STABILITY OF THIN PLATES WITH FUNCTIONALLY GRADED STRUCTURE

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This note presents the problem of stability of thin plates with functionally graded structure. To describe this problem the tolerance modelling is applied, cf. Woźniak, Michalak and Jędrysiak (eds.) [12] and Woźniak et al. (eds.) [13]. The tolerance averaging technique leads from the equation with non-continuous, tolerance-periodic, highly oscillating coefficients to the system of differential equations with slowly-varying coefficients.

1. INTRODUCTION

The main objects under consideration are thin plates with a functionally graded macrostructure in planes parallel to the plate midplane interacting with a heterogeneous Winkler's foundation. However, it is assumed that the microstructure is tolerance-periodic in these planes, cf. Fig. 1. Plates of this kind are consisted of many small elements, where adjacent elements are almost identical but distant one can be very different.

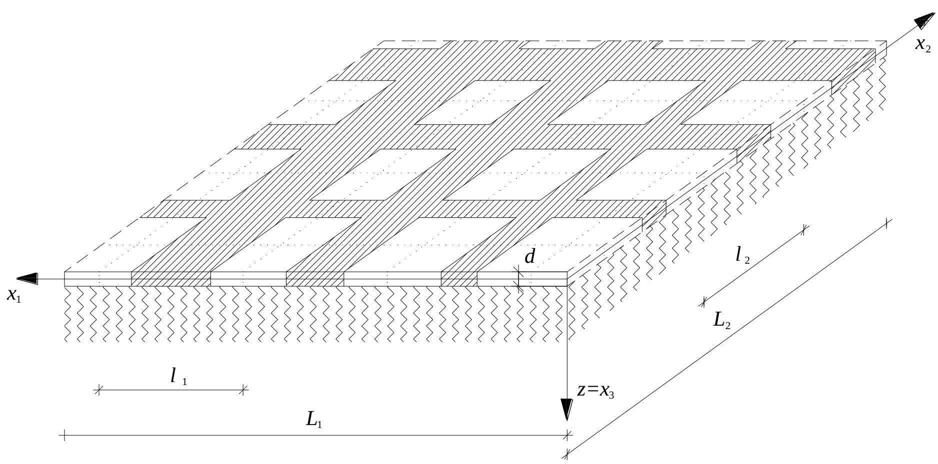

Fig. 1. A tolerance-periodic plate interacting with heterogeneous Winkler's foundation

Every element is treated as a thin plate. In various problems of these plates the effect of the microstructure cannot be neglected.

To describe plates of this kind there are used partial differential equations with highly oscillating, tolerance-periodic, non-continuous coefficients. They cannot be a proper tool to investigate special problems of such plates. Hence, there are proposed various averaging techniques, which make it possible to pass from the above equations to equations with smooth, slowly-varying coefficients.

Functionally graded structures are usually described by averaging approaches, which are developed for macroscopically homogeneous structures, e.g. periodic. Some of these are discussed by Suresh and Mortensen [10]. It can be mentioned these models, which are based on the asymptotic homogenization, cf. Bensoussan, Lions and Papanicolau [1]. Unfortunately, governing equations of them neglect the effect of the microstructure size.

The formulation of the proposed macroscopic models for analysis of functionally graded structures is based on the tolerance averaging technique, cf. Woźniak, Michalak and Jędrysiak (eds.) [12] and Woźniak et al. (eds.) [13]. Some applications of this method to the modelling of stability of various elastic periodic composites are given in series of publications, e.g. Jędrysiak [3-4], Michalak [8], Tomczyk [11], Domagalski and Jędrysiak [2]. The tolerance modelling was also adopted to functionally graded structures, e.g. for transversally tolerance-periodic plates by Jędrysiak [5], Jędrysiak and Michalak [6], Kaźmierczak and Jędrysiak [7] and for longitudinally functionally graded plates and structures by Michalak [9].

In this paper governing equations of the tolerance model of stability of thin plates with functionally graded structure are derived.

