bent axis), p-p is the tangent plane to the deformed cross section (at the neutral axis)

The total slope of the bent axis is

$$\frac{\partial y}{\partial x} = \psi + \gamma \tag{2.1}$$

where $y(x,t)$ is the transverse displacement. The longitudinal displacement of a point at a distance z from the neutral axis and the longitudinal deformations are expressed through the angle ψ: $u = -z\psi$, $\varepsilon_x = -z\partial\psi/\partial x$.

Constitutive relations are assumed according to the Voigt law for normal stresses as well as for shear ones in the form

$$\begin{aligned} \sigma_x &= E\varepsilon_x + k_1\frac{\partial \varepsilon_x}{\partial t} = E\Big(1 + \mu_1\frac{\partial}{\partial t}\Big)\varepsilon_x \\ \tau &= G\gamma + k_2\frac{\partial \gamma}{\partial t} = G\Big(1 + \mu_2\frac{\partial}{\partial t}\Big)\gamma \end{aligned} \tag{2.2}$$

(the stresses depend not only on deformations but also on the velocity of deformations), here $k_{1,2}$ and $\mu_1 = k_1/E$, $\mu_2 = k_2/G$ are the viscosity parameters. Further we assume $\mu_1 = \mu_2 = \mu$.

Then the bending moment and the transverse shear force in the cross section are specified by the known expressions

$$\begin{aligned} M &= -EJ\Big(1+\mu\frac{\partial}{\partial t}\Big)\frac{\partial\psi}{\partial x} \\ Q &= k'A\tau = k'AG\Big(1+\mu\frac{\partial}{\partial t}\Big)\Big(\frac{\partial y}{\partial x}-\psi\Big) \end{aligned} \tag{2.3}$$

where k' is the coefficient which depends upon the cross section shape, see e.g., Timoshenko (1955), A and J are the cross section area and the moment of inertia, E and G are moduli of elasticity in tension and shear, respectively.

The equations of balance of forces for the beam loaded by a distributed load $q_0(x,t)$ with account of the rotary inertia are as follows

$$\frac{\partial Q}{\partial x} - \rho A\frac{\partial^2 y}{\partial t^2} + q_0(x,t) = 0 \qquad -\rho J\frac{\partial^2\psi}{\partial t^2} + Q - \frac{\partial M}{\partial x} = 0 \tag{2.4}$$

These equations with regard to the above relations result in two differential equations of motion in y and ψ

$$\begin{aligned} &k'GA\Big(1+\mu\frac{\partial}{\partial t}\Big)\frac{\partial}{\partial x}\Big(\frac{\partial y}{\partial x}-\psi\Big) - \rho A\frac{\partial^2 y}{\partial t^2} + q_0(x,t) = 0 \\ &EJ\Big(1+\mu\frac{\partial}{\partial t}\Big)\frac{\partial^3\psi}{\partial x^3} - \rho J\frac{\partial^3\psi}{\partial x\partial t^2} + \rho A\frac{\partial^2 y}{\partial t^2} - q_0(x,t) = 0 \end{aligned} \tag{2.5}$$

After excluding the angle ψ, a single equation with respect to the displacement $y(x,t)$ is obtained

$$\begin{aligned} &EJ\Big(1+\mu\frac{\partial}{\partial t}\Big)^2\frac{\partial^4 y}{\partial x^4} - \rho J\Big(1+\frac{E}{k'G}\Big)\Big(1+\mu\frac{\partial}{\partial t}\Big)\frac{\partial^4 y}{\partial x^2\partial t^2} + \frac{\rho^2 J}{k'G}\frac{\partial^4 y}{\partial t^4} + \\ &+\rho A\Big(1+\mu\frac{\partial}{\partial t}\Big)\frac{\partial^2 y}{\partial t^2} = \Big(1+\mu\frac{\partial}{\partial t}\Big)q_0 + \frac{\rho J}{k'GA}\frac{\partial^2 q_0}{\partial t^2} - \frac{EJ}{k'GA}\Big(1+\mu\frac{\partial}{\partial t}\Big)\frac{\partial^2 q_0}{\partial x^2} \end{aligned} \tag{2.6}$$

In particular cases of the Euler-Bernoulli (E-B) viscoelastic beam and Timoshenko elastic beam, this equation is reduced to the well known equations.

