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Dedicated to Professor J. Klamka
on the occasion of his 70th birthday

Remarks on five equivalent forms of the fractional – order backward – difference

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Abstract. In this paper a fractional-order backward-difference/sum (FOBD/S) equivalent formulae are considered. From the Grünwald-Letnikov (GL – FOBD) definition formula and its Horner equivalent form one derives the Riemann-Liouville FOBD (RL – FOBD). Also the Caputo and polynomial-like forms are defined. All forms may be useful in real-time calculations (in the evaluation of digital control strategies) due to the reduction of fractional orders. The investigations are illustrated by a numerical example.

Key words: discrete systems, fractional – order backward difference/sum.

1. Introduction

In a discrete version of the Fractional Calculus the fractional-order derivatives and integrals [1–4] are replaced by fractional-order differences and sums [5]. As in the case of fractional-order derivatives and integrals several equivalent (under some assumptions) forms may be considered. The most common in use are: Grünwald-Letnikov fractional-order differ-integral (GL-FOD/S), FOD/S Horner form (H-FOD/S), Riemann-Liouville FOD/S (RL-FOD/S), FOD/S Caputo form [3, 4] and FOD/S Polynomial-Like form. The FO dynamical systems (FOS) described by FO difference equations (FODE) or state-space forms [6] reveal new unknown dynamical properties [7]. The FOS stability was analyzed among others publications [8, 9]. Another important property - controllability was considered in [10]. Selected control strategies were proposed in [11–15].

In practical applications of the Fractional Calculus [2] in its discrete-time version where the fractional-order derivatives and integrals are approximated by fractional-order differences and sums in microprocessor calculations occurs so called “calculation tail problem”. It is caused by finite system memory and constant finite sampling time. This problem is also called “a short memory principle” [4]. Since the problem has not yet been satisfactorily solved, it induces attempts to find an “optimal” formula for fractional order backward difference/sum real-time evaluation. The application of an “optimal FOBD/S form”, due to a chosen optimality criterion and assumed fractional order, may reduce calculation errors. The BD of the fractional order ν may be evaluated directly or as the n-th order classical difference of the FOBS of order $n-\nu$ or as the FOBS of order $n-\nu$ of the n-th order difference. Here n denotes an integer part of ν. This paper shows that the order of operations (i.e. the chosen FOBD/S form) is important in the calculations.

The paper is organized as follows. First, the basic equivalent five definitions of the FOBD/FOBS [5] are given. In Sec. 3 the main result – the equivalence of the introduced forms are proved. They may serve in microprocessor fractional order derivative/integral evaluation. In Sec. 4 a numerical example is given.

1.1. Mathematical background and notation. In the presented paper, the following notations are applied. The elements of a set of non-negative integers Z_+ are denoted by Latin letters i, j, k, m, n whereas the elements of a set of non-negative non-integers $\mathsf{R}_+/\mathsf{Z}_+$ are denoted by Greek letters ν, μ, ξ. Discrete – variable functions of a real discrete-variable k (sequences of real numbers) are denoted by f, g, h. Hence any function f is equivalently expressed as $f = f(k) = \{f_{k_0}, f_{k_0+1}, \cdots, f_{k-1}, f_k\}$, a lower index k_0 may be positive or negative.

2. Grünwald-Letnikov backward difference/sum equivalent forms

Consider a discrete-variable bounded real function $f(k)$ defined over a discrete-time interval $[k_0, k]$. In this Section five equivalent (under some conditions) FOBD/S forms are defined. One starts withe fundamental one known as the Grünwald-Letnikov form.

Definition 2.1. (Grünwald-Letnikov fractional-order backward-difference GL-FOBD).

A GL-FOBD of order $\nu \in \mathsf{R}_+/\mathsf{Z}_+$ is defined as a finite sum

$$ {}_{k_0}^{GL}\Delta_k^{(\nu)} f(k) = \sum_{i=k_0}^{k} a_{i-k_0}^{(\nu)} f_{k-i+k_0} \tag{1}$$

with coefficients $a_i^{(\nu)}$

$$a_i^{(\nu)} = \begin{cases} 0 & \text{for} \quad i<0 \\ 1 & \text{for} \quad i=0 \\ (-1)^i \frac{\nu(\nu-1)\cdots(\nu-i+1)}{i!} & \text{for} \quad i=1,2,\cdots \end{cases} . \tag{2}$$

