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ON STABILITY OF THIN PERIODICALLY, DENSELY STIFFENED CYLINDRICAL SHELLS

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The aim of this contribution is to propose a new averaged nonasymptotic model of stationary stability problems for thin linear-elastic cylindrical shells reinforced by stiffeners which are periodically, densely spaced along one direction tangent to the shell midsurface. As a tool of modelling we shall apply *the tolerance averaging technique.* The resulting equations have constant coefficients in the periodicity direction. Moreover, in contrast with models obtained by *the asymptotic homogenization technique*, the proposed one makes it possible to describe the effect of the periodicity cell size on the global shell stability (*a length-scale effect*). It will be shown that this effect plays an important role in the shell stability analysis and cannot be neglected.

Key words: shell, stiffeners, modelling, stability, cell

1. Introduction

In this paper, a new model of stability analysis for cylindrical shells having periodic structure (a periodically varying thickness and/or periodically varying elastic properties) along one direction tangent to the undeformed shell midsurface $\mathcal{M}$ is presented. This situation is mainly oriented towards cylindrical shells reinforced by periodically spaced dense system of ribs as shown in Fig.1. Shells with a periodic structure along one direction tangent to $\mathcal{M}$ are termed *uniperiodic*.

We restrict our considerations to those uniperiodic cylindrical shells, which are composed of a large number of identical elements. Moreover, every such element is treated as a shallow shell. It means that the period of inhomogeneity is very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius, as well as the smallest characteristic length dimension of the shell midsurface in the periodicity direction.

It should be noted that in the general case, on the shell midsurface we deal with not periodic but with what is called a "locally periodic structure" in directions tangent to $\mathcal{M}$. Following Woźniak (1999), by a locally periodic shell we mean a shell which, in subregions of the shell midsurface $\mathcal{M}$ much smaller than $\mathcal{M}$ can be approximately regarded as periodic. Hence, a locally periodic shell is made of a large number of not identical but similar elements. However, for cylindrical shells the Gaussian curvature is equal to zero and hence on the developable cylindrical surface we can separate a cell which can be referred to as representative cell for the whole shell midsurface. It means that on cylindrical surface we deal not with locally periodic but with a periodic structure.

Problems of periodic (or locally periodic) structures are investigated by means of different methods. The exact analysis of shells and plates of this kind within solid mechanics can be carried out only for a few special problems. In the most cases, exact equations of the shell (plate) theory are too complicated to constitute the basis for investigations of most of the engineering problems because they involve highly oscillating and often discontinuous coefficients. Thus many different approximate modelling methods for periodic (locally periodic) shells and plates have been formulated.

Structures of this kind are usually described using homogenized models derived by means of asymptotic methods. These models from a formal point of view represent certain equivalent structures with constant or slowly varying stiffnesses and averaged mass densities. Unfortunately, in most cases, the asymptotic procedures are restriced to the first approximation, which leads to averaged models neglecting the effect of the periodicity cell length dimensions on the global structure behavior, called *the length-scale effect*, cf. Caillerie (1984), Kohn and Vogelius (1984), Lutoborski (1985), Lewiński and Telega (1988, 2000), Kolpakov (2000).

The periodically ribbed plates and shells are also modelled as homogeneous orthotropic structures, cf. Ambartsumyan (1974), Grigolyuk and Kabanow (1978). These orthotropic models are not able to describe the length-scale effect on the overall shell stability, being independent of the period of inhomogeneity.

The nonasymptotic modelling procedure based on the notion of tolerance and leading to the so-called length-scale (or tolerance) models of dynamic and stationary problems for micro-periodic (or locally periodic) structures was proposed by Woźniak in a series of papers, e.g. Woźniak (1993, 1999), Wierzbicki and Woźniak (2002), Woźniak and Wierzbicki (2000). These tolerance models have constant or slowly varying coefficients and take into account the effect of inhomogeneity period lengths on the global body behavior (the length-scale effect). This effect is described by means of certain extra unknowns called *internal or fluctuation variables* and by known functions which represent oscillations inside the periodicity cell, and are obtained either as approximate solutions to special eigenvalue problems for free vibrations on the separated cell with periodic boundary conditions or by using the finite element discretization of the cell. The averaged models of this kind have been applied to analyze certain dynamic and stability problems of periodic structures, e.g. for Hencky-Reissner periodic plates (Baron, 2003), for Kirchhoff periodic plates (Jędrysiak, 2000), for periodic beams (Mazur-Śniady, 1993), for periodic wavy-plates (Michalak, 1998), for thin periodically ribbed plates (Nagórko and Woźniak, 2002), for periodic cylindrical shells (Tomczyk, 1999, 2003) and others.

The mentioned above tolerance model for cylindrical shells with a periodic structure in both directions tangent to the shell midsurface $\mathcal{M}$ (Tomczyk, 1999) and that for cylindrical shells having periodic structure in only one direction tangent to $\mathcal{M}$ (Tomczyk, 2003), can be applied to investigations of dynamic problems but they cannot be used to analyze stability problems of the shells under consideration. That is why, in this paper the tolerance model of stationary stability problems for uniperiodically (i.e. periodically along one direction tangent to $\mathcal{M}$) densely stiffened cylindrical shells will be derived and discussed.

It has to be mentioned that an extremely extensive literature deals with elastic stability of thin cylindrical shells reinforced by widely spaced stiffeners. Contrary to the shells with densely spaced ribs, which are objects of considerations in this paper, those having widely spaced stiffeners are analyzed with allowance for the discreteness in the arrangement of the ribs. It means that the stability problems of such shells are considered within the framework of discrete models, while the stability analysis of periodically, densely ribbed cylindrical shells investigated in this paper is carried out within continuum models. The discrete approach is in detail discussed in monographs by Amiro and Zarutsky (1980) and Gavrylenko (1989). Moreover, in the mentioned monographs can be found an extensive review of papers and books dealing with stability problems of widely ribbed shells, as well as of densely stiffened shells treated as homogeneous orthotropic structures.

It is well known that stability problems of thin cylindrical shells being homogeneous or weekly heterogeneous have to be investigated by using the geometrically nonlinear shell theory, cf. Kármán and Tsien (1941), Volmir (1967), Brush and Almroth (1975), Pietraszkiewicz (1989). However, in the case of the highly heterogeneous structures considered here (i.e. densely ribbed shells) which are described by using continuum models, we are interested in the upper state of critical forces and hence we can use the geometrically linear stability theory for thin linear-elastic cylindrical Kirchhoff-Love type shells.

The aim of this contribution is three-fold:

- First, to formulate an averaged model of a uniperiodically densely stiffened cylindrical shell which has constant coefficients in the direction of periodicity and describes the effect of the cell size on the global shell stability. This model will be derived by using *the tolerance averaging procedure* proposed by Woźniak and Wierzbicki, (2000) and hence will be called *the tolerance fluctuation variable model of stability problems for uniperiodically densely stiffened cylindrical shells.*

- Second, to derive a simplified (homogenized) model in which the length-scale effect is neglected.

- Third, to show that the length-scale effect plays a crucial role in the shell stability problems and cannot be neglected. In order to illustrate this thesis, the critical forces of a special case of cylindrical shell will be determined and investigated by using both the tolerance and homogenized models.

Basic denotations, preliminary concepts and starting equations will be presented in Section 2. The general line of the tolerance averaging approach will be shown in Section 3. The tolerance model for stationary stability problems in linear-elastic thin cylindrical shells with a periodic structure (i.e. with periodically densely spaced stiffeners) along one direction tangent to $\mathcal{M}$ and slowly varying or constant structure along the perpendicular direction tangent to $\mathcal{M}$ will be proposed and discussed in Section 4. For comparison, the governing equations of a certain homogenized model will be given in Section 5. In the subsequent section, in order to evaluate the length-scale effect in stability problems, both the obtained tolerance and homogenized models will be applied to investigations of critical forces in open circular cylindrical shells reinforced by ribs, which are densely and periodically spaced along the lines of principal curvature of the shell midsurface. Final remarks will be formulated in the last section.

2. Preliminaries

In this paper we will investigate thin linear-elastic cylindrical shells periodically, densely ribbed along one direction tangent to $\mathcal{M}$. Cylindrical shells of this kind will be termed *uniperiodic or uniperiodically stiffened.* At the same time, the stiffened shells under consideration have slowly varying or constant structure (i.e. slowly varying or constant geometrical and/or material properties) along the direction tangent to $\mathcal{M}$ and perpendicular to the direction of periodicity. Examples of such shells are presented in Fig. 1.

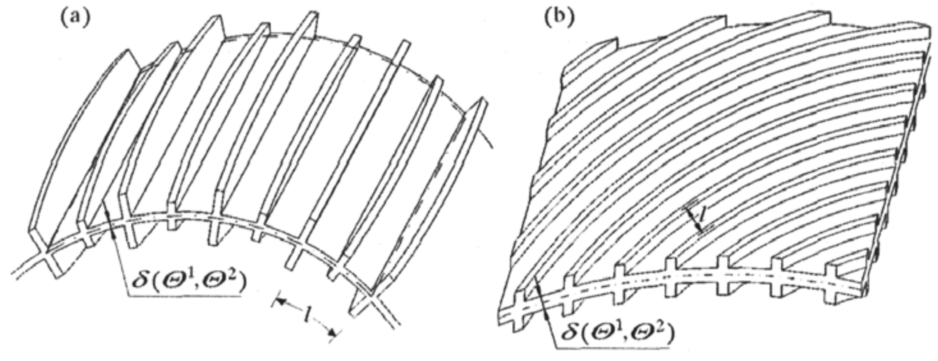

Fig. 1. Examples of uniperiodically stiffened shells

Denote by $\Omega \subset R^2$ a regular region of points $\boldsymbol{\Theta} \equiv (\Theta^1, \Theta^2)$ on the $O\Theta^1\Theta^2$-plane, Θ^1, Θ^2 being the Cartesian orthogonal coordinates on this plane and let E^3 be the physical space parametrized by the Cartesian orthogonal coordinate system $Ox^1x^2x^3$. Let us introduce the orthogonal parametric representation of the undeformed smooth cylindrical shell midsurface $\mathcal{M}$ by means of $\mathcal{M} := \{\boldsymbol{x} \equiv (x^1, x^2, x^3) \in E^3 : \ \boldsymbol{x} = \boldsymbol{x}(\Theta^1, \Theta^2), \ \boldsymbol{\Theta} \in \Omega\}$, where $\boldsymbol{x}(\Theta^1, \Theta^2)$ is the position vector of a point on $\mathcal{M}$ having coordinates Θ^1, Θ^2.