The boundary conditions for set (2.5) in variables y and ψ can be derived by making use of Hamilton's principle and are given, e.g., by Anderson (1953), Dolph (1954), Trail-Nash and Collar (1953). In particular, for the hinged end, one immediately gets from (2.3)

$$y = 0 \qquad \frac{\partial\psi}{\partial x} = 0 \tag{2.7}$$

2.2. Dimensionless equations

The obtained governing equation, (2.6), is not convenient for the analysis since it includes many parameters. In order to facilitate the general analysis let us introduce dimensionless variables and parameters

$$\begin{array}{lll} \xi = \dfrac{x}{r_0} & Y = \dfrac{y}{r_0} & \tau = \dfrac{c}{r_0}t \\ c^2 = \dfrac{E}{\rho} & r_0^2 = \dfrac{J}{A} & \chi = \dfrac{E}{k'G} \\ \mu^* = \dfrac{c}{r_0}\mu & q = \dfrac{q_0 r_0}{EA} & \end{array} \tag{2.8}$$

Here c is the sound velocity in the beam material, r_0 is the cross section radius of gyration, χ is the shear deformability parameter, μ^* is the dimensionless viscosity parameter. Note that for the classical Euler-Bernoulli model $\chi = 0$, which corresponds to an infinitely large shear stiffness.

In variables (2.8), equations (2.5) take the form

$$\begin{aligned} &\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial}{\partial\xi}\Big(\frac{\partial Y}{\partial\xi}-\psi\Big)-\chi\frac{\partial^2 Y}{\partial\tau^2}+\chi q(\xi,\tau)=0 \\ &\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^3\psi}{\partial\xi^3}-\frac{\partial^3\psi}{\partial\xi\partial\tau^2}+\frac{\partial^2 Y}{\partial\tau^2}-q(\xi,\tau)=0 \end{aligned} \tag{2.9}$$

and equation (2.7) transforms into

$$\begin{aligned} &\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)^2\frac{\partial^4 Y}{\partial\xi^4}-(1+\chi)\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^4 Y}{\partial\xi^2\partial\tau^2}+\chi\frac{\partial^4 Y}{\partial\tau^4}+ \\ &+\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2 Y}{\partial\tau^2}=\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)q+\chi\frac{\partial^2 q}{\partial\tau^2}-\chi\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2 q}{\partial\xi^2} \end{aligned} \tag{2.10}$$

This equation includes only two generalized parameters characterising the shear deformability and the viscosity, respectively. For a particular case of E-B viscoelastic beam ($\chi = 0$) with the rotational inertia (Rayleigh's model) and internal friction included, this equation reduces to the following one

$$\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^4 Y}{\partial\xi^4}-\frac{\partial^4 Y}{\partial\xi^2\partial\tau^2}+\frac{\partial^2 Y}{\partial\tau^2}=q \tag{2.11}$$

with the single parameter μ^*, and for the Timoshenko beam made of an elastic material ($\mu^* = 0$) to equation

$$\frac{\partial^4 Y}{\partial\xi^4}-(1+\chi)\frac{\partial^4 Y}{\partial\xi^2\partial\tau^2}+\frac{\partial^2 Y}{\partial\tau^2}+\chi\frac{\partial^4 Y}{\partial\tau^4}=q+\chi\frac{\partial^2 q}{\partial\tau^2}-\chi\frac{\partial^2 q}{\partial\xi^2} \tag{2.12}$$

with the single parameter χ. These equations are apparently preferable in comparison with often used dimensionless equations with several parameters.

The angle ψ can be expressed through Y using equation $(2.9)_2$. For the derivative $\psi_\xi \equiv \partial\psi/\partial\xi = r^0\partial\psi/\partial x$ (which enters in boundary conditions (2.7)) one has a relationship

$$\frac{\partial^2 Y}{\partial\tau^2} = \frac{\partial^2\psi_\xi}{\partial\tau^2} - \Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2\psi_\xi}{\partial\xi^2} + q(\xi,\tau) \tag{2.13}$$

The shear angle $\gamma = (\partial y/\partial x) - \psi = (\partial Y/\partial\xi) - \psi$ and its derivative γ_ξ is expressed through dimensionless variables Y and ψ_ξ

$$\gamma_\xi \equiv \frac{\partial\gamma}{\partial\xi} = \frac{\partial^2 Y}{\partial\xi^2} - \psi_\xi \tag{2.14}$$