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One can easily check that substituting ν by $n \in \mathsf{Z}_+$ in (1) one immediately gets a classical backward – differences of order n

$$\begin{aligned} &{}_{k_0}^{GL}\Delta_k^{(n)} f(k) = \Delta^{(n)} f(k) \\ &= \begin{cases} f_k & \text{for} \quad n = 0 \\ f_k - f_{k-1} & \text{for} \quad n = 1 \\ \Delta^{(n-1)} f(k) - \Delta^{(n-1)} f(k-1) & \text{for} \quad n > 1 \end{cases} \end{aligned} \tag{3}$$

Definition 2.2. (GL-FOBS).

A GL-FOBS of order $\nu \in \mathsf{R}_+/\mathsf{Z}_+$ is defined as a finite sum

$${}_{k_0}^{GL}\Delta_k^{(-\nu)} f(k) = {}_{k_0}^{GL}\Sigma_k^{(\nu)} f(k) = \sum_{i=k_0}^{k} a_{i-k_0}^{(-\nu)} f_{k-i+k_0}. \tag{4}$$

An n-fold sum of the discrete function $f(k)$ (integer-order backward-sum – IOBS) is obtained by a substitution of ν by $-n$ [16]

$${}_{k_0}^{GL}\Sigma_k^{(n)} f(k) = \sum_{i_1=k_0}^{k} \sum_{i_2=k_0}^{i_1} \cdots \sum_{i_n=k_0}^{i_{n-1}} f_{i_n} \tag{5}$$

and is treated as IOBD of negative order $-n < 0$ [5].

One can realise that

$$\begin{aligned} {}_{k_0}^{GL}\Delta_k^{(-\nu)} f(k) &= {}_{k_0}^{GL}\Sigma_k^{(\nu)} f(k), \\ {}_{k_0}^{GL}\Delta_k^{(\nu)} f(k) &= {}_{k_0}^{GL}\Sigma_k^{(-\nu)} f(k). \end{aligned} \tag{6}$$

To simplify the notation, in the IOBD/S formulae subscripts defining an operation range are omitted. Hence

$${}_{k_0}^{GL}\Delta_k^{(n)} f(k) = {}_{k-n}^{GL}\Delta_k^{(n)} f(k) = \Delta^{(n)} f(k). \tag{7}$$

Also a simplified notation will be used

$$\Delta^{(n)} f(k)\Big|_{k=l} = \Delta^{(n)} f_l. \tag{8}$$

It is worth to mention that the Grünwald-Letnikov FOBD/S is naturally related to the Grünwald-Letnikov fractional – order left – sided derivative/integral. One considers a continuous-time function $f(t)$ defined over interval $[t_0, t]$.

Definition 2.3. (Grünwald-Letnikov fractional – order left – sided derivative).

For a positive integer k and a real number h satisfying equality $hk = t - t_0$ the Grünwald-Letnikov fractional-order (left) derivative of a function $f(t)$ is defined by the infinite sum

$$\begin{aligned} {}_{t_0}^{GL}D_t^{(\nu)} f(t) &= \lim_{\substack{h \to 0^+ \\ k = \left[\frac{t-t_0}{h}\right]}} \left\{ \frac{1}{h^\nu} {}_{k_0}\Delta_k^{(\nu)} f(kh) \right\} \\ &= \lim_{\substack{h \to 0^+ \\ k = \left[\frac{t-t_0}{h}\right]}} \left\{ \frac{1}{h^\nu} \sum_{i=k_0}^{k} a_{i-k_0}^{(\nu)} f_{k-i+k_0} \right\}, \end{aligned} \tag{9}$$

where $[x]$ denotes an integer part of x.

Definition 2.4. (H-FOB/D).

The Horner form of the FOBD is expressed as

$$\begin{aligned} &{}_0^H\Delta_k^{(\nu)} y(k) \\ &c_0^{(\nu)}[y(k) + c_1^{(\nu)}[y(k-1) + c_2^{(\nu)}[y(k-2) + \ldots \\ &+ c_(k-2)^{(\nu)}[y(2) + c_(k-1)^{(\nu)}[y(1)] + c_k^{(\nu)} y(0)] \ldots]]] \end{aligned} \tag{10}$$

with coefficients

$$c_i^{(\nu)} = \begin{cases} 1 & \text{for} \quad i = 0 \\ \dfrac{i-1-\nu}{i} & \text{for} \quad i = 1, 2, 3, \ldots \end{cases} \tag{11}$$

Up to now the Horner's form of the GL-FOBD/S doesn't have its counterpart of the continuous operators: derivative and integral.