2. MODELLING FOUNDATIONS

Denote by $Ox_1x_2x_3$ the orthogonal Cartesian coordinate system. Setting $\mathbf{x}=(x_1,x_2)$ and $z=x_3$ it is assumed that the undeformed plate occupies the region $\Omega\equiv\{(\mathbf{x},z):-d(\mathbf{x})/2\leq z\leq d(\mathbf{x})/2,\mathbf{x}\in\Pi\}$, where Π is the midplane and $d(\cdot)$ is the plate thickness. Let ∂_α be derivatives of x_α, and also $\partial_{\alpha\ldots\delta}\equiv\partial_\alpha\ldots\partial_\delta$. The "basic cell" on Ox_1x_2 is denoted by $\Omega\equiv[-l_1/2,l_1/2]\times[-l_2/2,l_2/2]$, where l_1, l_2 are cell length dimensions along the x_1-, x_2-axis, respectively. The diameter of cell Ω is denoted by $l\equiv[(l_1)^2+(l_2)^2]^{1/2}$. It is called the microstructure parameter and satisfies condition $d_{\max}<<l<<\min(L_1,L_2)$. Thickness $d(\cdot)$ can be a tolerance-periodic function in $\mathbf{x}$ and also elastic moduli $a_{ijlm}=a_{ijlm}(\cdot,z)$ can be tolerance-periodic functions in $\mathbf{x}$ and even functions in z. Denote by $w(\mathbf{x},t)$ ($\mathbf{x}\in\Pi$, $t\in(t_0,t_1)$) a plate deflection and by p total loadings in the z-axis direction. We also assume that properties of a Winkler's foundation are described by a tolerance-periodic function in $\mathbf{x}$ - a Winkler's coefficient k. Introduce also the plate bending stiffnesses $d_{\alpha\beta\gamma\delta}$ defined as:

$$d_{\alpha\beta\gamma\delta}\equiv\int_{-d/2}^{d/2}z^2c_{\alpha\beta\gamma\delta}dz$$

being tolerance-periodic in $\mathbf{x}$. Moreover, the plate is subjected to in-plane forces $n_{\alpha\beta}$.

From the Kirchhoff-type plates theory assumptions applied to functionally graded plates, the fourth order partial differential equation for deflection $w(\mathbf{x},t)$ is derived:

$$\partial_{\alpha\beta}(d_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}w) - \partial_{\beta}(n_{\alpha\beta}\partial_{\alpha}w) + kw = p. \tag{1}$$

Coefficients of the above equation are highly oscillating, non-continuous, tolerance-periodic functions in **x**.

3. MODELLING CONCEPTS AND ASSUMTIONS

Averaged equations of functionally graded plates can be obtained by applying the tolerance averaging technique, cf. Woźniak et al. (eds.) [13], Woźniak, Michalak and Jędrysiak (eds.) [12]. In this books basic concepts of the tolerance modelling procedure are defined and explained, e.g. an averaging operator, a tolerance-periodic function, a slowly-varying function, a highly oscillating function. Below some of them are reminded.

Let $\Omega(\mathbf{x}) \equiv \mathbf{x} + \Omega$, $\Pi_{\Omega} = \{\mathbf{x} \in \Pi: \ \Omega(\mathbf{x}) \subset \Pi\}$, be a cell at $\mathbf{x} \in \Pi_{\Omega}$. The known averaging operator for an arbitrary integrable function *f* is defined by

$$< f > (\mathbf{x}) = \frac{1}{l_1 l_2} \int_{\Omega(\mathbf{x})} f(y_1, y_2) dy_1 dy_2, \quad \mathbf{x} \in \Pi_{\Omega}. \tag{2}$$

If function *f* is tolerance-periodic in **x**, then averaged value by (2) is a slowly-varying function in **x**.