Boundary conditions (2.7) in dimensionless variables take the form

$$Y = 0 \qquad\qquad \psi_\xi = 0 \tag{2.15}$$

One can also obtain an equation for ψ, or (more convenient) equation for ψ_ξ. Excluding Y from set (2.9) leads to an equation with the same operator in the left hand side but differing in the right hand side

$$\begin{gathered}\Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)^2\frac{\partial^4\psi_\xi}{\partial\xi^4} - (1+\chi)\Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^4\psi_\xi}{\partial\xi^2\partial\tau^2} + \chi\frac{\partial^4\psi_\xi}{\partial\tau^4} + \\ + \Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2\psi_\xi}{\partial\tau^2} = \Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2 q}{\partial\xi^2}\end{gathered} \tag{2.16}$$

Combining equations (2.9) and (2.14) results in the equation for the shear angle or for the derivative γ_ξ (again with the same left hand side)

$$\begin{gathered}\Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)^2\frac{\partial^4\gamma_\xi}{\partial\xi^4} - (1+\chi)\Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^4\gamma_\xi}{\partial\xi^2\partial\tau^2} + \chi\frac{\partial^4\gamma_\xi}{\partial\tau^4} + \\ + \Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2\gamma_\xi}{\partial\tau^2} = \chi\frac{\partial^4 q}{\partial\tau^2\partial\xi^2} - \chi\Big(1 + \mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^4 q}{\partial\xi^4}\end{gathered} \tag{2.17}$$

3. Free oscillations of a hinged-hinged beam

3.1. Solution

For free oscillations $q=0$ in (2.10), one arrives at the equation

$$\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)^2\frac{\partial^4 Y}{\partial\xi^4}-(1+\chi)\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^4 Y}{\partial\xi^2\partial\tau^2}+\Big(1+\mu^*\frac{\partial}{\partial\tau}\Big)\frac{\partial^2 Y}{\partial\tau^2}+\chi\frac{\partial^4 Y}{\partial\tau^4}=0 \tag{3.1}$$

and an identical one for ψ_ξ (from (2.16)). Here only the simplest case of a hinged-hinged beam is considered, for which the solution is sought in the form

$$Y(\xi,\tau)=\mathrm{e}^{\mathrm{i}\omega\tau}\sin k\xi \tag{3.2}$$

The parameter k is determined from boundary conditions (2.15): $k_n=n\pi r_0/l$, $(n=1,2,\ldots)$. Substitution (3.2) into (3.1) gives a frequency equation for ω_n with complex coefficients

$$(1+\mathrm{i}\omega_n\mu^*)^2k_n^4-\omega_n^2(1+\chi)(1+\mathrm{i}\omega_n\mu^*)k_n^2-(1+\mathrm{i}\omega_n\mu^*-\chi\omega_n^2)\omega_n^2=0 \tag{3.3}$$

So the natural frequency is a complex quantity. After introducing the denotation

$$Z_n=\frac{\omega_n^2}{1+\mathrm{i}\omega_n\mu^*} \tag{3.4}$$

equation (3.3) reduces to an equation for Z_n with real coefficients

$$\chi Z_n^2-Z_n[1+(1+\chi)k_n^2]+k_n^4=0 \tag{3.5}$$

whence two roots Z_n are

$$Z_{n,1,2}=\frac{1}{2\chi}\Big[(1+k_n^2+\chi k_n^2)\mp\sqrt{(1+k_n^2+\chi k_n^2)^2-4\chi k_n^4}\Big] \tag{3.6}$$

It is easily seen that both $Z_{n,1,2}$ values are real and positive. Then for each $Z_{n,1,2}$ value one gets from (3.4) the following equation specifying the natural frequencies ω_n

$$\omega_n^2-\mathrm{i}\omega_n\mu^*Z_{n,1,2}-Z_{n,1,2}=0 \tag{3.7}$$

with a pair of complex natural frequencies which differ with sign of the real parts

$$\omega_{n,1,2}=\alpha_{n,1,2}+\mathrm{i}\beta_{n,1,2} \qquad\qquad \omega_{n,3,4}=-\alpha_{n,1,2}+\mathrm{i}\beta_{n,1,2} \tag{3.8}$$

where

$$\alpha_{n,1,2}=\sqrt{Z_{n,1,2}-\frac{(\mu^*Z_{n,1,2})^2}{4}} \qquad\qquad \beta_{n,1,2}=\frac{\mu^*Z_{n,1,2}}{2} \tag{3.9}$$