Definition 2.5. (Riemann-Liouville fractional – order backward difference (RL-FOBD)).

A RL-FOBD of order $\nu \in \mathsf{R}_+/\mathsf{Z}_+$ is defined as a finite sum

$$\begin{aligned} {}_{k_0}^{RL}\Delta_k^{(\nu)} f(k) &= \Delta^{(n)} \left[{}_{k_0}^{GL}\Sigma_k^{(n-\nu)} f(k) \right] \\ &= \Delta^{(n)} \left[{}_{k_0}^{GL}\Delta_k^{(\nu-n)} f(k) \right]. \end{aligned} \tag{12}$$

As in the Grünwald-Letnikov fractional-order left-hand derivative the RL-FOBD has its continuous equivalent known as the Riemann-Liouville fractional – order left – sided derivative of a continuous-time function $f(t)$ defined over interval $[t_0, t]$.

Definition 2.6. (Riemann-Liouville fractional – order left – sided derivative).

For a positive integer n satisfying equality $0 \le n - 1 < \nu < n$ the Reimann-Liouville fractional-order (left) derivative of a function $f(t)$ is defined as an n-th order (classical) derivative of the fractional integral of order $n - \nu$

$$\begin{aligned} &{}_{t_0}^{RL}D_t^{(\nu)} f(t) \\ &= \left(\frac{d}{dt}\right)^n \left[\frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} \frac{f(x)}{(t-x)^{\nu-n+1}} dx \right]. \end{aligned} \tag{13}$$

The next form is known as the Caputo FOBD/S.

Definition 2.7. (Caputo fractional-order backward-difference/sum C-FOBD).

Let $[k_0\, k]$ be a finite interval of a discrete variable and let ${}_0^{RL}\Delta_k^{(\nu)} f(k)$ exists. For $n = \lfloor \nu \rfloor + 1$ the Caputo FOBD/S ${}_0^C\Delta_k^{(\nu)} f(k)$ is defined as

$${}_{k_0}^{C}\Delta_k^{(\nu)} f(k) = {}_{k_0}^{GL}\Delta_k^{(\nu-n)} \left[\Delta^{(n)} f(k) \right]. \tag{14}$$

For the continuous function the Caputo derivative is defined as the fractional-order integral of the classical n-the order derivative.

Definition 2.8. (Caputo fractional – order left – sided derivative).

For a positive integer n satisfying equality $0 \leq n-1 < \nu < n$ the Caputo fractional-order (left) derivative of a function $f(t)$ is defined as the fractional-order $n-\nu$ integral of n-th order (classical) derivative

$$ {}_{t_0}^{RL}D_t^{(\nu)} f(t) = \frac{1}{\Gamma(n-\nu)} \int\limits_{t_0}^{t} \frac{f^{(n)}(x)}{(t-x)^{\nu-n+1}} dx. \quad (15)$$

The final fifth form of the FOBD/S may be expressed as the FOBD/S polynomial-like form. The nomenclature of the form presented below origins the linear time-invariant system polynomial description [7]. Formula (1) for $k_0 = 0$ can be expressed in a form

$$ {}_0^{GL}\Delta_k^{(\nu)} f_k = \left[\mathbf{a}_{k+1}^{(\nu)}\right]^{\mathrm{T}} {}_0\mathbf{f}_k, \quad (16)$$

where

$$\left[{}_0\mathbf{a}_k^{(\nu)}\right]^{\mathrm{T}} = \begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \dots & a_{k-1}^{(\nu)} & a_k^{(\nu)} \end{bmatrix}, \quad (17)$$

$$ {}_0\mathbf{f}_k = \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_1 \\ f_0 \end{bmatrix}. \quad (18)$$

