Opuscula Math. 34, no. 4 (2014), 725–732
http://dx.doi.org/10.7494/OpMath.2014.34.4.725 Opuscula Mathematica

This work is dedicated to Professor Leon Mikołajczyk
on the occasion of his 85th birthday.

ON A NEW CRITICAL POINT THEOREM AND SOME APPLICATIONS TO DISCRETE EQUATIONS

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Communicated by Vicentiu D. Radulescu

Abstract. Using the Fenchel-Young duality we derive a new critical point theorem. We illustrate our results with solvability for certain discrete BVP. Multiple solutions are also considered.

Keywords: critical point, multiplicity, discrete equation, Fenchel-Young transform.

Mathematics Subject Classification: 49J27, 39A12, 39A10.

1. INTRODUCTION

In this work we provide a type of critical point theorem with applications to boundary value problems, both continuous and discrete ones. We base our work on the methodology connected with the Fenchel-Young transform and Gâteaux differentiability. Our result allows us to obtain a critical point for a functional investigated on a set which need not be open.

Let us introduce the problem under consideration and the main assumptions. Let E be a real reflexive Banach space. Given two Gâteaux differentiable mappings Φ, $H : E \to \mathbb{R}$ with derivatives φ, $h : E \to E^*$ respectively, we undertake the existence of solutions to

$$\varphi(u) = h(u), \quad u \in E \tag{1.1}$$

under some geometric conditions related to convexity and the existence of a minimizer over a certain set. We denote by $J : E \to \mathbb{R}$ the action functional connected with (1.1), i.e. $J(u) = \Phi(u) - H(u)$. Therefore we link solutions to (1.1) with critical points of J.

Let us mention [9] for some recent results concerning a general type of critical point theorem. The ideas employed in the proof of our main result originate from [8], where these had a very complicated nature. Later several authors worked on improving the methodology contained in [8], see for example [3, 4] and references therein. The result which we present in this work seems to provide a most applicable version of all those mentioned and connected with the so called dual method. Combining it with a classical variational result, we obtain the existence of at least two distinct solutions in the discrete setting. We illustrate our results by examining the solvability of a Dirichlet problem with discrete p-Laplacian. Such problems have been investigated lately by many authors, see for example [5, 7, 10] for the most recent results. For a background on variational methods we refer to [6, 11] while for a background on difference equations to [1].

2. PRELIMINARIES

We recall some information about the Fenchel-Young transform, see [2]. We recall that the Fenchel-Young dual for a Gâteaux differentiable function $H : E \to \mathbb{R}$, see [2], reads

$$H^*(v) = \sup_{u \in E} \{\langle u, v \rangle - H(u)\}, \quad H^* : E^* \to \mathbb{R},$$

$\langle u, v \rangle$ stands for the duality pairing. Note that H^* and H^{**}, where H^{**} is defined in an obvious manner, are always convex l.s.c. functionals irrespective of what properties H has. When $H^{**}(u) = H(u)$ the derivative of H at u is the subdifferential of H at u in the sense of convex analysis. Note that a differentiable and non-convex function can otherwise have this subdifferential empty. Moreover, if the subdifferential of H in the sense of convex analysis is nonempty at some point u, then $H^{**}(u) = H(u)$, see [2]. When $H^{**}(u) = H(u)$, H^{**} is finite and Gâteaux differentiable at u, we also have the following relation

$$H(u) + H^*(v) = \langle u, v \rangle \Longleftrightarrow v = h(u). \tag{2.1}$$

Relation (2.1) can be viewed as a kind of a convexity at a point, and as it appears it is enough in our reasoning. When H is convex and differentiable on E, than (2.1) holds for all u. Another useful property of the Fenchel-Young transform is the Fenchel-Young inequality

$$\langle p, u \rangle \leq H(u) + H^*(p)$$

which is valid for any $p \in E^*, u \in E$ and any functional H defined on E irrespective of its convexity.

3. A CRITICAL POINT THEOREM

Theorem 3.1. *Let there exist $u, v \in E$ satisfying $\varphi(v) = h(u)$ and such that $J(u) \leq J(v)$. If additionally $H^{**}(u) = H(u)$, $\Phi^{**}(v) = \Phi(v)$, $\Phi^{**}(u) = \Phi(u)$, Φ^{**} and H^{**} are finite and Gâteaux differentiable at v and u respectively, then u is a critical point to J, and thus it solves* (1.1).