Each complex frequency corresponds to a pair of particular solutions, for any n

$$Y_{nj}(\xi,\tau) = Y_{0,nj}\mathrm{e}^{-\beta_{nj}\tau}\mathrm{e}^{\mathrm{i}(\alpha_{n\,j}\tau+\theta_{nj})}\sin k_n\xi \qquad j=1,2 \tag{3.10}$$

or, in the real form

$$Y_{nj}(\xi,\tau) = Y_{0,nj}\mathrm{e}^{-\beta_{nj}\tau}\cos(\alpha_{nj}\tau+\theta_{nj})\sin k_n\xi \qquad j=1,2 \tag{3.11}$$

with constants $Y_{0,nj}$ and θ_{nj} (amplitude and initial phase of the particular solution). They present decaying oscillations. The real parts of the complex eigenfrequencies give the cyclic frequencies α_{n1} and α_{n2}, the imaginary parts give the damping factors β_{n1} and β_{n2}.

The general solution is a linear combination of the particular solutions for all n with arbitrary constants.

The above formulas determine frequencies and damping of oscillations in the dimensionless variables ξ,τ. These quantities in the initial variables can be obtained with account of (2.8). Denoting with lower indexes x,t the quantities computed in the initial variables, one has instead of (3.2)

$$y(x,t) = y_0\mathrm{e}^{\mathrm{i}\omega_{xt}t}\sin k_{xt}x$$

As $k_{xt}x = k_{xt}r_0\xi$, $\omega_{xt}t = \omega_{xt}r_0\tau/c$ we get: $k_{xt}r_0 = k$, $\omega_{xt}r_0/c = \omega$. Hence, having the dependence $\omega = f(k)$, one obtains the corresponding relationship in the initial variables in the form

$$\omega_{xt} = \frac{c\omega}{r_0} = \frac{c}{r_0}f(k_{xt}r_0) \tag{3.12}$$

(for the real and imaginary parts of ω_n, i.e. for $\alpha_{n1,2}$ and $\beta_{n1,2}$, the translation formulas are similar).

3.2. Analysis of the solution for the elastic Timoshenko beam

Consider first the case of a Timoshenko beam made of an elastic material. The free oscillation problem for the elastic Timoshenko beam was studied in numerous works, but our aim here is to demonstrate merits of the proposed equations in performing general analysis.

For $\mu^* = 0$, $\chi \neq 0$ from (3.4) one has $Z_n = \omega_n^2$. As both roots $Z_{n,1,2}$ (3.6) are real and positive, the eigenfrequencies $\omega_{n,1,2}$ are real, $\beta_{n1,2} = 0$, $\omega_{n,1,2} = \pm\alpha_{n1,2} = \pm\sqrt{Z_{n,1,2}}$.

Let us consider first *approximate formulas and asymptotics* for the natural frequencies. Let us rewrite $Z_n = \omega_n^2$ (3.6) as follows

$$\omega_{n,1,2}^2 = \frac{1 + k_n^2 + \chi k_n^2}{2\chi}\Big[1 \mp \sqrt{1 - \frac{4\chi k_n^4}{(1 + k_n^2 + \chi k_n^2)^2}}\Big] \tag{3.13}$$

Elementary analysis shows that $4\chi k_n^4 < (1 + k_n^2 + \chi k_n^2)^2$ for any k_n and χ. So one may expand the expression under the root in (3.13) into power series. Keeping only two terms, one obtains

$$\omega_{n,1,2}^2 \approx \frac{1 + k_n^2 + \chi k_n^2}{2\chi}\Big[1 \mp \Big(1 - \frac{2\chi k_n^4}{(1 + k_n^2 + \chi k_n^2)^2}\Big)\Big] \tag{3.14}$$

So, approximately, the first branch of the solution, corresponding to sign "−", is

$$\omega_{n,1}^2 \approx \frac{k_n^4}{1 + k_n^2(1 + \chi)} \tag{3.15}$$

and the second branch (sign "+")

$$\omega_{n,2}^2 \approx \frac{1 + k_n^2 + \chi k_n^2}{\chi}\Big(1 - \frac{\chi k_n^4}{(1 + k_n^2 + \chi k_n^2)^2}\Big) \tag{3.16}$$