It is valid for consecutive time instants $k,\ k-1, \cdots, 1, 0$. A collection of expressions (16) evaluated for mentioned earlier time instants in a vector-matrix form yields

$$ {}_0^{P}\Delta_k^{(\nu)} {}_0\mathbf{f}_k = {}_0\mathbf{A}_k^{(\nu)} {}_0\mathbf{f}_k, \quad (19)$$

where

$$ {}_0\mathbf{A}_k^{(\nu)} = \begin{bmatrix} \left[{}_0\mathbf{a}_k^{(\nu)}\right]^{\mathrm{T}} \\ \left[{}_0\mathbf{a}_{k-1}^{(\nu)}\right]^{\mathrm{T}} \\ \left[{}_0\mathbf{a}_{k-2}^{(\nu)}\right]^{\mathrm{T}} \\ \vdots \\ \left[{}_0\mathbf{a}_1^{(\nu)}\right]^{\mathrm{T}} \\ \left[{}_0\mathbf{a}_0^{(\nu)}\right]^{\mathrm{T}} \end{bmatrix} \quad (20)$$

$$ = \begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & a_2^{(\nu)} & \cdots & a_{k-1}^{(\nu)} & a_k^{(\nu)} \\ 0 & a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-2}^{(\nu)} & a_{k-1}^{(\nu)} \\ 0 & 0 & a_0^{(\nu)} & \cdots & a_{k-3}^{(\nu)} & a_{k-2}^{(\nu)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_0^{(\nu)} & a_1^{(\nu)} \\ 0 & 0 & 0 & \cdots & 0 & a_0^{(\nu)}) \end{bmatrix}$$

is a $(k+1) \times (k+1)$ matrix. Matrix (19) belongs to a very important class of matrices. It is the upper triangular band matrix. Coefficient $a_0^{(\nu)} \neq$ so the matrix is always invertible.

Definition 2.7. (PL-FOBD/S).

Polynomial-like FOBD/S ${}_0^{P}\Delta_k^{(\nu)} {}_0\mathbf{f}_k$ is defined by formula (19).

3. Equivalence of the FOBD forms

The equivalence of the defined in Sec. 2 forms will be presented in forms of the following theorems.

Theorem 3.1. The GL-VOBD/S is equivalent to the H-FOBD/S.

Proof. A substitution of coefficients (2) by (11) in formula (1) after simple rearrangements gives formula (10).

For $\nu \in \mathsf{R}_+$ one defines $n \in \mathsf{Z}_+$

$$ n = \begin{cases} \nu & \text{for} \quad \nu \in \mathsf{Z}_+ \\ [\nu]+1 & \text{for} \quad \nu \notin \mathsf{Z}_+ \end{cases}. \quad (21)$$

This is equivalent to the following non-equalities

$$0 \leq n-1 < \nu < n. \quad (22)$$

Lemma 3.1. The classical backward differences of consecutive integer orders $f(k)i = 1, 2, \dots$ of the discrete – variable function $f(k)$ and its shifted values are related by the following vector – matrix relation

$$\begin{bmatrix} \Delta^{(0)} f(k) \\ \Delta^{(1)} f(k) \\ \Delta^{(2)} f(k) \\ \Delta^{(3)} f(k) \\ \Delta^{(4)} f(k) \\ \Delta^{(5)} f(k) \\ \vdots \end{bmatrix} = \mathbf{D}_k \begin{bmatrix} f_k \\ f_{k-1} \\ f_{k-2} \\ f_{k-3} \\ f_{k-4} \\ f_{k-5} \\ \vdots \end{bmatrix}, \quad (23)$$

where

$$\mathbf{D}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & -1 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}. \quad (24)$$

Formula (23) may be derived from formula (3) by putting $n = 0, 1, 2, \dots$. It is also valid for $k = k_0$. Hence

$$\left.\begin{bmatrix} \Delta^{(0)} f(k) \\ \Delta^{(1)} f(k) \\ \Delta^{(2)} f(k) \\ \Delta^{(3)} f(k) \\ \Delta^{(4)} f(k) \\ \Delta^{(5)} f(k) \\ \vdots \end{bmatrix}\right|_{k=k_0} = \begin{bmatrix} \Delta^{(0)} f(k_0) \\ \Delta^{(1)} f(k_0) \\ \Delta^{(2)} f(k_0) \\ \Delta^{(3)} f(k_0) \\ \Delta^{(4)} f(k_0) \\ \Delta^{(5)} f(k_0) \\ \vdots \end{bmatrix} = \mathbf{D}_k \begin{bmatrix} f_{k_0} \\ f_{k_0-1} \\ f_{k_0-2} \\ f_{k_0-3} \\ f_{k_0-4} \\ f_{k_0-5} \\ \vdots \end{bmatrix}. \quad (25)$$