Throughout the paper indices $\alpha, \beta, \ldots$ run over 1,2 and are related to the midsurface parameters Θ^1, Θ^2; indices $A, B, \ldots$ run over $1, 2, \ldots, N$, summation convention holds for all aforesaid indices.

To every point $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega$ we assign a covariant base vectors $\boldsymbol{a}_\alpha = \boldsymbol{x}_{,\alpha}$ and covariant midsurface first and second metric tensors denoted by $a_{\alpha\beta}$, $b_{\alpha\beta}$, respectively, which are given as follows: $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$, $b_{\alpha\beta} = \boldsymbol{n} \cdot \boldsymbol{a}_{\alpha,\beta}$, where $\boldsymbol{n}$ is a unit vector normal to $\mathcal{M}$.

Let $\delta(\boldsymbol{\Theta})$ stand for the shell thickness.

Taking into account that coordinate lines $\Theta^2 =$ const are parallel on the $O\Theta^1\Theta^2$-plane and that Θ^2 is an arc coordinate on $\mathcal{M}$, we define l as the period of shell structure in Θ^2-direction. The period l is assumed to be sufficiently large compared with the maximum shell thickness and sufficiently small as compared with the midsurface curvature radius R as well as the characteristic length dimension L of the shell midsurface along the direction of shell periodicity, i.e. $\sup\delta(\cdot) \ll l \ll \min\{R, L\}$. Under the given above assumptions for period l, the shell under consideration will be referred to as *a mesostructured shell*, cf. Woźniak (1999), and the period l will be called *the mesostructure length parameter*.

We shall denote by $\Lambda \equiv \{0\} \times (-l/2, l/2)$ the straight line segment on the $O\Theta^1\Theta^2$-plane along the $O\Theta^2$-axis direction, which can be taken as a representative cell of the periodic shell structure (the periodicity cell). To every $\boldsymbol{\Theta} \in \Omega$ an arbitrary cell on $O\Theta^1\Theta^2$-plane will be defined by means of: $\Lambda(\boldsymbol{\Theta}) \equiv \boldsymbol{\Theta} + \Lambda$, $\boldsymbol{\Theta} \in \Omega_\Lambda$, $\Omega_\Lambda := \{\boldsymbol{\Theta} \in \Omega : \ \Lambda(\boldsymbol{\Theta}) \subset \Omega\}$, where the point $\boldsymbol{\Theta} \in \Omega_\Lambda$ is the center of a cell $\Lambda(\boldsymbol{\Theta})$ and Ω_Λ is a set of all the cell centers which are inside Ω.

A function $f(\boldsymbol{\Theta})$ defined on Ω_Λ will be called Λ-periodic if for arbitrary but fixed Θ^1 and arbitrary Θ^2, $\Theta^2 \pm l$ it satisfies the condition: $f(\Theta^1, \Theta^2) = f(\Theta^1, \Theta^2 \pm l)$ in the whole domain of its definition and it is not constant.

It is assumed that the cylindrical shell thickness as well as its material properties are Λ-periodic functions of Θ^2 and slowly varying functions of Θ^1. Shells like that are called *uniperiodic*, moreover under the given above assumptions for period l they are referred to as *the mesostructured shells*.

The above periodic heterogeneities can be also interpreted as those caused by a periodically spaced dense system of ribs, as shown in Fig. 1.

For an arbitrary integrable function $\varphi(\cdot)$ defined on Ω, following Woźniak and Wierzbicki (2000), we define *the averaging operation*, given by

$$\langle\varphi\rangle(\boldsymbol{\Theta}) \equiv \frac{1}{l} \int\limits_{\Lambda(\boldsymbol{\Theta})} \varphi(\Theta^1, \Psi^2)\, d\Psi^2 \qquad \boldsymbol{\Theta} = (\Theta^1, \Theta^2) \in \Omega_\Lambda \tag{2.1}$$

For a function φ, which is Λ-periodic in Θ^2, formula (2.1) leads to $\langle\varphi\rangle(\Theta^1)$. If the functions φ is Λ-periodic in Θ^2 and is independent of Θ^1, its averaged value obtained from (2.1) is constant.

Our considerations will be based on the simplified linear Kirchhoff-Love second-order theory of thin elastic shells in which terms depending on the second metric tensor of $\mathcal{M}$ are neglected in the formulae for curvature changes. Below, we quote the general formulations of the theory under consideration.

2.1. The Kirchhoff-Love shell equations

Let $u_\alpha(\boldsymbol{\Theta})$, $w(\boldsymbol{\Theta})$ stand for the midsurface shell displacements in directions tangent and normal to $\mathcal{M}$, respectively. We denote by $\varepsilon_{\alpha\beta}(\boldsymbol{\Theta})$, $\kappa_{\alpha\beta}(\boldsymbol{\Theta})$

the membrane and curvature strain tensors and by $n^{\alpha\beta}(\boldsymbol{\Theta})$, $m^{\alpha\beta}(\boldsymbol{\Theta})$ the stress resultants and stress couples, respectively. The properties of the shell are described by 2D-shell stiffness tensors $D^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$, $B^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$. Let $f_\alpha(\boldsymbol{\Theta})$, $f(\boldsymbol{\Theta})$ be external force components per midsurface unit area, respectively tangent and normal to $\mathcal{M}$.

Functions $D^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$, $B^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$ and $\delta(\boldsymbol{\Theta})$ are Λ-periodic functions of Θ^2 and are assumed to be slowly varying functions of Θ^1.

We denote by $\overline{N}^{\alpha\beta}$ the constant compressive membrane forces in the shell midsurface, which satisfy the following equations of equilibrium: $\overline{N}^{\alpha\beta}_{,\alpha} + f^\beta = 0$, $b_{\alpha\beta}\overline{N}^{\alpha\beta} + f = 0$.

The simplified linear Kirchhoff-Love second-order theory of thin elastic cylindrical shells is governed by:

— the strain-displacement equations

$$\varepsilon_{\gamma\delta} = u_{(\gamma,\delta)} - b_{\gamma\delta}w \qquad\qquad \kappa_{\gamma\delta} = -w_{,\gamma\delta} \tag{2.2}$$

— the stress-strain relations

$$n^{\alpha\beta} = D^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \qquad\qquad m^{\alpha\beta} = B^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \tag{2.3}$$

— the equations of equilibrium

$$\begin{aligned} &n^{\alpha\beta}_{,\alpha} = 0 \qquad\qquad \overline{N}^{\alpha\beta} = \text{const} \\ &m^{\alpha\beta}_{,\alpha\beta} + b_{\alpha\beta}n^{\alpha\beta} - \overline{N}^{\alpha\beta}w_{,\alpha\beta} = 0 \end{aligned} \tag{2.4}$$

These equations take also into account the dense system of ribs.

In the above equations the displacements $u_\alpha = u_\alpha(\boldsymbol{\Theta})$ and $w = w(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega$, are the basic unknowns.

For uniperiodically densely ribbed shells, $D^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$ and $B^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega$, are non-continuous highly oscillating Λ-periodic functions; that is why equations (2.2)-(2.4) cannot be directly applied to the numerical analysis of special problems. From (2.2)-(2.4) an averaged model of uniperiodic cylindrical shells under consideration having coefficients, which are independent of the Θ^2-midsurface parameter and are slowly varying functions of Θ^1 as well as describing the cell size effect on critical forces, will be derived. In order to derive it, *the tolerance averaging procedure* given by Woźniak and Wierzbicki (2000) will be applied. To make the analysis more clear, in the next section we shall outline the basic concepts and the main assumptions of this approach, following the monograph by Woźniak and Wierzbicki (2000).

3. Modelling concepts and assumptions

Following the monograph by Woźniak and Wierzbicki (2000), we outline below the basic concepts and assumptions which will be used in the course of modelling procedure.

3.1. Basic concepts

The fundamental concepts of the tolerance averaging approach are those of a certain *tolerance system, locally slowly varying functions, periodic-like functions and periodic-like oscillating functions.* These functions will be defined with respect to the Λ-periodic shell structure defined in the foregoing section.

By *a tolerance system* we shall mean a pair $T = (\mathcal{F}, \varepsilon(\cdot))$, where $\mathcal{F}$ is a set of real-valued bounded functions $F(\cdot)$ defined on $\overline{\Omega}$ and their derivatives (including also time derivatives), which represent the unknowns in the problem under consideration (such as unknown shell displacements tangent and normal to $\mathcal{M}$) and for which the tolerance parameters ε_F being positive real numbers and determining the admissible accuracy related to computations of values of $F(\cdot)$ are given; by ε is denoted the mapping $\mathcal{F} \ni F \to \varepsilon_F$.

A continuous bounded differentiable function $F(\boldsymbol{\Theta}, t)$ defined on $\overline{\Omega}$ is called *locally slowly varying* (or *Λ-slowly varying*) with respect to the cell Λ and the tolerance system T, $F \in SV_\Lambda(T)$ if, roughly speaking, it can be treated (together with its derivatives) as constant on an arbitrary periodicity cell Λ.

The continuous function $\varphi(\cdot)$ defined on $\overline{\Omega}$ will be termed a *Λ-periodic-like function*, $\varphi(\cdot) \in PL_\Lambda(T)$, with respect to the cell Λ and the tolerance system T, if for every $\boldsymbol{\Theta} = (\Theta^1, \Theta^2) \in \Omega_\Lambda$ there exists a continuous Λ-periodic function $\varphi_{\boldsymbol{\Theta}}(\cdot)$ such that $(\forall \boldsymbol{\Psi} = (\Theta^1, \Psi^2))[\|\boldsymbol{\Theta} - \boldsymbol{\Psi}\| \leqslant l \Rightarrow \varphi(\boldsymbol{\Psi}) \cong \varphi_{\boldsymbol{\Theta}}(\boldsymbol{\Psi})]$, $\boldsymbol{\Psi} \in \Lambda(\boldsymbol{\Theta})$, and if similar conditions are also fulfilled by all its derivatives. It means that the values of a periodic-like function $\varphi(\cdot)$ in an arbitrary cell $\Lambda(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega_\Lambda$, can be approximated, with sufficient accuracy, by the corresponding values of a certain Λ-periodic function $\varphi_{\boldsymbol{\Theta}}(\cdot)$. The function $\varphi_{\boldsymbol{\Theta}}(\cdot)$ will be referred to as a Λ-periodic approximation of $\varphi(\cdot)$ on $\Lambda(\boldsymbol{\Theta})$.