For qualitative considerations, the last expression may be further simplified

$$\omega_{n,2}^2 \approx \frac{1 + k_n^2 + \chi k_n^2}{\chi} \tag{3.17}$$

In the plane (k_n, ω_n), expression (3.17) determines a hyperbola (with parameter χ). The above formulas lead to the following asymptotic expressions for cases $k_n \to 0$ and $k_n \to \infty$. The first branch for small k_n is the parabola (from (3.15))

$$\omega_{n,1} \approx k_n^2 \tag{3.18}$$

and for large k_n it is a straight line

$$\omega_{n,1} \approx \frac{k_n}{\sqrt{1 + \chi}} \tag{3.19}$$

The second branch for small k_n is close to $\omega_{n,2}^2 \approx 1/\chi$, and for large k_n is the straight line

$$\omega_{n,2} \approx k_n\sqrt{\frac{1 + \chi}{\chi}} \tag{3.20}$$

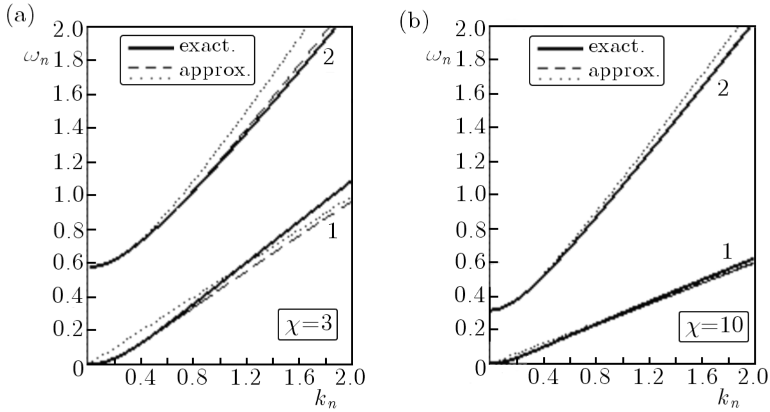

Fig. 2. Natural frequency vs. wave number k_n for the elastic Timoshenko beam at $\chi = 3$ (a) and $\chi = 10$ (b). 1, 2 – the first and second frequency spectrum, solid curves – exact solution, dashed and dotted lines – approximate solutions by (3.15), (3.16) and (3.19), (3.20), respectively

In Fig. 2, ω_n-k_n relationships are presented for $\chi = 3$ and $\chi = 10$, Fig. 2a and Fig. 2b, respectively (value $\chi = 3$ corresponds approximately to an isotropic material, see expression for χ (2.8)). Two the branches are given (curves 1, 2, respectively), solid curves – exact predictions by (3.13), dashed curves – approximate values by (3.15) and (3.16), and dotted lines – values by approximations (3.19) and (3.20), for the first and second branches, respectively. The error of the approximate formulas is rather small for $\chi = 3$ and practically disappears for $\chi = 10$ in the whole k_n range considered. For larger χ values, simple formulas (3.15) and (3.17) give practically exact predictions for the both frequency branches at any k_n.

Note that parabola (3.18) is the exact solution to the E-B beam. When taking into account the rotatory inertia, but disregarding the shear flexibility (Rayleigh model, $\chi = 0$), the first branch, (3.15), reduces to

$$\omega_{n,1}^2 = \frac{k_n^4}{1 + k_n^2}$$

For the first branch both these relationships give the correct asymptotics for small k_n (large wavelength), but not for large k_n (short wavelength). Note that the second branch is absent in the E-B model as well as in the Rayleigh model. For real materials, one may put $\chi > 3$, so neglecting χ in relationship (3.19) is not acceptable.

In Fig. 3a, natural frequencies vs. wave number $k_n = n\pi r_0/l$ are presented for three values of the shear parameter, solid curves – the first frequency branch, dotted curves – the second branch (all curves computed by exact formulas (3.13)).

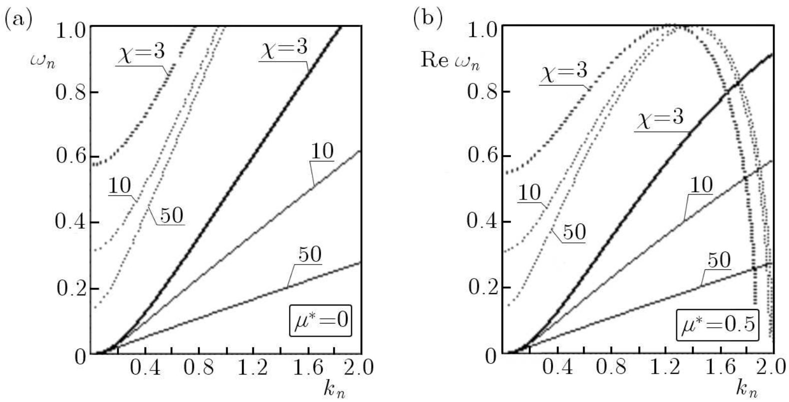

Fig. 3. Natural frequency vs. wave number k_n for the Timoshenko beam at three values of shear parameter χ (a) elastic material; (b) visco-elastic material (solid curves – the first branch, dotted curves – the second one)