Simple calculations of a product $\mathbf{D}_k\mathbf{D}_k$ reveal that

$$\mathbf{D}_k = [\mathbf{D}_k]^{-1}, \tag{26}$$

and

$$\begin{bmatrix} f_{k_0} \\ f_{k_0-1} \\ f_{k_0-2} \\ f_{k_0-3} \\ f_{k_0-4} \\ f_{k_0-5} \\ \vdots \end{bmatrix} = \mathbf{D}_k \begin{bmatrix} \Delta^{(0)} f(k_0) \\ \Delta^{(1)} f(k_0 t) \\ \Delta^{(2)} f(k_0) \\ \Delta^{(3)} f(k_0) \\ \Delta^{(4)} f(k_0) \\ \Delta^{(5)} f(k_0) \\ \vdots \end{bmatrix}. \tag{27}$$

One can also comment the meaning of the result (27). Though, by assumption the function $f(k) = 0$ for $k < k_0$ its non-zero values $f(k_0)$, $f(k_0 - 1), \ldots$ should be treated as initial conditions. In accordance of the above definitions one may state a following theorem.

Theorem 3.1. Let $\nu \in \mathsf{R}_+/\mathsf{Z}_+$ and $n = [\nu] + 1$ where [.] denotes an integer part. If $\Delta^{(n)} f(k)$ exists, then the RL-FOBD can be represented in the form

$$\begin{aligned} {}_{k_0}^{RL}\Delta_k^{(\nu)} f(k) &= {}_{k_0}^{GL}\Sigma_k^{(n-\nu)} \left[\Delta^{(n)} f(k)\right] \\ &+ \sum_{i=0}^{n-1} a_{k-k_0}^{(\nu-1-i)} \Delta^{(i)} f(k_0 - 1). \end{aligned} \tag{28}$$

Proof. The three terms in formula (28) will be denoted as L, R_1, R_2 respectively. By definition 2.5 and formula (26) L may be expressed as

$$L = {}_{k_0}^{RL}\Delta_k^{(\nu)} f(k) = \Delta^{(n)} \left[{}_{k_0}^{GL}\Delta_k^{(\nu-n)} f(k)\right]$$

$$= \begin{bmatrix} a_0^{(n)} & a_1^{(n)} & \cdots & a_{n-1}^{(n)} & a_n^{(n)} \end{bmatrix} \begin{bmatrix} {}_{k_0}^{GL}\Delta_k^{(\nu-n)} f(k) \\ {}_{k_0}^{GL}\Delta_{k-1}^{(\nu-n)} f(k-1) \\ \vdots \\ {}_{k_0}^{GL}\Delta_{k-n+1}^{(\nu-n)} f(k-n+1) \\ {}_{k_0}^{GL}\Delta_{k-n}^{(\nu-n)} f(k-n) \end{bmatrix}$$

$$= \begin{bmatrix} a_0^{(n)} & a_1^{(n)} & \cdots & a_{n-1}^{(n)} & a_n^{(n)} \end{bmatrix} \begin{bmatrix} a_0^{(\nu-n)} & a_1^{(\nu-n)} & \cdots & a_{n-1}^{(\nu-n)} & a_n^{(\nu-n)} & \cdots & a_{k-k_0}^{(\nu-n)} \\ 0 & a_0^{(\nu-n)} & \cdots & a_{n-2}^{(\nu-n)} & a_{n-1}^{(\nu-n)} & & a_{k-1-k_0}^{(\nu-n)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_0^{(\nu-n)} & a_1^{(\nu-n)} & \cdots & a_{k-n+1-k_0}^{(\nu-n)} \\ 0 & 0 & \cdots & 0 & a_0^{(\nu-n)} & \cdots & a_{k-n-k_0}^{(\nu-n)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0-1}^{(\nu)} & a_{k-k_0}^{(\nu)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{bmatrix}.$$