Let $\mu(\cdot)$ be a positive Λ-periodic function. *The periodic-like function φ is* called *Λ-oscillating* (with the weight μ), $\varphi(\cdot) \in PL^\mu_\Lambda(T)$, provided that the condition $\langle \mu\varphi \rangle(\boldsymbol{\Theta}) \cong \mathbf{0}$ holds for every $\boldsymbol{\Theta} \in \Omega_\Lambda$ in the special case $\mu = \text{const}$ the oscillating periodic-like function satisfies condition $\langle \varphi \rangle(\boldsymbol{\Theta}) \cong \mathbf{0}$, $\boldsymbol{\Theta} \in \Omega_\Lambda$; in this case we shall write $\varphi \in PL^1_\Lambda(T)$.

3.2. Modelling assumptions

The tolerance averaging technique is based on two modelling assumptions. The first of them is strictly related to the concept of locally slowly varying and periodic-like functions.

Tolerance Averaging Assumption. *If $F \in SV_\Lambda(T)$, $\varphi(\cdot) \in PL_\Lambda(T)$ and $\varphi_{\boldsymbol{\Theta}}(\cdot)$ is a Λ-periodic approximation of $\varphi(\cdot)$ on $\Lambda(\boldsymbol{\Theta})$ then, for every Λ-periodic bounded function $f(\cdot)$ and every continuous Λ-periodic differentiable function $h(\cdot)$ such that $\sup\{|h(\Psi^1,\Psi^2)|,\ (\Psi^1,\Psi^2) \in \Lambda\} \leqslant l$, the following tolerance averaging relations determined by the pertinent tolerance parameters hold for every $\boldsymbol{\Theta} \in \Omega_\Lambda$:*

$$\textbf{(T1)} \qquad \langle fF\rangle(\boldsymbol{\Theta}) \cong \langle f\rangle(\boldsymbol{\Theta})F(\boldsymbol{\Theta})$$

$$\textbf{(T2)} \qquad \langle f(hF)_{,2}\rangle(\boldsymbol{\Theta}) \cong \langle fFh_{,2}\rangle(\boldsymbol{\Theta})$$

$$\textbf{(T3)} \qquad \langle f\varphi\rangle(\boldsymbol{\Theta}) \cong \langle f\varphi_{\boldsymbol{\Theta}}\rangle(\boldsymbol{\Theta})$$

$$\textbf{(T4)} \qquad \langle h(f\varphi)_{,2}\rangle(\boldsymbol{\Theta}) \cong -\langle f\varphi h_{,2}\rangle(\boldsymbol{\Theta})$$

It means that in the course of averaging, the left-hand sides of formulae **(T1)**-**(T4)** can be approximated by their right-hand sides, respectively.

The second modelling assumption is based on heuristic premises.

Conformability Assumption. *In every periodic solid the displacement fields have to conform to the periodic structure of this solid. It means that the displacement fields are periodic-like functions and hence can be represented by a sum of averaged displacements, which are locally slowly varying, and by highly oscillating periodic-like disturbances, caused by the periodic structure of the solid.*

The aforementioned *Conformability Assumption* together with the *Tolerance Averaging Assumption* constitute the foundations of *the tolerance averaging technique*. Using this technique, *the tolerance model of stationary stability problems for uniperiodically densely stiffened cylindrical shells* will be derived in the subsequent section. It can be mentioned that this tolerance averaging method has been used to derive *a tolerance model of dynamic problems for uniperiodic cylindrical shells* in the paper by Tomczyk (2003). However, this model cannot be applied to the investigation of stability problems.

4. The tolerance model of stability problems for uniperiodically densely stiffened cylindrical shells

4.1. Modelling procedure

Let us assume that there is a certain tolerance system $T = (\mathcal{F}, \varepsilon(\cdot))$, where the set $\mathcal{F}$ consists of the unknown shell displacements tangent and normal to $\mathcal{M}$ and their derivatives.

The tolerance averaging approach to Eqs. (2.2)-(2.4) will be realized in five steps.

Step 1. From the *Conformability Assumption* it follows that the unknown shell displacements $u_\alpha(\boldsymbol{\Theta})$, $w(\boldsymbol{\Theta})$ in Eqs. (2.2)-(2.4) have to satisfy the conditions: $u_\alpha(\boldsymbol{\Theta}) \in PL_\Lambda(T)$, $w(\boldsymbol{\Theta}) \in PL_\Lambda(T)$. It means that in every cell $\Lambda(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega_\Lambda$, the displacement fields can be represented, within a tolerance, by their periodic approximations. A simple consequence of the *Conformability Assumption* is co called the *modelling decomposition*

$$\begin{aligned} &u_\alpha(\boldsymbol{\Theta}) = U_\alpha(\boldsymbol{\Theta}) + d_\alpha(\boldsymbol{\Theta}) && w(\boldsymbol{\Theta}) = W(\boldsymbol{\Theta}) + p(\boldsymbol{\Theta}) \\ &U_\alpha(\boldsymbol{\Theta}), W(\boldsymbol{\Theta}) \in SV_\Lambda(T) && d_\alpha(\boldsymbol{\Theta}), p(\boldsymbol{\Theta}) \in PL^1_\Lambda(T) \end{aligned} \tag{4.1}$$

which appears under the normalizing condition $\langle d_\alpha(\boldsymbol{\Theta})\rangle = \langle p(\boldsymbol{\Theta})\rangle = 0$.

It can be shown, cf. Woźniak and Wierzbicki (2000), that the unknown locally slowly varying functions $U_\alpha(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$ in (4.1) are given by: $U_\alpha(\boldsymbol{\Theta}) \equiv \langle u_\alpha\rangle(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta}) \equiv \langle w\rangle(\boldsymbol{\Theta})$. Functions $U_\alpha(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$ represent the averaged parts of displacements $u_\alpha(\boldsymbol{\Theta})$, $w(\boldsymbol{\Theta})$, respectively and are called *macrodisplacements*.

The unknown displacement disturbances $d_\alpha(\boldsymbol{\Theta})$, $p(\boldsymbol{\Theta})$ in (4.1) being oscillating periodic-like functions are caused by the highly oscillating character of the shell mesostructure.

Step 2. Substituting the right-hand side of (4.1) into (2.4) and after the tolerance averaging of the resulting equations, we arrive at the equations

$$\begin{aligned} &[\langle D^{\alpha\beta\gamma\delta}\rangle(\Theta^1)(U_{\gamma,\delta} - b_{\gamma\delta}W) + \langle D^{\alpha\beta\gamma\delta}d_{\gamma,\delta}\rangle(\boldsymbol{\Theta}) - b_{\gamma\delta}\langle D^{\alpha\beta\gamma\delta}p\rangle(\boldsymbol{\Theta})]_{,\alpha} = 0 \\ &[\langle B^{\alpha\beta\gamma\delta}\rangle(\Theta^1)W_{,\gamma\delta} + \langle B^{\alpha\beta\gamma\delta}p_{,\gamma\delta}\rangle(\boldsymbol{\Theta})]_{,\alpha\beta} - b_{\alpha\beta}[\langle D^{\alpha\beta\gamma\delta}\rangle(\Theta^1)(U_{\gamma,\delta} - b_{\gamma\delta}W) + \\ &+\langle D^{\alpha\beta\gamma\delta}d_{\gamma,\delta}\rangle(\boldsymbol{\Theta}) - b_{\gamma\delta}\langle D^{\alpha\beta\gamma\delta}p\rangle] + \overline{N}^{\alpha\beta}W_{,\alpha\beta} = 0 \end{aligned} \tag{4.2}$$

which must hold for every $\boldsymbol{\Theta} \in \Omega_\Lambda$.

Step 3. Multiplying Eqs. $(2.4)_1$ and $(2.4)_2$ by arbitrary Λ-periodic test functions d^*, p^*, respectively, such that $\langle d^*\rangle = \langle p^*\rangle = 0$, integrating these equations over $\Lambda(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega_\Lambda$, and using the *Tolerance Averaging Assumption*, as well as denoting by $\widetilde{d}_\alpha$, $\widetilde{p}$ the Λ-periodic approximations of d_α, p, respectively, on $\Lambda(\boldsymbol{\Theta})$, we obtain the periodic problem on $\Lambda(\boldsymbol{\Theta})$ for functions $\widetilde{d}_\alpha(\Theta^1, \Psi^2)$, $\widetilde{p}(\Theta^1, \Psi^2)$, $(\Theta^1, \Psi^2) \in \Lambda(\boldsymbol{\Theta}) = \Lambda(\Theta^1, \Theta^2)$, given by the following variational conditions

$$\begin{aligned}
&-\langle d_{,2}^{*}\, D^{2\beta\gamma\delta}\widetilde{d}_{\gamma,\delta}\rangle + \langle d^*(D^{1\beta\gamma\delta}\widetilde{d}_{\gamma,\delta})_{,1}\rangle - b_{\gamma\delta}[-\langle d_{,2}^{*}\, D^{2\beta\gamma\delta}\widetilde{p}\rangle + \langle d^*(D^{1\beta\gamma\delta}\widetilde{p})_{,1}\rangle] = \\
&= \langle d_{,\alpha}^{*}\, D^{\alpha\beta\gamma\delta}\rangle(U_{\gamma,\delta} - b_{\gamma\delta}W) - [\langle d^* D^{1\beta\gamma\delta}\rangle(U_{\gamma,\delta} - b_{\gamma\delta}W)]_{,1} \\
&\langle p_{,22}^{*}\, B^{22\gamma\delta}\widetilde{p}_{,\gamma\delta}\rangle - 2\langle p_{,2}^{*}\,(B^{21\gamma\delta}\widetilde{p}_{,\gamma\delta})_{,1}\rangle + \langle p^*(B^{11\gamma\delta}\widetilde{p}_{,\gamma\delta})_{,11}\rangle + \\
&-b_{\alpha\beta}[\langle p^* D^{\alpha\beta\gamma\delta}\widetilde{d}_{\gamma,\delta}\rangle - b_{\gamma\delta}\langle p^* D^{\alpha\beta\gamma\delta}\widetilde{p}\rangle] + \overline{N}^{11}\langle p^*\widetilde{p}_{,11}\rangle + 2\overline{N}^{12}\langle p^*\widetilde{p}_{,12}\rangle + \\
&+\overline{N}^{22}\langle p^*\widetilde{p}_{,22}\rangle = b_{\alpha\beta}\langle p^* D^{\alpha\beta\gamma\delta}\rangle(U_{\gamma,\delta} - b_{\gamma\delta}W) + \\
&-\langle p_{,22}^{*}\, B^{22\lambda\delta}\rangle W_{,\gamma\delta} + 2[(\langle p_{,2}^{*}\, B^{21\gamma\delta}\rangle_{,1} - \langle p_{,21}^{*}\, B^{21\gamma\delta}\rangle)W_{,\gamma\delta} + \langle p_{,2}^{*}\, B^{21\gamma\delta}\rangle W_{,\gamma\delta 1}] + \\
&-\{[(\langle p^* B^{11\gamma\delta}\rangle_{,1} - 2\langle p_{,1}^{*}\, B^{11\lambda\delta}\rangle)_{,1} + \langle p_{,11}^{*}\, B^{11\gamma\delta}\rangle]W_{,\gamma\delta} + 2(\langle p^* B^{11\gamma\delta}\rangle_{,1} + \\
&-\langle p_{,1}^{*}\, B^{11\gamma\delta}\rangle)W_{,\gamma\delta 1} + \langle p^* B^{11\gamma\delta}\rangle W_{,\gamma\delta 11}\}
\end{aligned} \tag{4.3}$$

Conditions $(4.3)_1$ and $(4.3)_2$ must hold for every Λ-periodic test function d^* and for every Λ-periodic test function p^*, respectively.