If $\chi \to 0$ then the second branch goes to infinity, therefore this branch is absent in the E-B and Rayleigh models. The second branch is always higher than the first one.

Now it is worthwhile to discuss the *physical sense of the second branch.* The existence of two values of natural frequency, $\omega_{n,1}$ and $\omega_{n,2}$, which relate to the same n value (the same wavelength) was noted already in the first investigations on dynamics of Timoshenko beams (Anderson, 1953; Trail-Nash and Collar, 1953; Uflyand, 1948), but the meaning of the "second spectrum" of eigenfrequencies has attracted particular attention of investigators only later, in the 70-s, and hitherto remains a topic of debate (Abbas and Thomas, 1977; Bhashyam and Prathap, 1981; Ekwaro-Osire *et al.*, 2001; Levinson and Cooke, 1982; Nesterenko, 1993; Prathap, 1983; Stephen, 1982, 2006; Stephen and Puchegger, 2006). Some of the investigators adhere to the opinion that "the second spectrum predictions of Timoshenko beam theory should be disregarded" (Stephen, 2006). But, in our opinion, the physical nature of the second branch has been brought to light already in the papers by Dolph (1954), Downs (1976), Huang (1961). It was there established that for the first branch the transverse deflections due to bending and shear are of the same phase and are summed to give the total transverse displacement. For the second branch, the bending deflection and the shear one are opposite in phase, and the total transverse displacement equals to their difference (these features can be ascertained on the base of the above solution and equations presented in p. 2.1).

In particular, Downs (1976) detected that the second branch for long waves includes a "shear mode" with vanishing total transverse deflection, and this mode has been also obtained from equations of the theory of elasticity.

3.3. Analysis of the solution for the viscoelastic Timoshenko beam

Let us consider now the general case of the viscoelastic Timoshenko beam. Results of computations by (3.6), (3.9) are presented in Fig. 3b. Dependencies of the natural frequency on the wave number k_n (at three values of the shear flexibility parameter χ) for the viscosity parameter $\mu^* = 0.5$ show that the influence of this parameter is essential for the second branch and exhibits itself mostly at k_n being of the order of unity (or larger). In distinction to the elastic material, when the dimensionless natural frequency monotonically increases with k_n, in the case of a viscoelastic material the natural frequency becomes to decrease at large k_n, vanishing at k_n close to 2. This means that modes of the second type are impossible to appear for very short wavelengths. For oscillations of the first type (prevailing bending modes), the influence of μ^* is relatively weaker.

So the conclusion may be drawn that *the internal friction eliminates the second branch for sufficiently short wavelengths, and there remains only the first branch.*

4. Forced oscillations of a simply supported beam

Let us consider now forced oscillations of the simply supported beam under harmonic excitation

$$q_{xt}(x,t) = \widehat{q}_x(x)\mathrm{e}^{\mathrm{i}\Omega t} \tag{4.1}$$

In dimensionless variables (2.8), we have $q(\xi,\tau) = \widehat{q}(\xi)\exp(\mathrm{i}\Omega_\tau \tau)$, where $\Omega_\tau = \Omega r_0/c$. Expanding the load and displacements into Fourier series (only stationary oscillations are considered here)

$$\begin{aligned} \widehat{q}(\xi) &= \sum_m q_m \sin\frac{m\pi r_0 \xi}{l} \\ Y(\xi,\tau) &= \sum_m Y_m \sin\frac{m\pi r_0 \xi}{l}\mathrm{e}^{\mathrm{i}\Omega_\tau \tau} \end{aligned} \tag{4.2}$$

one obtains from (2.10) the following equation for the amplitude y_m of each harmonics