The first term of the right – hand side of (29) equals

$$R_1 = {}_{k_0}^{GL}\Sigma_k^{(n-\nu)}\left[\Delta^{(n)} f(k)\right]$$

$$= \begin{bmatrix} a_0^{(\nu-n)} & a_1^{(\nu-n)} & \cdots & a_{k-k_0-1}^{(\nu-n)} & a_{k-k_0}^{(\nu-n)} \end{bmatrix} \begin{bmatrix} {}_{k-n}\Delta_k^{(n)} f(k) \\ {}_{k-n-1}\Delta_{k-1}^{(n)} f(k-1) \\ \vdots \\ {}_{k_0-n+1}\Delta_{k_0+1}^{(n)} f(k_0+1) \\ {}_{k_0-n}\Delta_{k_0}^{(n)} f(k_0) \end{bmatrix}$$

$$= \begin{bmatrix} a_0^{(\nu-n)} & \cdots & a_{k-k_0}^{(\nu-n)} \end{bmatrix} \begin{bmatrix} a_0^{(n)} & a_1^{(n)} & \cdots & a_n^{(n)} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & a_0^{(n)} & \cdots & a_{n-1}^{(n)} & a_n^{(n)} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a_0^{(n)} & \cdots & a_n^{(n)} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & a_{n-1}^{(n)} & a_n^{(n)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k_0} \\ f_{k_0-1} \\ \vdots \\ f_{k_0-n_0} \end{bmatrix}$$

$$= \begin{bmatrix} a_0^{(\nu-n)} & \cdots & a_{k-k_0}^{(\nu-n)} \end{bmatrix} \begin{bmatrix} a_0^{(n)} & a_1^{(n)} & \cdots & 0 & 0 \\ 0 & a_0^{(n)} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_0^{(n)} & a_1^{(n)} \\ 0 & 0 & \cdots & 0 & a_0^{(n)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{bmatrix} \quad (30)_a$$

$$+ \begin{bmatrix} a_0^{(\nu-n)} & \cdots & a_{k-k_0}^{(\nu-n)} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ a_n^{(n)} & 0 & \cdots & 0 & 0 \\ a_{n-1}^{(n)} & a_n^{(n)} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_2^{(n)} & a_3^{(n)} & \cdots & a_n^{(n)} & 0 \\ a_1^{(n)} & a_2^{(n)} & \cdots & a_{n-1}^{(n)} & a_n^{(n)} \end{bmatrix} \begin{bmatrix} f_{k_0-1} \\ f_{k_0-2} \\ \vdots \\ f_{k_0-n} \end{bmatrix}$$

$$= \begin{bmatrix} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0-1}^{(\nu)} & a_{k-k_0}^{(\nu)} \end{bmatrix} \begin{bmatrix} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{bmatrix}$$

$$+\left[\begin{array}{ccc} a_0^{(\nu-n)} & \cdots & a_{k-k_0}^{(\nu-n)} \end{array}\right]\left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ a_n^{(n)} & 0 & \cdots & 0 & 0 \\ a_{n-1}^{(n)} & a_n^{(n)} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_2^{(n)} & a_3^{(n)} & \cdots & a_n^{(n)} & 0 \\ a_1^{(n)} & a_2^{(n)} & \cdots & a_{n-1}^{(n)} & a_n^{(n)} \end{array}\right]\mathbf{D}_{k_0-1}\mathbf{D}_{k_0-1}\left[\begin{array}{c} f_{k_0-1} \\ f_{k_0-2} \\ \vdots \\ f_{k_0-n} \end{array}\right]$$

$$=\left[\begin{array}{ccccc} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0-1}^{(\nu)} & a_{k-k_0}^{(\nu)} \end{array}\right]\left[\begin{array}{c} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{array}\right]$$

$$+\left[\begin{array}{ccc} a_0^{(\nu-n)} & \cdots & a_{k-k_0}^{(\nu-n)} \end{array}\right]\left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ a_n^{(n)} & 0 & \cdots & 0 & 0 \\ a_{n-1}^{(n)} & a_n^{(n)} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_2^{(n)} & a_3^{(n)} & \cdots & a_n^{(n)} & 0 \\ a_1^{(n)} & a_2^{(n)} & \cdots & a_{n-1}^{(n)} & a_n^{(n)} \end{array}\right]\mathbf{D}_{k_0-1}\left[\begin{array}{c} \Delta^{(0)}f(k_0-1) \\ \Delta^{(1)}f(k_0-1) \\ \vdots \\ \Delta^{(n-2)}f(k_0-1) \\ \Delta^{(n-1)}f(k_0-1) \end{array}\right]$$