Proof. We put $p = \varphi(v) = h(u)$. Since $\Phi^{**}(v) = \Phi(v)$, $H^{**}(u) = H(u)$ and since $\frac{d}{du}\Phi = \varphi$, $\frac{d}{du}H = h$ we have by the definition of p

$$\Phi(v) = \langle v, p\rangle - \Phi^*(p) \quad \text{and} \quad H(u) = \langle u, p\rangle - H^*(p). \tag{3.1}$$

By the Fenchel-Young inequality $-H(v) \leq H^*(p) - \langle p, v\rangle$ and by the first relation in (3.1), we have

$$J(v) = \Phi(v) - H(v) = \langle v, p\rangle - \Phi^*(p) - H(v) \leq H^*(p) - \Phi^*(p).$$

Therefore, by the above, we get

$$\Phi(u) - H(u) \leq J(u) \leq J(v) \leq H^*(p) - \Phi^*(p)$$

and so by the Fenchel-Young inequality and the second relation in (3.1), we see

$$\langle p, u\rangle \leq \Phi(u) + \Phi^*(p) \leq H^*(p) + H(u) = \langle p, u\rangle,$$

Therefore

$$\langle p, u\rangle = \Phi(u) + \Phi^*(p),$$

hence $p = \frac{d}{du}\Phi(u) = \varphi(u)$. Hence recalling definition of p we see that $p = \varphi(v) = h(u) = \varphi(u)$. This means that equation (1.1) is satisfied. □

The following corollaries are direct consequences of the main result.

Corollary 3.2. *Assume that H and Φ are convex and that there exist $u, v \in E$ satisfying $\varphi(v) = h(u)$ and such that $J(u) \leq J(v)$. Then u is a critical point to J, and thus it solves* (1.1).

Corollary 3.3. *Let $X \subset E$. Assume that H and Φ are convex on E. Let there exist $u \in E$, $v \in X$ satisfying $\varphi(v) = h(u)$ and such that $J(u) \leq \inf_{x\in X} J(x)$. Then u is a critical point to J, and thus it solves* (1.1).

Note that X could be a closed set. Assuming that $u \in X$, we see that a minimizer of J over a closed set is its critical point provided that some additional convexity conditions are fulfilled.

We finish with a simple multiplicity result which works in the setting of finite dimensional spaces and therefore is applicable to discrete BVPs.

Theorem 3.4. *Let E be a finite dimensional space and let $X \subset E$ contain at least two points. Assume that H and Φ are convex on E. Let there exist $u \in E$, $v \in X$ satisfying $\varphi(v) = h(u)$ such that $J(u) \leq \inf_{x\in X} J(x)$. Then u is a critical point to J, and thus it solves* (1.1). *If, moreover, J is anti-coercive, then* (1.1) *has another solution different from u.*

Proof. The first solution exists by Theorem 3.1. Since J is anti-coercive and continuous, it has an argument of a maximum which we denote by w. Since J is differentiable in the sense of Gâteaux, w is a critical point. If $w \notin X$ we are done. Assume that $w \in X$. Note that $J(u) \leq \inf_{x \in X} J(x) \leq J(w)$. Thus we have to consider two cases. When $J(u) = J(w)$, we see that J is constant on X. Since X contains at least two points, we get the second assertion of the theorem. When $J(u) < J(w)$ it is obvious that $w \neq u$. □

The above result improves the multiplicity theorem from [3], since it does not require equation $\varphi(v) = h(u)$ to have a solution u for certain fixed v.

4. APPLICATION TO DISCRETE EQUATIONS

Consider the following discrete problem:

$$\begin{aligned} &-\Delta\left(\phi_p\left(\Delta x(k-1)\right)\right) = \lambda f\left(k, x(k)\right), \quad k \in \mathbb{N}(1,T), \\ &x(0) = x(T+1) = 0, \end{aligned} \tag{4.1}$$

where $\lambda > 0$ is a numerical parameter, $\phi_p(t) = |t|^{p-2}t,\ p \geq 2$, Δ is the forward difference operator defined by $\Delta x(k) = x(k+1) - x(k)$. For fixed a, b such that $a < b < \infty$, $a \in \mathbb{N} \cup \{0\}$, $b \in \mathbb{N}$ we denote $\mathbb{N}(a,b) = \{a, a+1, \ldots, b-1, b\}$. Let $F(t,\xi) = \int_0^\xi f(t,s)ds$ for $(t,\xi) \in (\mathbb{N}(1,T+1) \times \mathbb{R})$. Note that the integration is with respect to the second variable. We will employ the following assumptions:

(H_0) $f \in C(\mathbb{N}(1,T+1) \times \mathbb{R}; \mathbb{R})$;

(H_1) there exist constants $\alpha > 0$ and $d > 0$ such that $x \to F(k,x)$ is convex on $\mathbb{R}$ for all $k \in \mathbb{N}(1,T)$ and

$$|f(k,d)| \leq \alpha d^{p-1}$$

for all $k \in \mathbb{N}(1,T)$;

(H_2) there exist constants $\mu > p$, $c_1 > 0$, $c_2 \in \mathbb{R}$ and $m > d$ such that

$$F(k,x) \geq c_1 |x|^\mu + c_2$$

for all $k \in \mathbb{N}(1,T)$ and all $|x| \geq m$.