Equations (4.2), (4.3) represent the basis for obtaining the tolerance model for analyzing stationary stability problems of linear elastic cylindrical shells reinforced by ribs periodically densely spaced in one direction tangent to $\mathcal{M}$.

Step 4. In order to obtain solutions to the periodic problems on $\Lambda(\boldsymbol{\Theta})$, given by the variational equations (4.3), we can apply the known orthogonalization method. Hence, for arbitrary Θ^1 and $(\Theta^1, \Psi^2) \in \Lambda(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} = (\Theta^1, \Theta^2) \in \Omega_\Lambda$ we can look for solutions to the periodic problem (4.3) in the form of the finite series

$$\begin{aligned}
&\widetilde{d}_\alpha(\Theta^1, \Psi^2) = h^A(\Theta^1, \Psi^2)Q_\alpha^A(\Theta^1, \Theta^2) \\
&\widetilde{p}(\Theta^1, \Psi^2) = g^A(\Theta^1, \Psi^2)V^A(\Theta^1, \Theta^2) \qquad A = 1, 2, \ldots, N
\end{aligned} \tag{4.4}$$

in which the choice of the number N of terms in the finite sums determines different degrees of approximations and where $Q_\alpha^A(\Theta^1, \Theta^2)$, $V^A(\Theta^1, \Theta^2)$ are new unknowns called *fluctuation variables*, being locally slowly varying functions in Θ^2, i.e. $Q_\alpha^A, V^A \in SV_\Lambda(T)$. Moreover, $h^A(\Theta^1, \Psi^2)$, $g^A(\Theta^1, \Psi^2)$, $A = 1, \ldots, N$, are known in every problem under consideration, linear-independent, l-periodic functions such that $h^A, lh_{,2}^{A}, l^{-1}g^A, g_{,2}^{A}, lg_{,22}^{A} \in \mathcal{O}(l)$,

$\max\|h^A(\Theta^1,\Psi^2)| \leqslant l$, $\max|g^A(\Theta^1,\Psi^2)| \leqslant l^2$, $\langle h^A\rangle(\Theta^1) = \langle g^A\rangle(\Theta^1) = 0$ for every A, $\langle h^A h^B\rangle(\Theta^1) = \langle g^A g^B\rangle(\Theta^1) = 0$ for every $A \neq B$.

Functions $h^A(\Theta^1,\Psi^2)$, $g^A(\Theta^1,\Psi^2)$, $A = 1,2,\ldots,N$, in (4.4) can be derived from the periodic Finite Element Method discretization of the cell and hence will be referred to as *the shape functions*. It can be observed that in many cases this discretization of the cell requires a large number of finite elements and consequently, the number N of extra unknowns Q^A_α, V^A in (4.4) is also large.

The functions $h^A(\Theta^1,\Psi^2)$, $g^A(\Theta^1,\Psi^2)$, $A = 1,\ldots,N$, can also be obtained as exact or approximate solutions to certain periodic eigenvalue problems on the cell describing free periodic vibrations of the stiffened shell. It means that the functions h^A, g^A represent the expected forms of free periodic vibration modes of an arbitrary cell and hence are referred to as the *mode-shape functions*. Following Tomczyk (2003), this periodic eigenvalue problem of finding Λ-periodic eigenfunctions $h_\alpha(\Theta^1,\Psi^2)$, $g(\Theta^1,\Psi^2)$, $(\Theta^1,\Psi^2) \in \Lambda(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} = (\Theta^1,\Theta^2) \in \Omega_\Lambda$ is given by the equations

$$
\begin{aligned}
&[D^{2\beta\gamma 2}(\Theta^1,\Psi^2)h_{\gamma,2}(\Theta^1,\Psi^2)]_{,2} + \mu(\Theta^1,\Psi^2)[\omega(\Theta^1)]^2 a^{\alpha\beta} h_\alpha(\Theta^1,\Psi^2) = 0 \\
&[B^{2222}(\Theta^1,\Psi^2)g_{,22}(\Theta^1,\Psi^2)]_{,22} - \mu(\Theta^1,\Psi^2)[\omega(\Theta^1)]^2 g(\Theta^1,\Psi^2) = 0
\end{aligned}
\tag{4.5}
$$

and by the periodic boundary conditions on the cell $\Lambda(\boldsymbol{\Theta})$ together with the continuity conditions inside $\Lambda(\boldsymbol{\Theta})$; by $\mu(\Theta^1,\Psi^2)$, $a^{\alpha\beta}$ and ω we have denoted the shell mass density per midsurface unit area, the contravariant midsurface first metric tensor and the free vibration frequency, respectively. By averaging the above equations over $\Lambda(\boldsymbol{\Theta})$ we obtain $\langle\mu h_\alpha\rangle(\Theta^1) = \langle\mu g\rangle(\Theta^1) = 0$.

Thus, $[h^1_\alpha(\Theta^1,\Psi^2), g^1(\Theta^1,\Psi^2)]$, $[h^2_\alpha(\Theta^1,\Psi^2), g^2(\Theta^1,\Psi^2)], \ldots$ is a sequence of eigenfunctions related to the sequence of eigenvalues $[\omega^2_\alpha,\omega^2]_1$, $[\omega^2_\alpha,\omega^2]_2, \ldots$. In the modelling procedure this sequence is restricted to the $N \geqslant 1$ eigenfunctions. Moreover, in most problems the analysis will be restricted to the simplest case $N = 1$ in which we take into account only the lowest natural vibration modes (in directions tangent and normal to $\mathcal{M}$) related to Eqs. (4.5). In this paper it is assumed that $h^A_1 = h^A_2$ and hence we denote $h^A \equiv h^A_1 = h^A_2$.

Step 5. Substituting the right-hand sides of (4.4) into (4.2) and (4.3) and setting $d^* = h^A(\Theta^1,\Psi^2)$, $p^* = g^A(\Theta^1,\Psi^2)$, $A = 1,2,\ldots,N$, in (4.3), on the basis of the *Tolerance Averaging Assumption we arrive at the tolerance fluctuation variable model of stability problems for uniperiodically densely stiffened cylindrical shells.* In the next subsection the equations of this model will be given and discussed.

4.2. Governing equations

In the previous subsection, applying the tolerance averaging of Kirchhoff-Love second-order shell equations we have arrived at the *tolerance fluctuation variable model of stationary stability problems for uniperiodically densely stiffened cylindrical shells.*

Under extra denotations

$$
\begin{array}{ll}
\widetilde{D}^{\alpha\beta\gamma\delta} \equiv \langle D^{\alpha\beta\gamma\delta} \rangle & D^{A\alpha\beta\gamma} \equiv \langle D^{\alpha\beta\gamma\delta} h,^{A}_{\delta} \rangle \\
\overline{D}^{A\alpha\beta\gamma} \equiv l^{-1} \langle D^{\alpha\beta\gamma 1} h^{A} \rangle & L^{A\alpha\beta} \equiv l^{-2} b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} g^{A} \rangle \\
\widetilde{B}^{\alpha\beta\gamma\delta} \equiv \langle B^{\alpha\beta\gamma\delta} \rangle & K^{A\alpha\beta} \equiv \langle B^{\alpha\beta\gamma\delta} g,^{A}_{\gamma\delta} \rangle \\
\overline{K}^{A\alpha\beta} \equiv l^{-1} \langle B^{\alpha\beta 1\delta} g,^{A}_{\delta} \rangle & \breve{K}^{A\alpha\beta} \equiv l^{-2} \langle B^{\alpha\beta 11} g^{A} \rangle \\
C^{AB\beta\gamma} \equiv \langle D^{\alpha\beta\gamma\delta} h,^{A}_{\alpha} h,^{B}_{\delta} \rangle & \overline{C}^{AB\beta\gamma} \equiv l^{-1} \langle D^{\alpha\beta\gamma 1} h,^{A}_{\alpha} h^{B} \rangle \\
F^{AB\beta} \equiv l^{-2} b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} h,^{A}_{\alpha} g^{B} \rangle & \widetilde{C}^{AB\beta\gamma} \equiv l^{-2} \langle D^{1\beta\gamma 1} h^{A} h^{B} \rangle \\
\overline{F}^{AB\beta} \equiv l^{-3} b_{\gamma\delta} \langle D^{1\beta\gamma\delta} h^{A} g^{B} \rangle & S^{AB} \equiv \langle B^{\alpha\beta\gamma\delta} g,^{A}_{\alpha\beta} g,^{B}_{\gamma\delta} \rangle \\
\overline{L}^{AB} \equiv l^{-4} b_{\alpha\beta} b_{\gamma\delta} \langle D^{\alpha\beta\gamma\delta} g^{A} g^{B} \rangle & \breve{R}^{AB} \equiv l^{-1} \langle B^{1\beta\gamma\delta} g,^{A}_{\beta} g,^{B}_{\gamma\delta} \rangle \\
\widetilde{R}^{AB} \equiv l^{-2} \langle B^{11\gamma\delta} g,^{A}_{\gamma\delta} g^{B} \rangle & \overline{R}^{AB} \equiv l^{-3} \langle B^{1\beta 11} g,^{A}_{\beta} g^{B} \rangle \\
\overset{\frown}{R}{}^{AB} \equiv l^{-4} \langle B^{1111} g^{A} g^{B} \rangle & \widetilde{S}^{AB} \equiv l^{-2} \langle B^{1\gamma 1\delta} g,^{A}_{\gamma} g,^{B}_{\delta} \rangle \\
T^{AB} \equiv l^{-2} \langle g,^{A}_{2} g,^{B}_{2} \rangle & \breve{T}^{AB} \equiv l^{-3} \langle g,^{A}_{2} g^{B} \rangle \\
\overset{\frown}{T}{}^{AB} \equiv l^{-1} \langle g,^{A}_{2} g,^{B}_{1} \rangle & \widetilde{\widetilde{T}}^{AB} \equiv l^{-2} \langle g,^{A}_{11} g^{B} \rangle \\
\widetilde{T}^{AB} \equiv l^{-2} \langle g,^{A}_{1} g^{B} \rangle & \overline{T}^{AB} \equiv l^{-4} \langle g^{A} g^{B} \rangle
\end{array}
\tag{4.6}
$$

this model is represented by:

– the constitutive equations

$$
\begin{aligned}
N^{\alpha\beta} &= \widetilde{D}^{\alpha\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + D^{B\alpha\beta\gamma} Q^{B}_{\gamma} + \underline{l\overline{D}^{B\alpha\beta\gamma}} Q^{B}_{\gamma,1} - \underline{l^{2} L^{B\alpha\beta}} V^{B} \\
M^{\alpha\beta} &= \widetilde{B}^{\alpha\beta\gamma\delta} W,_{\gamma\delta} + K^{B\alpha\beta} V^{B} + \underline{2l\overline{K}^{B\alpha\beta}} V,^{B}_{1} + \underline{l^{2}\breve{K}^{B\alpha\beta}} V,^{B}_{11} \\
H^{A\beta} &= D^{A\beta\gamma\delta}(U_{\gamma,\delta} - b_{\gamma\delta}W) + C^{AB\beta\gamma} Q^{B}_{\gamma} + \underline{l\overline{C}^{AB\beta\gamma}} Q^{B}_{\gamma,1} - \underline{l^{2} F^{AB\beta}} V^{B} \\
\overline{H}^{A\beta} &\equiv \underline{l\overline{D}^{A\beta\gamma\delta}}(U_{\gamma,\delta} - b_{\gamma\delta}W) + \underline{l\overline{C}^{AB\beta\gamma}} Q^{B}_{\gamma} + \underline{l^{2}\widetilde{C}^{AB\beta\gamma}} Q^{B}_{\gamma,1} - \underline{l^{3}\overline{F}^{AB\beta}} V^{B}
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
G^A &\equiv -\underline{l^2 L^{A\gamma\delta}}(U_{\gamma,\delta} - b_{\gamma\delta}W) + K^{A\alpha\beta}W_{,\alpha\beta} - \underline{l^2 F^{AB\gamma}}Q^B_\gamma - \underline{l^3 \overline{F}^{AB\gamma}}Q^B_{\gamma,1} + \\
&\quad + (S^{AB} + \underline{l^4 \overline{L}^{AB}})V^B + 2\underline{l\breve{R}^{AB}}V^B_{,1} + \underline{l^2 \widetilde{R}^{AB}}V^B_{,11} \\
\widetilde{G}^A &= \underline{l^2 \breve{K}^{A\alpha\beta}}W_{,\alpha\beta} + \underline{l^2 \widetilde{R}^{AB}}V^B + 2\underline{l^3 \overline{R}^{AB}}V^B_{,1} + \underline{l^4 \widehat{R}^{AB}}V^B_{,11} \\
\overline{G}^A &= \underline{l\overline{K}^{A\alpha\beta}}W_{,\alpha\beta} + \underline{l\breve{R}^{AB}}V^B + 2\underline{l^2 \widetilde{S}^{AB}}V^B_{,1} + \underline{l^3 \overline{R}^{AB}}V^B_{,11}
\end{aligned}
$$

– the system of three averaged partial differential equations of equilibrium for macrodisplacements $U_\alpha(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$

$$
\begin{aligned}
&N^{\alpha\beta}_{,\alpha} = 0 \\
&M^{\alpha\beta}_{,\alpha\beta} - b_{\alpha\beta}N^{\alpha\beta} + \overline{N}^{\alpha\beta}W_{,\alpha\beta} = 0
\end{aligned}
\tag{4.8}
$$

— the system of $3N$ partial differential equations for the fluctuation variables $Q^B_\alpha(\boldsymbol{\Theta})$, $V^B(\boldsymbol{\Theta})$, $B = 1,2,\ldots,N$

$$
\begin{aligned}
&H^{A\beta} - \overline{H}^{A\beta}_{,1} = 0 \\
&G^A + \widetilde{G}^A_{,11} - 2\overline{G}^A_{,1} + \overline{N}^{11}(\underline{l^2 \widetilde{\widetilde{T}}^{AB}}V^B + 2\underline{l^2 \widetilde{T}^{AB}}V^B_{,1} + \underline{l^4 \overline{T}^{AB}}V^B_{,11}) + \\
&\quad + 2\overline{N}^{12}(\underline{l^3 \breve{T}^{AB}}V^B_{,1} + \underline{l\widehat{T}}V^B) - \overline{N}^{22}\underline{l^2 T^{AB}}V^B = 0 \qquad A,B = 1,2,\ldots,N
\end{aligned}
\tag{4.9}
$$

where some terms depend explicitly on the mesostructure length parameter l.

The above model has a physical sense provided that the basic unknowns $U_\alpha(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$, $Q^A_\gamma(\boldsymbol{\Theta})$, $V^A(\boldsymbol{\Theta}) \in SV_\Lambda(T)$, $A = 1,2,\ldots,N$, i.e. they are locally slowly varying functions of Θ^2-midsurface parameter.

It can be observed that in the tolerance model equations (4.8), (4.9) we deal with $\overline{N}^{\alpha\beta} > 0$ if $\overline{N}^{\alpha\beta}$ are compressive forces.

Taking into account (4.1) and (4.4), the shell displacement fields can be approximated by means of the formulae

$$
\begin{aligned}
&u_\alpha(\boldsymbol{\Theta}) \simeq U_\alpha(\boldsymbol{\Theta}) + h^A(\Theta^1, \Psi^2)Q^A_\alpha(\boldsymbol{\Theta}) \\
&w(\boldsymbol{\Theta}) \simeq W(\boldsymbol{\Theta}) + g^A(\Theta^1, \Psi^2)V^A(\boldsymbol{\Theta}) \qquad A = 1,2,\ldots,N
\end{aligned}
\tag{4.10}
$$

where the approximation $\simeq$ depends on the number of terms $h^A(\cdot)Q^A_\alpha(\cdot)$, $g^A(\cdot)V^A(\cdot)$.

The characteristic features of the derived model are:

- The model takes into account the effect of the cell size on the overall shell stability; this effect is described by coefficients dependent on the mesostructure length parameter l.

- The model equations involve averaged coefficients which are independent of Θ^2-midsurface parameter (i.e. they are constant in the direction of periodicity) and are slowly varying functions of Θ^1.

- The number and form of boundary conditions for macrodisplacements $U_\alpha(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$ are the same as in the classical shell theory governed by equations (2.2)-(2.4). The boundary conditions for the fluctuation variables $Q_\gamma^A(\boldsymbol{\Theta})$, $V^A(\boldsymbol{\Theta})$ should be defined only on the boundaries $\Theta^1 = \text{const}$.

- It is easy to see that in order to derive the governing equations (4.7)-(4.9), we have to postulate *a priori* periodic shape functions $h^A(\Theta^1, \Psi^2)$, $g^A(\Theta^1, \Psi^2)$, $A = 1, 2, \ldots, N$, which can be derived from the periodic finite element method discretization of the cell or obtained as solutions to the periodic eigenvalue problem describing free vibrations of the shell, given by (4.5). Moreover, for uniperiodic shells the shape (mode- shape) functions are periodic in only one direction; in this work they are l-periodic functions only of Θ^2-midsurface parameter.

Assuming that the cylindrical shell under consideration has material and geometrical properties independent of Θ^1 we obtain the governing equations (4.7)-(4.9) with constant averaged coefficients. Moreover, in this case the shape functions h^A, g^A, $A = 1, 2, \ldots, N$, are also independent of Θ^1-midsurface parameter.

For a homogeneous shell $D^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$ and $B^{\alpha\beta\gamma\delta}(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \in \Omega$, are constant and because $\langle h^A \rangle = \langle g^A \rangle = 0$, we obtain $\langle h,_{\alpha}^{A} \rangle = \langle g,_{\alpha}^{A} \rangle = \langle g,_{\alpha\beta}^{A} \rangle = 0$. In this case equations (4.8) reduce to the well-known linear-elastic shell equations of equilibrium for macrodisplacements $U_\alpha(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$ and independently for $Q_\alpha^A(\boldsymbol{\Theta})$, $V^A(\boldsymbol{\Theta})$, we arrive at a system of N differential equations, which has only trivial solution $Q_\alpha^A = V^A = 0$. Hence the constitutive equations (4.7) and equations of equilibrium (4.8) reduce to the starting equations (2.3) and (2.4), respectively. However, it has to be emphasized that the starting equations (2.2)-(2.4) governed by the geometrically linear Kirchhoff-Love second-order shell theory cannot be used for investigations of stability problems for homogeneous shells; it is well known that the stability analysis of homogeneous shells has to be carried out within the framework of geometrically nonlinear shell theory, cf. the references in the Introduction.

In the next section the homogenized model of uniperiodic cylindrical shells under consideration will be derived as a special case of Eqs. (4.7)-(4.9).