$$+\left[\begin{array}{ccccc} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0-1}^{(\nu)} & a_{k-k_0}^{(\nu)} \end{array}\right]\left[\begin{array}{c} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{array}\right]-\left[\begin{array}{ccccc} a_{k-k_0}^{(\nu-1t)} & a_{k-k_0}^{(\nu-2)} & \cdots & a_{k-k_0}^{(\nu-n+1)} & a_{k-k_0}^{(\nu-n)} \end{array}\right]\left[\begin{array}{c} \Delta^{(0)}f(k_0-1) \\ \Delta^{(1)}f(k_0-1) \\ \vdots \\ \Delta^{(n-2t)}f(k_0-1) \\ \Delta^{(n-1)}f(k_0-1) \end{array}\right]$$

$$=\left[\begin{array}{ccccc} a_0^{(\nu)} & a_1^{(\nu)} & \cdots & a_{k-k_0-1}^{(\nu)} & a_{k-k_0}^{(\nu)} \end{array}\right]\left[\begin{array}{c} f_k \\ f_{k-1} \\ \vdots \\ f_{k-n+1} \\ f_{k-n} \\ \vdots \\ f_{k_0} \end{array}\right]-\sum_{i=0}^{n-1} a_{k-k_0}^{(\nu-i)}\Delta^{(i)}f(k_0-1)=L-R_2. \qquad (30)_b$$

Thus it was proved that $R_1 = L - R_2$ what ends the proof.

One should realize that the GL-FOBD formula (30) is similar to the well-known form of the Riemann-Liouville fractional-order derivative [4]

$$ {}_{t_0}^{RL}D_t^{(\nu)} f(t) = \frac{1}{\Gamma(n-\nu)} \int_{t_0}^{t} (t-\tau)^{n-1-\nu} f^{(n)}(\tau) d\tau + \sum_{i=0}^{n-1} \frac{(t-t_0)^{i-\nu}}{\Gamma(i+1-\nu)} f^{(i)}(t_0). \tag{31} $$

A comparison of formulae (14) and (28) yields

$$ {}_{k_0}^{C}\Delta_k^{(\nu)} f(k) = {}_{k_0}^{RL}\Delta_k^{(\nu)} f(k) - \sum_{i=0}^{n-1} a_{k-k_0}^{(\nu-1-i)} \Delta^{(i)} f(k_0 - 1). \tag{32} $$

Assuming that $\Delta^{(i)} f(k_0 - 1) = 0$ for $i = 0, 1, 2, \ldots, n-1$ one may state that

$$ {}_{k_0}^{RL}\Delta_k^{(\nu)} f(k) = {}_{k_0}^{GL}\Delta_k^{(\nu)} f(k) = {}_{k_0}^{H}\Delta_k^{(\nu)} f(k) = {}_{k_0}^{C}\Delta_k^{(\nu)} f(k). \tag{33} $$

The numerical example confirms different FOBD evaluation accuracy due to the application of formulae considered above. Two forms are compared: the GL-FOBD and RL-FOBD. The FOBD of the discrete Dirac pulse [16] is calculated.

4. Numerical example

Firstly, one evaluates some FOBDs of a discrete Dirac pulse

$$ \delta(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \neq 0 \end{cases} \tag{34} $$

Immediate calculations show that by Definition 2.1

$$ {}_0\Delta_k^{(-1)} \delta(k) = \sum_{i=0}^{k} a_i^{(-1)} \delta_{k-i} = a_k^{(-1)} = \mathbf{1}(k) = \begin{cases} 0 & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k \geq 0 \end{cases} \tag{35} $$

$$ {}_0\Delta_k^{(-2)} \delta(k) = \sum_{i=0}^{k} a_i^{(-2)} \delta_{k-i} = a_k^{(-2)} = (-1)^k \frac{(-2)(-3)\cdots(-k)(-k-1)}{1 \cdot 2 \cdots (k-1)k} \mathbf{1}(k) = \frac{(k+1)}{1} \mathbf{1}(k), \tag{36} $$

$$ {}_0\Delta_k^{(-3)} \delta(k) = a_k^{(-3)} = (-1)^k \frac{(-3)(-4)\cdots(-k-1)(-k-2)}{1 \cdot 2 \cdots (k-1)k} \mathbf{1}(k) = \frac{(k+1)(k+2)}{1 \cdot 2} \mathbf{1}(k). \tag{37} $$