Example 4.1. There are many functions satisfying both (H_1) and (H_2). Let us mention the following.

(a) Take

$$f(x) = \begin{cases} e^x & \text{for } x \geq 0, \\ -e^{-x} & \text{for } x < 0. \end{cases}$$

Then $F(x) = e^{|x|}$. Taking $\alpha \geq \frac{e^d}{d^{p-1}}$ for any $d > 0$ we see that (H_1) and (H_2) hold.

(b) Let $\mu > p$ be an odd number, define $f(x) = \alpha x^{\mu}$. Taking $d > 1$ and $\alpha \geq d^{\mu-p}$ we see that (H_1) and (H_2) hold.

By a solution x of (4.1) we mean such a function $x : \mathbb{N}(0, T+1) \to \mathbb{R}$ which satisfies the given equation on $\mathbb{N}(1, T)$ and the boundary conditions. Solutions to (4.1) will be investigated in the space E of functions $x : \mathbb{N}(0, T+1) \to \mathbb{R}$ such that $x(0) = x(T+1) = 0$. The space E is considered with the norm

$$\|x\| = \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^p \right)^{\frac{1}{p}}.$$

We may also consider E with the following norm

$$\|x\|_0 = \left(\sum_{k=1}^{T} |x(k)|^p \right)^{\frac{1}{p}}.$$

Since E is finite dimensional, there exist constants $c_b = \frac{1}{2}$ and $c_a = \left(T(T+1)^{p-1}\right)^{1/p}$ such that

$$c_b \|x\| \leq \|x\|_0 \leq c_a \|x\| \text{ for all } x \in E. \tag{4.2}$$

Solutions to (4.1) correspond to the critical points to the following functional $\mathcal{I} : E \to R$

$$\mathcal{I}(u) = \sum_{k=1}^{T+1} \tfrac{1}{p} |\Delta x(k-1)|^p - \lambda \sum_{k=1}^{T} F(k, x(k)).$$

Lemma 4.2. *Assume that (H_0), (H_2) are satisfied. Then for any $\lambda > 0$ functional $\mathcal{I}$ is anticoercive, i.e. $\mathcal{I}(x) \to -\infty$ as $\|x\| \to +\infty$.*

Proof. From Hölder's inequality we get

$$\sum_{k=1}^{T} |x(k)|^p \leq \left(\sum_{k=1}^{T} |x(k)|^p|^{\frac{\mu}{p}} \right)^{\frac{p}{\mu}} \left(\sum_{k=1}^{T} |1|^{\frac{1}{1-\frac{p}{\mu}}} \right)^{1-\frac{p}{\mu}} = \left(\sum_{k=1}^{T} |x(k)|^{\mu} \right)^{\frac{p}{\mu}} T^{\frac{\mu-p}{\mu}}.$$

By (4.2), we have

$$\sum_{k=1}^{T} |x(k)|^{\mu} \geq T^{\frac{p-\mu}{p}} \|x\|_0^{\mu} \geq T^{\frac{p-\mu}{p}} (c_b)^{\mu} \|x\|^{\mu}. \tag{4.3}$$

By (H_2) and by (4.3), we obtain for any $x \in E$

$$\mathcal{I}(x) \leq \tfrac{1}{p}\|x\|^p - \lambda c_1 \sum_{k=1}^{T} |x(k)|^{\mu} - \lambda c_2 T \leq \tfrac{1}{p}\|x\|^p - \lambda c_1 T^{\frac{p-\mu}{p}} (c_b)^{\mu} \|x\|^{\mu} - c_2 T.$$

Hence $\mathcal{I}(x) \to -\infty$ as $\|x\| \to +\infty$ $\square$

Note that

$$|x(k)| \le \sum_{i=1}^{k-1} |\Delta x(i)| \quad \text{for} \quad k \in \mathbb{N}(1,T).$$

Therefore, for $k \in \mathbb{N}(1,T)$,

$$|x(k)|^p \le \left(\sum_{i=0}^{k-1} |\Delta x(i)|\right)^p \le \left(\sum_{i=0}^{T} |\Delta x(i)|\right)^p \le \left(\left(\sum_{i=0}^{T} 1^q\right)^{\frac{1}{q}} \cdot \left(\sum_{i=0}^{T} |\Delta x(i)|^p\right)^{\frac{1}{p}}\right)^p$$

$$= (T+1)^{\frac{p}{q}} \cdot \sum_{i=1}^{T+1} |\Delta x(i-1)|^p.$$

So

$$\|x\|_C = \max_{k\in\mathbb{N}(1,T)} |x(k)| \le \sqrt[q]{T+1}\, \|x\| . \tag{4.4}$$

With the above results we can formulate and prove the following theorem.