5. Homogenized model

The simplified model of uniperiodically densely ribbed cylindrical shells can be derived directly from the tolerance model (4.7)-(4.9) by a limit passage $l \to 0$, i.e. by neglecting the underlined terms which depend on the mesostructure length parameter l. Hence, we arrive at the homogenized shell model governed by:

— equilibrium equations

$$
\begin{aligned}
&D^{\alpha\beta\gamma\delta}_{eff}(U_{\gamma,\delta\alpha} - b_{\gamma\delta}W_{,\alpha}) = 0 \\
&B^{\alpha\beta\gamma\delta}_{eff}W_{,\alpha\beta\gamma\delta} - b_{\alpha\beta}D^{\alpha\beta\gamma\delta}_{eff}(U_{\gamma,\delta} - b_{\gamma\delta}W) + \overline{N}^{\alpha\beta}W_{,\alpha\beta} = 0
\end{aligned}
\tag{5.1}
$$

— constitutive equations

$$
\begin{aligned}
N^{\alpha\beta} &= D^{\alpha\beta\gamma\delta}_{eff}(U_{\gamma,\delta} - b_{\gamma\delta}W) \\
M^{\alpha\beta} &= B^{\alpha\beta\gamma\delta}_{eff}W_{,\gamma\delta}
\end{aligned}
\tag{5.2}
$$

where

$$
\begin{aligned}
D^{\alpha\beta\gamma\delta}_{eff} &\equiv \widetilde{D}^{\alpha\beta\gamma\delta} - D^{A\alpha\beta\eta}G^{AB}_{\eta\xi}D^{B\xi\gamma\delta} \\
B^{\alpha\beta\gamma\delta}_{eff} &\equiv \widetilde{B}^{\alpha\beta\gamma\delta} - K^{A\alpha\beta}E^{AB}K^{B\gamma\delta}
\end{aligned}
$$

with $G^{AB}_{\alpha\beta}$ and E^{AB} defined by

$$
G^{AB}_{\alpha\beta}C^{BC\beta\gamma} = \delta^{\gamma}_{\alpha}\delta^{AC} \qquad\qquad E^{AB}R^{BC} = \delta^{AC}
$$

The homogenized model obtained above governed by Eqs. (5.1), (5.2) is not able to describe the length-scale effect on the overall shell behavior being independent of the mesostructure length parameter l.

In order to show differences between the results obtained from the tolerance uniperiodic shell model (4.7)-(4.9) and from the homogenized model (5.1), (5.2), critical forces of a special case of uniperiodic cylindrical shell will be analyzed in the next section.

6. Applications

The objective of this Section is to determine and investigate the critical forces of an open circular cylindrical shell with L_1, L_2 as its axial length and arc length along the lines of principal curvature of the shell midsurface, respectively, and with δ, R as its constant thickness and its midsurface curvature radius, respectively. The shell is reinforced by two families of densely spaced ribs, which are parallel to the generatrix of cylindrical surface and are periodically distributed along the lines of the shell midsurface principal curvature, cf. Fig. 2. The stiffeners of both kinds are assumed to have constant rectangular cross-sections with A_1, A_2 as their areas and with I_1, I_2 as their moments of inertia. Moreover, the gravity centers of the stiffener cross-sections are situated on the shell midsurface. It is assumed that both the shell and stiffeners are made of homogeneous isotropic materials and let us denote by E, ν the Young's modulus and Poisson's ratio of the shell material, respectively, and by E_1, E_2 the Young's moduli of the rib materials, cf. Fig. 3.

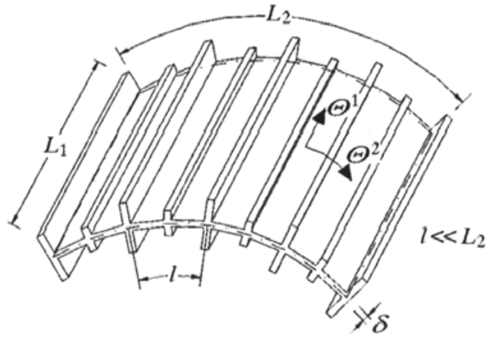

Fig. 2. A shell with two families of uniperiodically spaced ribs

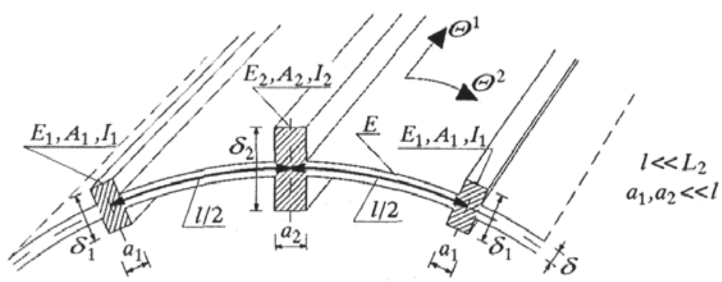

Fig. 3. A fragment of the stiffened shell cross-section

Let Θ^1, Θ^2 be the axial and arc coordinates on the shell midsurface $\mathcal{M}$, respectively, and let Θ^2-coordinate lines coincide with the lines of principal curvature of this surface.

It is assumed that the edges of the shell lie on the coordinate lines $\Theta^1 = 0$, $\Theta^1 = L_1$ and $\Theta^2 = 0$, $\Theta^2 = L_2$ and that all four edges are simply supported.

In agreement with the considerations in Section 2, on $O\Theta^1\Theta^2$-plane we define l as the period of the stiffened shell structure in the Θ^2-direction, which represents the distance (i.e. the arc length measured along the lines of midsurface principal curvature) between axes of two neighboring ribs belonging to the same family, cf. Fig. 2 and Fig. 3. It means that the axes of undeformed stiffeners are situated on the lines $\Theta^2 = n_1 l$, $n_1 = 0,1,2,\ldots,M$, and $\Theta^2 = n_2 l + l/2$, $n_2 = 0,1,2,\ldots,(M-1)$, $L_2 = (M-1)l$, where $(2M-1)$ is the number of stiffeners, cf. Fig. 2.

The period l has to satisfy the conditions $\delta \ll l \ll L_2$. It means that the number of stiffeners has to be very large. We also assume that $L_1 \geqslant L_2$; it follows that l satisfies the condition $l \ll L_1$.

Denoting by a_1, a_2 the widths of the ribs (cf. Fig. 3) we assume that $a_1, a_2 \ll l$ and hence the torsional rigidity of stiffeners can be neglected.

We define the periodicity cell Λ on the $O\Theta^1\Theta^2$-plane by means of $\Lambda \equiv (-l/2, l/2)$, $\Lambda(\Theta^1,\Theta^2) \equiv (\Theta^1, \Theta^2 - l/2, \Theta^1, \Theta^2 + l/2)$, $(\Theta^1,\Theta^2) \in \Omega_\Lambda$, $\Omega_\Lambda := \{\Theta \in \Omega, \Lambda(\Theta) \in \Omega\}$. The cell Λ is shown in Fig. 4. Setting $\Psi^2 \in \langle -l/2, l/2\rangle$, we assume that the cell Λ has a symmetry axis for $\Psi^2 = 0$.

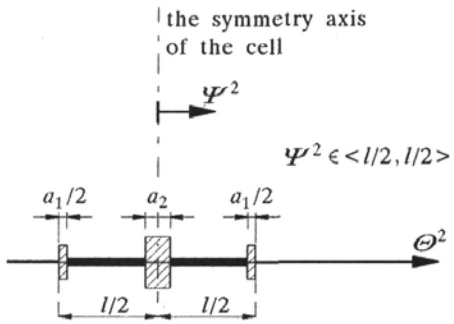

Fig. 4. A periodicity cell along the $O\Theta^2$-axis direction on the $O\Theta^1\Theta^2$-plane, $a_1, a_2 \ll l$

The periodically ribbed shell under consideration will be treated as a nonstiffened shell with constant thickness δ, made of a certain non-homogeneous and orthotropic material. The shell's tensile and bending stiffnesses in the axial direction are l-periodic functions in Θ^2, being independent of Θ^1 and are different from tensile and bending rigidities in circumferential direction, being constant functions.

Denote $D = E\delta/(1-\nu^2)$, $B = E\delta^3/[12(1-\nu^2)]$, $D, B = \text{const}$, $H^{\alpha\beta\gamma\delta} = 0.5[a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma} + \nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma})]$ with $a^{\alpha\gamma}$, $\epsilon^{\alpha\gamma}$ as contravariant first midsurface tensor and Ricci bivector, respectively. After some manipulations we obtain the following expressions for the nonzero components of tensor $H^{\alpha\beta\gamma\delta}$: $H^{1111} = H^{2222} = 1$, $H^{1122} = H^{2211} = \nu$, $H^{1212} = H^{1221} = H^{2121} = H^{2112} = H^{2112} = (1-\nu)/2$.

Under the assumption that the torsional rigidity of stiffeners is neglected, the components of the shell stiffness tensors $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$, except for D^{1111}, B^{1111}, are constant and given by: $D^{\alpha\beta\gamma\delta} = DH^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta} = BH^{\alpha\beta\gamma\delta}$. The tensile rigidity $D^{1111}(\Psi^2)$ and bending rigidity $B^{1111}(\Psi^2)$ are l-periodic functions in Ψ^2 and take the following form

$$D^{1111}(\Psi^2) = \begin{cases} DH^{1111} = D & \text{for} \quad \Psi^2 \in (-l/2, l/2) - \{0\} \\ E_1A_1/2 & \text{for} \quad \Psi^2 = -l/2 \ \text{ and } \ \Psi = l/2 \\ E_2A_2 & \text{for} \quad \Psi^2 = .0 \end{cases} \tag{6.1}$$

$$B^{1111}(\Psi^2) = \begin{cases} BH^{1111} = B & \text{for} \quad \Psi^2 \in (-l/2, l/2) - \{0\} \\ E_1I_1/2 & \text{for} \quad \Psi^2 = -l/2 \ \text{ and } \ \Psi = l/2 \\ E_2I_2 & \text{for} \quad \Psi^2 = 0 \end{cases} \tag{6.2}$$

Taking into account definition (2.1) we obtain for functions $D^{1111}(\Psi^2)$, $B^{1111}(\Psi^2)$ given above the following averaged values

$$\begin{aligned} \widetilde{D}^{1111} &\equiv \langle D^{1111} \rangle = D + \frac{1}{l}(E_1A_1 + E_2A_2) \\ \widetilde{B}^{1111} &\equiv \langle B^{1111} \rangle = B + \frac{1}{l}(E_1I_1 + E_2I_2) \end{aligned} \tag{6.3}$$

Let the shell be compressed in axial direction by the constant forces $\overline{N}^{11}$ and at the same time let it be extended in direction of Θ^2 by the constant forces $\overline{N}^{22} = -\xi\overline{N}^{11}$, $\xi > 0$. Moreover it is assumed that $\overline{N}^{12} = \overline{N}^{21} = 0$.