Following the aforementioned books let us denote a set of tolerance-periodic functions by $TP_{\delta}^{\alpha}(\Pi,\Omega)$, a set of slowly-varying functions by $SV_{\delta}^{\alpha}(\Pi,\Omega)$, a set of highly oscillating functions by $HO_{\delta}^{\alpha}(\Pi,\Omega)$, where $\alpha \geq 0$ and δ is a tolerance parameter. Let a highly oscillating function $h(\cdot)$, $h \in HO_{\delta}^{2}(\Pi,\Omega)$, defined on $\overline{\Pi}$, have continuous gradient $\partial^1 h$ and have a piecewise continuous and bounded gradient $\partial^2 h$. Function $h(\cdot)$ is called *the fluctuation shape function* of the 2-nd kind. It depends on *l* as a parameter and satisfies conditions: $\partial^m h \in O(l^{2-m})$ for $m=0,1,2$, $\partial^0 h \equiv h$, and $<\mu h>(\mathbf{x}) \approx 0$ for every $\mathbf{x} \in \Pi_{\Omega}$, where μ is a positive, constant function.

4. TOLERANCE MODELLING

In the tolerance modelling three fundamental modelling assumptions are introduced. The first of them concerns *the micro-macro decomposition*:

$$w(\mathbf{x},t) = U(\mathbf{x},t) + h^A(\mathbf{x})Q^A(\mathbf{x},t), \quad A = 1,\ldots,N, \quad \mathbf{x} \in \Pi, \tag{3}$$

where:
$U(\cdot,t)$ – a kinematic unknown called *the macrodeflection*;
$Q^A(\cdot,t)$ – kinematic unknowns called *the fluctuation amplitudes*;
$h^A(\cdot)$ – the known *fluctuation shape functions*.

The second assumption is *the tolerance averaging approximation*, i.e. it is assumed that terms of an order of $O(\delta)$ are negligibly small, e.g. for $f \in TP_{\delta}^{2}(\Pi,\Omega)$, $F \in SV_{\delta}^{2}(\Pi,\Omega)$, $h^A \in FS_{\delta}^{2}(\Pi,\Omega)$, these terms can be neglected in formulas:

$$
\begin{aligned}
&< f >(\mathbf{x}) = < \bar{f} >(\mathbf{x}) + O(\delta), \\
&< fF >(\mathbf{x}) = < f >(\mathbf{x}) F(\mathbf{x}) + O(\delta), \\
&< f\partial_\alpha (h^A F) >(\mathbf{x}) = < f\partial_\alpha h^A >(\mathbf{x}) F(\mathbf{x}) + O(\delta).
\end{aligned} \tag{4}
$$

The third modelling assumption is *the in-plane forces restriction*, in which in-plane forces $n_{\alpha\beta}$ (α, β=1, 2) in the midplane are assumed to be tolerance-periodic functions in **x**, which can be decomposed as $n_{\alpha\beta} = N_{\alpha\beta} + \tilde{n}_{\alpha\beta}$, with $N_{\alpha\beta} \in SV_\delta^2(\Pi,\Omega)$ as the averaged part ($N_{\alpha\beta} \equiv < n_{\alpha\beta} >$); and $\tilde{n}_{\alpha\beta} \in TP_\delta^2(\Pi,\Omega)$ as the fluctuating part ($< \tilde{n}_{\alpha\beta} >= 0$). Moreover, terms with the fluctuating part of in-plane forces $\tilde{n}_{\alpha\beta}$ are neglected as small comparing to the terms involving the averaged part $N_{\alpha\beta}$ (cf. Jędrysiak and Michalak [6]).