Continuing this procedure one derives a general formula

$$ {}_0\Delta_k^{(-n)} \delta(k) = a_k^{(-n)} = \frac{1}{n!} \prod_{i=1}^{n} (k+i) \mathbf{1}(k) \text{ for } n = 1, 2, \cdots \tag{38} $$

The coefficients defined by formula (2) may be also treated as discrete – variable functions $a^{(\nu)}(k)$. One can easily prove that

$$ {}_{k_0}^{GL}\Delta_k^{(\mu)} \left[a^{(\nu)}(k) \right] = \sum_{i=k_0}^{k} a_{i-k_0}^{(\mu)} a_{k-i+k_0}^{(\nu)} = \begin{bmatrix} a_0^{(\mu)} & a_1^{(\mu)} & \cdots & a_{k-1-k_0}^{(\mu)} & a_{k-k_0}^{(\mu)} \end{bmatrix} \begin{bmatrix} a_k^{(\nu)} \\ a_{k-1}^{(\nu)} \\ \vdots \\ a_{k_0+1}^{(\nu)} \\ a_{k_0}^{(\nu)} \end{bmatrix} = a_{k-k_0}^{(\mu+\nu)}. \tag{39} $$

For different orders ν satisfying $n = [\nu] + 1$ one evaluates a performance index

$$ \boldsymbol{I}(\boldsymbol{\nu}) = 20 \log_{10} \left[\max{}_{k \in [\mathbf{0}, \mathbf{2000}]} \boldsymbol{e}(\boldsymbol{k}, \boldsymbol{\nu}) \right] \tag{40} $$

defined for an error function

$$ e(k, \nu) = \frac{{}_0^{GL}\Delta_k^{(\nu)} \delta(k) - {}_0^{RL}\Delta_k^{(\nu)} \delta(k)}{{}_0^{GL}\Delta_k^{(\nu)} \delta(k)} 100. \tag{41} $$

The performance index is plotted in Fig. 1. The plot reveals which of the proposed forms: GL-FOBD, RL-FOBD or C-FOBD should be applied to minimize the calculation errors assuming the same number representation in the computing device.

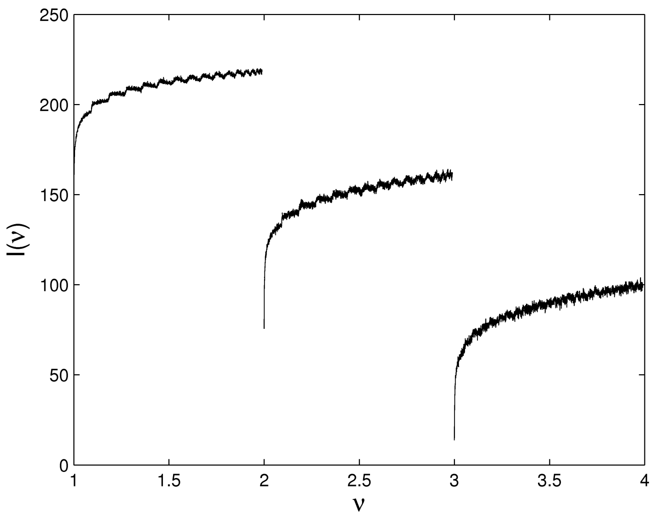

Fig. 1. The performance index vs. order

5. Conclusions

The aforementioned example shows that by applying two equivalent forms of the FOBD with the same calculation precision one obtains slightly different results which increase due to the deterioration of numbers representation. The values of the coefficients $a_i^{(\nu)}$ are crucial in the FOBD calculation process. For $0 < \nu < 1$ the coefficients (2) tend sharply to zero whereas the coefficients (11) tend to 1. The precision of $a_i^{(\nu)}$ and $c_i^{(\nu)}$ evaluation depends on the order ν. An appropriate change of $a_i^{(\nu)}$ via the proposed equivalent forms may diminish errors in a real-time calculation of FOBD. One should also realize that in the FOBD/Ss concatenation operations one may mix proposed forms (bearing in mind the initial conditions).

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