For the sake of simplicity, we restrict our considerations to the first terms in series $h^A(\cdot)Q^A_\alpha(\cdot, t)$, $g^A(\cdot)V^A(\cdot, t)$, $A = 1, 2, \ldots, N$, i.e. $A = N = 1$. Hence, we introduce only two l-periodic shape functions $h(\Psi^2) \equiv h^1(\Psi^2)$, $g(\Psi^2) \equiv g^1(\Psi^2)$, $\Psi^2 \in \langle -l/2, l/2 \rangle$, which have to satisfy condition $\langle h \rangle = \langle g \rangle = 0$ and the values of which are of order $\mathcal{O}(l)$ and $\mathcal{O}(l^2)$, respectively. Functions $h(\Psi^2)$, $g(\Psi^2)$ can be derived from the periodic finite element method discretization of the cell or obtained as solutions to the periodic eigenvalue problem on the cell given by equation (4.5).

Taking into account the symmetric form of the cell, cf. Fig. 4, we assume that the shape function $h(\Psi^2)$ is antisymmetric on the cell Λ while the shape function $g(\Psi^2)$ is symmetric.

Taking into account the fact that, except for D^{1111}, B^{1111}, the components of the shell stiffness tensors $D^{\alpha\beta\gamma\delta}$, $B^{\alpha\beta\gamma\delta}$ are constant and that the functions $h(\Psi^2)$, $g(\Psi^2)$ are independent of Θ^1 as well as bearing in mind the symmetric form of the cell and the symmetric form of function $g(\Psi^2)$ as well as anti-symmetric form of function $h(\Psi^2)$, it can be shown that only the following averages in (4.6) are different from zero: $\widetilde{D}^{\alpha\beta\gamma\delta}$, $\widetilde{B}^{\alpha\beta\gamma\delta}$, $\breve{K}^{A11}$, C^{AB11}, C^{AB22}, $\widetilde{C}^{AB11}$, $\widetilde{C}^{AB22}$, F^{AB2}, S^{AB}, $\overline{L}^{AB}$, $\widetilde{R}^{AB}$, $\overset{\frown}{R}^{AB}$, $\widetilde{S}^{AB}$, T^{AB}, $\overline{T}^{AB}$, $A,B=1$. Under the assumption $A=B=N=1$ we introduce the following denotations for these non-zero averages

$$
\begin{array}{llll}
\breve{K}^{11} \equiv \breve{K}^{A11} & C^{11} \equiv C^{AB11} & C^{22} \equiv C^{AB22} & \\
\widetilde{C}^{11} \equiv \widetilde{C}^{AB11} & \widetilde{C}^{22} \equiv \widetilde{C}^{AB22} & F^{2} \equiv F^{AB2} & \\
S \equiv S^{AB} & \overline{L} \equiv \overline{L}^{AB} & \widetilde{R} \equiv \widetilde{R}^{AB} & (6.4) \\
\overset{\frown}{R} \equiv \overset{\frown}{R}^{AB} & \widetilde{S} \equiv \widetilde{S}^{AB} & T \equiv T^{AB} & \\
\overline{T} \equiv \overline{T}^{AB} & A,B=1 & &
\end{array}
$$

We also denote $Q_1(\boldsymbol{\Theta}) \equiv Q_1^1(\boldsymbol{\Theta})$, $Q_2(\boldsymbol{\Theta}) \equiv Q_2^1(\boldsymbol{\Theta})$, $V(\boldsymbol{\Theta}) \equiv V^1(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \equiv (\Theta^1, \Theta^2)$.

Bearing in mind the conditions and notations given above we will derive below the formulae for critical forces of the considered uniperiodic shell by using both the tolerance model given by Eqs. (4.7)-(4.9) and the homogenized model presented by Eqs. (5.1),(5.2).

6.1. The tolerance model

Now, the governing equations (4.8), (4.9) of the tolerance model is separated into the independent equation for $Q_1(\boldsymbol{\Theta})$: $C^{11}Q_1 - l^2\widetilde{C}^{11}Q_{1,11} = 0$, which yields $\Theta_1 = 0$, and the system of five equations for macrodisplacements $U_1(\boldsymbol{\Theta})$, $U_2(\boldsymbol{\Theta})$, $W(\boldsymbol{\Theta})$ and fluctuation variables $Q_2(\boldsymbol{\Theta})$, $V(\boldsymbol{\Theta})$, $\boldsymbol{\Theta} \equiv (\Theta^1, \Theta^2)$, being locally slowly-varying functions of Θ^2

$$
\begin{aligned}
&\widetilde{D}^{1111}U_{1,11} + D[(1-\nu)2^{-1}U_{1,22} + (1+\nu)2^{-1}U_{2,12} + \nu R^{-1}W_{,1}] = 0 \\
&D[(1+\nu)2^{-1}U_{1,12} + (1-\nu)2^{-1}U_{2,11} + U_{2,22} + R^{-1}W_{,2}] = 0
\end{aligned}
$$

$$
\begin{aligned}
&D(\nu R^{-1}U_{1,1} + R^{-1}U_{2,2} + R^{-2}W) + \widetilde{B}^{1111}W_{,1111} + B(2W_{,1122} + W_{,2222}) + \\
&\quad + \overline{N}^{11}(W_{,11} - \xi W_{,22}) + l^2\breve{K}^{11}V_{,1111} = 0, \qquad\qquad (6.5)\\
&C^{22}Q_2 - l^2\widetilde{C}^{22}Q_{2,11} - l^2F^2V = 0\\
&l^2\breve{K}^{11}W_{,1111} - l^2F^2Q_2 + (S + l^4\overline{L} + \overline{N}^{11}\xi l^2 T)V + [2l^2(\widetilde{R} - 2\widetilde{S}) + \\
&\quad + \overline{N}^{11}l^4\overline{T}]V_{,11} + l^4\widehat{R}V_{,1111} = 0 \qquad\qquad \xi > 0
\end{aligned}
$$

where some terms depend explicitly on the mesostructure length parameter l; the averages $\widetilde{D}^{1111}$, $\widetilde{B}^{1111}$ are defined by (6.3) and the remaining ones are given by (6.4) and (4.6).

It is easy to see that all coefficients of the above equations are constant.

Solutions to Eqs. (6.5) satisfying the boundary conditions for a simply supported shell can be assumed in the form (see Ambartsumyan, 1974)

$$
\begin{aligned}
U_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(\alpha_m\Theta^1)\sin(\beta_n\Theta^2)\\
U_2 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin(\alpha_m\Theta^1)\cos(\beta_n\Theta^2)\\
W &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin(\alpha_m\Theta^1)\sin(\beta_n\Theta^2) \qquad\qquad (6.6)\\
Q_2 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} D_{mn}\sin(\alpha_m\Theta^1)\sin(\beta_n\Theta^2)\\
V &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn}\sin(\alpha_m\Theta^1)\sin(\beta_n\Theta^2)
\end{aligned}
$$

where $\alpha_m = m\pi/L_1$, $\beta_n = n\pi/L_2$, $m, n = 1, 2, \ldots$; m, n represent the numbers of buckling half-waves in directions of Θ^1- and Θ^2-coordinate lines, respectively.

Substituting the right-hand sides of Eqs. (6.6) into Eqs. (6.5) we obtain the system of five linear homogeneous algebraic equations for A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn}. For a nontrivial solution, the determinant of the coefficients of these equations must equal zero. In this manner we arrive at the characteristic equation for the critical force $\overline{N}^{11}$. Setting $(\overline{N}^{11}_{cr})^{tm} \equiv \overline{N}^{11}$ and introducing the following notations

$$
\begin{aligned}
\eta_{1m} &\equiv l^4(\breve{K}^{11})^2[1+\alpha_m^2 l^2 \widetilde{C}^{22}(C^{22})^{-1}] \\
\eta_{2m} &\equiv l^4(F^2)^2(C^{22})^{-1} - [1+\alpha_m^2 l^2 \widetilde{C}^{22}(C^{22})^{-1}](S+l^4\overline{L})\cdot \\
&\quad \cdot[1-2\alpha_m^2 l^2(\widetilde{R}-2\widetilde{S})(S+l^4\overline{L})^{-1} + l^4\alpha_m^4\widehat{R}(S+l^4\overline{L})^{-1}] \\
\eta_{3m} &\equiv l^2 T[1+l^2\alpha_m^2\overline{T}(-\xi T)^{-1}]
\end{aligned}
\tag{6.7}
$$

and

$$
\begin{aligned}
\chi_1 &\equiv D^2\{(1-\nu)2^{-1}(\alpha_m^4+\beta_n^4)+\alpha_m^2\beta_n^2(1+\nu)+D^{-1}(E_1A_1+E_2A_2)l^{-1}\cdot \\
&\quad \cdot [\alpha_m^4(1-\nu)2^{-1}+\alpha_m^2\beta_n^2] \\
\chi_2 &\equiv [\widetilde{B}^{1111}\alpha_m^4+B\beta_n^2(2\alpha_m^2+\beta_n^2)]\chi_1+D^3R^{-2}[2\alpha_m^2\beta_n^2+ \\
&\quad + \alpha_m^4(1-\nu)^2(1+\nu)2^{-1}+D^{-1}(E_1A_1+E_2A_2)l^{-1}\alpha_m^4(1-\nu)2^{-1}]
\end{aligned}
\tag{6.8}
$$

this equation has the following form

$$
\begin{aligned}
&[(\overline{N}_{cr}^{11})^{tm}]^2(\alpha_m^2-\xi\beta_n^2)\chi_1+(\overline{N}_{cr}^{11})^{tm}[(\alpha_m^2-\xi\beta_n^2)\chi_1(-\eta_{2m})(\xi\eta_{3m})^{-1}-\chi_2]+ \\
&+\eta_{1m}(\xi\eta_{3m})^{-1}\chi_1+\eta_{2m}(\xi\eta_{3m})^{-1}\chi_2=0 \qquad \xi>0
\end{aligned}
\tag{6.9}
$$

It should be noted that the right-hand sides in (6.8) are always positive, i.e. $\chi_1 > 0$ and $\chi_2 > 0$.