$$\begin{aligned}(1+\mathrm{i}\Omega_\tau\mu^*)^2k_m^4Y_m-(1+\chi)(1+\mathrm{i}\Omega_\tau\mu^*)k_m^2\Omega_\tau^2Y_m+\\ -(1+\mathrm{i}\Omega_\tau\mu^*-\chi\Omega_\tau^2)\Omega_\tau^2Y_m=\\ =(1+\mathrm{i}\Omega_\tau\mu^*)q_m-\chi\Omega_\tau^2q_m+\chi(1+\mathrm{i}\Omega_\tau\mu^*)k_m^2q_m\end{aligned} \tag{4.3}$$

where $k_m = m\pi r_0/l$. Introducing notation

$$Z=\frac{\Omega_\tau^2}{1+\mathrm{i}\Omega g_\tau\mu^*} \tag{4.4}$$

one gets from (4.3)

$$Y_m=\frac{1-\chi Z+\chi k_m^2}{k_m^4-(1+\chi)k_m^2Z-(1-\chi Z)Z}\frac{q_m}{1+\mathrm{i}\Omega_\tau\mu^*} \tag{4.5}$$

Functions of the dynamic amplification factor $k_{dyn} = k_m^4|Y_m|/|q_m|$ vs. the ratio Ω_τ/k_m^2 for several values of the shear flexibility parameter χ and two k_m values are presented in Fig. 4a,b (for $\mu^* = 1.0$).

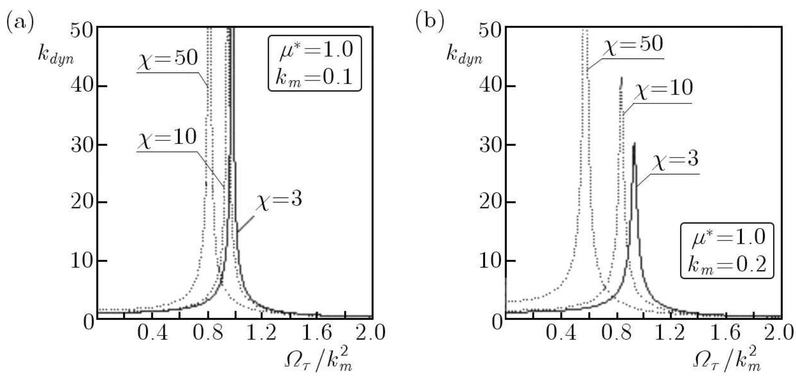

Fig. 4. Dynamic amplification factor k_{dyn} vs. Ω_τ/k_m^2 for several values of shear parameter χ and two values of $k_m = m\pi r_0/l$

It is seen that the shear flexibility, which perceptibly decreases the natural frequency of the beam, results in a shift of the resonance peak. This shift becomes considerable for relatively short wavelengths (not too small k_m values). Simultaneously, with a decrease of the frequency, a rise of the resonance peak is observed in comparison with the classical E-B beam.

5. Conclusions

Dynamical analysis of a Timoshenko beam made of an elastic and viscoelastic material has been carried out based on the proposed dimensionless equations of motion, and depending only on two generalized parameters. For the elastic Timoshenko beam, a simple and complete analytical description has been given for natural frequencies in the case of hinged-hinged edges. For the viscoelastic Timoshenko beam, a frequency equation with complex coefficients has been obtained, and solutions have been derived for two branches describing two possible types of free oscillations. The effect of the viscous internal friction parameter on free oscillations has been studied, and it was shown that the internal friction eliminates the second branch for sufficiently short wavelengths, and there remains only the first branch.

For the forced oscillation problem numerical results have been presented. It was shown that the shear flexibility parameter can perceptibly influence the frequency response curves, decreasing the natural frequency of the beam and shifting the resonance peak.

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Drgania własne i wymuszone belki Timoshenko wykonanej z materiału lepko-sprężystego

Streszczenie

W pracy analizowano dynamikę belki Timoshenko wykonanej z materiału lepko-sprężystego. Wyprowadzono bezwymiarowe równania ruchu zależne jedynie od dwóch parametrów: sztywności ścinania i współczynnika lepkiego tarcia wewnętrznego. Korzyści zaproponowanych równań przedstawiono na przypadkach drgań swobodnych i wymuszonych belki przegubowo podpartej na obu końcach. Analizowano wpływ sztywności ścinania belki i współczynnika lepkiego tarcia wewnętrznego na częstości drgań i dynamiczny współczynnik wzmocnienia.

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