5. MODEL EQUATIONS

Using micro-macro decomposition (3) in equation (1), making some manipulations and introducing denotations:

$$
\begin{aligned}
&D_{\alpha\beta\gamma\delta} \equiv < d_{\alpha\beta\gamma\delta} >, && D_{\alpha\beta}^A \equiv < d_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} h^A >, && D^{AB} \equiv < d_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} h^A \partial_{\gamma\delta} h^B >, \\
&K \equiv < k >, && K^A \equiv l^{-2} < k h^A >, && K^{AB} \equiv l^{-4} < k h^A h^B >, \\
&H_{\alpha\beta}^{AB} \equiv l^{-2} < \partial_{\alpha\beta} h^A h^B >, && P \equiv < p >, && P^A \equiv l^{-2} < p h^A >,
\end{aligned} \tag{5}
$$

we arrive at the system of equations:

$$
\begin{aligned}
&\partial_{\alpha\beta}(D_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} U + D_{\alpha\beta}^A Q^A) - \partial_\alpha (N_{\alpha\beta} \partial_\beta U) + KU + \underline{l^2 K^A Q^A} = P, \\
&D_{\gamma\delta}^A \partial_{\gamma\delta} U + D^{AB} Q^B - \underline{l^2 N_{\alpha\beta} H_{\alpha\beta}^{AB} Q^B} + \underline{l^2 K^A U} + \underline{l^4 K^{AB} Q^B} = \underline{l^2 P^A}.
\end{aligned} \tag{6}
$$

Equations (6) with micro-macro decomposition (3) and specified fluctuation shape function h^A, A=1,…,N, stand *the tolerance model of stability for thin functionally graded plates resting on a Winkler's foundation*. This is a system of N+1 partial and ordinary differential equations, with slowly-varying functions in **x**. These model equations involve terms, being underlined, with the microstructure parameter l. Hence, the tolerance model makes it possible to analyse the effect of the microstructure size on stability problems of these plates under consideration. It can be observed that boundary conditions have to formulate only for macrodeflection U.

In order to evaluate and compare results obtained in the framework of the above model a simplified model, called *the asymptotic model* is introduced. Governing equations of this model can be derived by using the formal asymptotic procedure, cf. Woźniak et al. [13], Jędrysiak [5], Kaźmierczak and Jędrysiak [7]. However, these equations can be also obtained from equations (6) by neglecting the underlined terms and they take the following form:

$$\partial_{\alpha\beta}(D_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}U + D^{A}_{\alpha\beta}Q^{A}) - \partial_{\alpha}(N_{\alpha\beta}\partial_{\beta}U) + KU = P,$$
$$D^{A}_{\gamma\delta}\partial_{\gamma\delta}U + D^{AB}Q^{B} = 0. \tag{7}$$

It can be observed that equations $(7)_2$ stand a system of algebraic equations from which the fluctuation amplitudes Q^A, A=1,…,N, can be calculated:

$$Q^{B} = -(D^{AB})^{-1}D^{A}_{\gamma\delta}\partial_{\gamma\delta}U. \tag{8}$$

After substituting the amplitudes obtained by (8) into equation $(7)_1$ and introducing so called effective stiffness

$$D^{eff}_{\alpha\beta\gamma\delta} \equiv D_{\alpha\beta\gamma\delta} - D^{A}_{\alpha\beta}(D^{AB})^{-1}D^{B}_{\gamma\delta}$$

we have only one differential equations for macrodeflection U:

$$\partial_{\alpha\beta}(D^{eff}_{\alpha\beta\gamma\delta}\partial_{\gamma\delta}U) - \partial_{\alpha}(N_{\alpha\beta}\partial_{\beta}U) + KU = P. \tag{9}$$

Equation (9) together with equations (8) and micro-macro decomposition (3) describe *the asymptotic model of stability for thin functionally graded plates resting on a Winkler's foundation*. The above equation neglects the effect of the microstructure size on stability problems of these plates.

6. REMARKS

Using the tolerance modelling to the known differential equation of Kirchhoff-type plates with a tolerance-periodic microstructure resting on a heterogeneous Winkler's foundation the averaged model equations are derived. Hence, instead of the governing equation with non-continuous, tolerance-periodic coefficients the system of differential equations with slowly-varying coefficients is obtained. These equations make it possible to analyse the effect of the microstructure size on stability problems of the plates under consideration.

Some other aspects of these problems and also some applications of the proposed tolerance model will be shown separately.

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