Theorem 4.3. *Assume that conditions* (H_0)–(H_2) *are satisfied. Assume that*

$$0 < \lambda \le \frac{1}{\alpha T (T+1)^{\frac{p}{q}}}.$$

Then problem (4.1) *has at least two solutions.*

Proof. Fix $\lambda \in \left(0, \frac{1}{\alpha T (T+1)^{\frac{p}{q}}}\right]$. We shall apply Theorem 3.4. By Lemma 4.2, the functional $\mathcal{I}$ is anticoercive. Put

$$\Phi(x) = \sum_{k=1}^{T+1} \tfrac{1}{p} |\Delta x(k-1)|^p, \quad H(x) = \lambda \sum_{k=1}^{T} F(k, x(k))$$

and note that these are convex C^1 functionals.

Let us define a set $D \subset E$ by

$$D = \{x \in E : |x(k)| \le d \text{ for } k \in \mathbb{N}(1,T)\}.$$

Since $\mathcal{I}$ is continuous and D is closed and bounded, we see that there exists an argument of a minimum of $\mathcal{I}$ over D, which we denote by u. Consider the auxiliary Dirichlet problem:

$$\begin{aligned} &-\Delta\left(\phi_p\left(\Delta x(k-1)\right)\right) = \lambda f\left(k, u(k)\right), \quad k \in \mathbb{N}(1,T), \\ &x(0) = x(T+1) = 0. \end{aligned} \tag{4.5}$$

Note that problem (4.5) is uniquely solvable by some $v \in E$. This follows since the action functional corresponding to (4.5)

$$J_p(x) = \sum_{k=1}^{T+1} \tfrac{1}{p} |\Delta x(k-1)|^p - \lambda \sum_{k=1}^{T} f(k, u(k)) x(k)$$

is coercive, C^1 and strictly convex. Multiplying

$$-\Delta\left(\phi_p\left(\Delta v(k-1)\right)\right) = \lambda f\left(k, u(k)\right), \quad k \in \mathbb{N}(1,T),$$

by v and summing from 1 to T we have what follows

$$\sum_{k=1}^{T+1} |\Delta v(k-1)|^p = \lambda \sum_{k=1}^{T} f(k,u(k))v(k).$$

By (4.4), we see that

$$\sum_{k=1}^{T+1} |\Delta v(k-1)|^p \geq (T+1)^{-\frac{p}{q}} \left(\|v\|_C\right)^p.$$

Note that since F is convex, so f is nondecreasing. Therefore, by (H_1), we see that $f(k,x) \leq \alpha d^{p-1}$ for all $x \in [-d,d]$. Thus

$$\lambda \sum_{k=1}^{T} f(k,u(k))v(k) \leq \lambda \alpha T d^{p-1} \|v\|_C.$$

Thus

$$(T+1)^{-\frac{p}{q}} \left(\|v\|_C\right)^p \leq \lambda \alpha T d^{p-1} \|v\|_C.$$

Note that $\lambda \leq \frac{1}{\alpha T (T+1)^{\frac{p}{q}}}$. Thus $\|v\|_C \leq d$ and $v \in D$. Hence Theorem 3.4 applies. $\square$

We finish by considering the case of the existence of at least one solution. The first of the corollaries follows by Theorem 3.1 and the arguments used in the above proof and the second is obtained since an anitcoercive C^1 functional in a finite dimensional setting has obviously at least one maximizer which is a critical point when a functional is sufficiently differentiable.

Corollary 4.4. *Assume that conditions (H_0), (H_1) are satisfied. Assume that*

$$0 < \lambda \leq \frac{1}{\alpha T (T+1)^{\frac{p}{q}}}.$$

Then problem (4.1) has at least one solution.

Corollary 4.5. *Assume that conditions (H_0), (H_2) are satisfied. Assume that $\lambda > 0$. Then problem (4.1) has at least one solution.*

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Received: February 23, 2014.
Revised: April 4, 2014.
Accepted: April 7, 2014.