Because the shell under consideration satisfies the condition $l/L_1 \ll 1$, i.e. $\alpha_m l \ll 1$, in the sequel the simplified form of equation (6.9) will be applied, in which the terms $(\alpha_m l)^2\widetilde{C}^{22}(C^{22})^{-1}$, $(\alpha_m l)^2(\widetilde{R}-2\widetilde{S})(S+l^4\overline{L})^{-1}$, $(\alpha_m l)^4\widehat{R}(S+l^4\overline{L})^{-1}$ and $(\alpha_m l)^2\overline{T}(-\xi T)^{-1}$ can be neglected as small in comparison with unity and then

$$
\begin{aligned}
\eta_{1m} &\approx \eta_1 \equiv l^4(\breve{K}^{11})^2 \qquad\qquad \eta_{3m}\approx\eta_3\equiv l^2T \\
\eta_{2m} &\approx \eta_2 \equiv l^4[(F^2)^2(C^{22})^{-1}-\overline{L}]-S
\end{aligned}
\tag{6.10}
$$

Taking into account (6.10) and using the notations

$$
\widetilde{b}_1 \equiv \eta_1(\xi\eta_3)^{-1} \qquad\qquad \widetilde{b}_2 \equiv \eta_2(\xi\eta_3)^{-1}
\tag{6.11}
$$

we obtain from Eq. (6.9), for each pair of values of m and n, $m,n=1,2,\ldots$, the following formulae for *fundamental lower critical force* $(\overline{N}_{cr1}^{11})^{tm}$ and for *the* *additional higher critical force* $(\overline{N}^{11}_{cr2})^{tm}$, caused by the uniperiodic structure of the shell under consideration

$$
\begin{aligned}
(\overline{N}^{11}_{cr1})^{tm} &= \frac{1}{2}[\widetilde{b}_2 + \chi_2\chi_1^{-1}(\alpha_m^2 - \xi\beta_n^2)^{-1}] + \\
&-\frac{1}{2}\sqrt{[\widetilde{b}_2 + \chi_2\chi_1^{-1}(\alpha_m^2 - \xi\beta_n^2)^{-1}]^2 - 4[\widetilde{b}_1\chi_1 + \chi_2\widetilde{b}_2](\alpha_m^2 - \xi\beta_n^2)^{-1}\chi_1^{-1}} \\
(\overline{N}^{11}_{cr2})^{tm} &= \frac{1}{2}[\widetilde{b}_2 + \chi_2\chi_1^{-1}(\alpha_m^2 - \xi\beta_n^2)^{-1}] + \\
&+\frac{1}{2}\sqrt{[\widetilde{b}_2 + \chi_2\chi_1^{-1}(\alpha_m^2 - \xi\beta_n^2)^{-1}]^2 - 4[\widetilde{b}_1\chi_1 + \chi_2\widetilde{b}_2](\alpha_m^2 - \xi\beta_n^2)^{-1}\chi_1^{-1}}
\end{aligned}
\tag{6.12}
$$

In (6.12) the period length l is contained in terms $\widetilde{b}_1$, $\widetilde{b}_2$.

The pairs of values of m and n corresponding to the smallest values of the lower $(\overline{N}^{11}_{cr1})^{tm}$ and higher $(\overline{N}^{11}_{cr2})^{tm}$ critical forces will be determined and discussed in a separate paper.

6.2. The homogenized model

In order to evaluate the obtained results, let us consider the above problem within the homogenized (i.e. asymptotic) model. From Eqs. (6.5), after neglecting the terms of orders $\mathcal{O}(l^2)$ and $\mathcal{O}(l^4)$, we obtain the following governing relations of the homogenized model

$$
\begin{aligned}
&\widetilde{D}^{1111}U_{1,11} + D[(1-\nu)2^{-1}U_{1,22} + (1+\nu)2^{-1}U_{2,12} + \nu R^{-1}W_{,1}] = 0 \\
&D[(1+\nu)2^{-1}U_{1,12} + (1-\nu)2^{-1}U_{2,11} + U_{2,22} + R^{-1}W_{,2}] = 0 \\
&D(\nu R^{-1}U_{1,1} + R^{-1}U_{2,2} + R^{-2}W) + \widetilde{B}^{1111}W_{,1111} + \\
&+B(2W_{,1122} + W_{,2222}) + \overline{N}^{11}(W_{,11} - \xi W_{,22}) = 0
\end{aligned}
\tag{6.13}
$$

The model obtained above is not able to describe the length-scale effect on the overall shell stability being independent of the period length l.

It is easy to see that there are not fluctuation variables in the asymptotic model (6.13) derived here. It means that $U_1 = u_1$, $U_2 = u_2$, $W = w$ and hence the governing equations (6.13) coincide with the well-known equations of stability problems for stringer-stiffened cylindrical shells; see Brush, Almroth (1975).

The solutions to Eqs. (6.13) can be assumed in the form $(6.6)_{1,2,3}$. Substituting the solutions to (6.13) we obtain the system of three linear homogeneous algebraic equations in A_{mn}, B_{mn}, C_{mn}. For $A_{mn} \neq 0$, $B_{mn} \neq 0$, $C_{mn} \neq 0$

we arrive at the formula for critical values of compressive forces $\overline{N}^{11}$. Setting $(\overline{N}^{11}_{cr})^{hm} \equiv \overline{N}^{11}$, this formula has the form

$$(\overline{N}^{11}_{cr})^{hm} = \chi_2 \chi_1^{-1} (\alpha_m^2 - \xi \beta_n^2)^{-1} \qquad \xi > 0 \tag{6.14}$$

where χ_1, χ_2 are given by (6.8), $\alpha_m = m\pi/L_1$, $\beta_n = n\pi/L_2$, $m, n = 1, 2, \ldots$, and ξ represents the relation between the tensile forces $\overline{N}^{22}$ and compressive forces $\overline{N}^{11}$.

It is easy to see that in the above formula the cell size is neglected and that in the framework of the asymptotic model it is not possible to determine the additional higher critical force, caused by the periodic structure of the shell.

In the next subsection a comparison of the results obtained in Sections 6.1 and 6.2 will be presented.

6.3. Comparison of results

Let us compare the lower critical force given by $(6.12)_1$, which has been derived from the tolerance model with that given by (6.14) obtained from the homogenized model.

Setting $N \equiv (\overline{N}^{11}_{cr1})^{tm}/(\overline{N}^{11}_{cr})^{hm}$ and using notations (6.11) we obtain from $(6.12)_1$ and (6.14) the following expression

$$\begin{aligned} N = {} & \frac{1}{2}\chi_2^{-1}[\widetilde{b}_2(\alpha_m^2 - \xi\beta_n^2)\chi_1 + \chi_2] + \\ & -\frac{1}{2}\chi_2^{-1}\sqrt{[\widetilde{b}_2(\alpha_m^2 - \xi\beta_n^2)\chi_1 + \chi_2]^2 - 4[\widetilde{b}_1\chi_1 + \chi_2\widetilde{b}_2](\alpha_m^2 - \xi\beta_n^2)\chi_1} \end{aligned} \tag{6.15}$$

From (6.15) it is seen that if the parameter ξ representing the relation between tensile forces $\overline{N}^{22}$ and compressive forces $\overline{N}^{11}$ is close to $(\alpha_m/\beta_n)^2 = (mL_2/nL_1)^2$ then N suddenly decreases. It means that the differences between the values of critical forces obtained from both the models under consideration are very large; the lower critical force $(6.12)_1$ derived from the tolerance model is much smaller than the critical force (6.14) obtained from the homogenized one.

Now, let us assume that ξ is not close to $(\alpha_m/\beta_n)^2$. It is easy to show that treating every average value in $(6.12)_1$, which is of order $\mathcal{O}(l^2)$, as a small parameter ε and then representing the square root in $(6.12)_1$ in the form of the power series with respect to $\varepsilon \in \mathcal{O}(l^2)$, we arrive at the interrelation

$$(\overline{N}^{11}_{cr1})^{tm} = (\overline{N}^{11}_{cr})^{hm} + \mathcal{O}(l^4) \tag{6.16}$$

between the values of critical forces $(\overline{N}^{11}_{cr1})^{tm}$ and $(\overline{N}^{11}_{cr})^{hm}$ obtained within the framework of the tolerance and homogenized models, respectively. It means that the differences between lower value of the critical force derived from the tolerance model and critical force obtained from the asymptotic one are negligibly small. Thus, in this case, the effect of the period length l on the values of critical forces can be neglected and we can use the asymptotic model represented by Eqs. (5.1), (5.2) instead of the non-asymptotic tolerance model given by Eqs. (4.7)-(4.9).

6.4. Conclusions

Summarizing the results obtained in this section it can be concluded that:

- Contrary to the homogenized (asymptotic) model, the proposed nonasymptotic one describes the effect of the period length l on the shell stability.
- In the framework of the non-asymptotic tolerance model proposed in this contribution, the fundamental lower and additional higher critical forces can be derived. The higher critical force, caused by a periodic structure of the stiffened shell, cannot be determined using the homogenized (i.e. asymptotic) model.
- The differences between the values of lower critical forces obtained from the proposed tolerance model and those from the asymptotic one are negligibly small; the critical force $(\overline{N}^{11}_{cr})^{hm}$ calculated from the asymptotic model is an approximation of order $\mathcal{O}(l^4)$ of the lower critical force $(\overline{N}^{11}_{cr1})^{tm}$ derived from the tolerance model, i.e. $(\overline{N}^{11}_{cr1})^{tm} = (\overline{N}^{11}_{cr})^{hm} + \mathcal{O}(l^4)$. Thus the effect of the period length l on the shell stability can be neglected and hence, the homogenized model given by (5.1), (5.2) is sufficient from the point of view of calculation for the problem of determining the critical forces in uniperiodically densely stiffened cylindrical shells under consideration. However, for certain values of the ξ-parameter representing the relation between tensile forces $\overline{N}^{22}$ and compressive forces $\overline{N}^{11}$, these differences are very large; critical forces related to the tolerance model are much smaller than those derived from homogenized one. It means that in this case, the length-scale effect plays an important role and cannot be neglected and hence only the non-asymptotic tolerance model represented by Eqs. (4.7)-(4.9) has to be used to analyze the critical forces of the ribbed shell under consideration.

7. Final remarks

The subject matter of this contribution is a thin linear-elastic cylindrical shell having a periodic structure (a periodically varying thickness and/or periodically varying elastic and inertial properties) in one direction, tangent to the undeformed shell midsurface $\mathcal{M}$. Shells of this kind are termed *uniperiodic*. Moreover, it is assumed that the uniperiodic cylindrical shells, being objects of our considerations, are composed of a very large number of identical elements and every such element is treated as a shallow shell. It means that the period of inhomogeneity is very large compared with the maximum shell thickness and very small as compared to the midsurface curvature radius as well as the smallest characteristic length dimension of the shell midsurface in the periodicity direction. This uniperiodic structure of the cylindrical shells considered here is related to the periodically spaced dense system of ribs as shown in Fig. 1.

For the uniperiodically densely stiffened cylindrical shells the known governing equations of the Kirchhoff-Love shell theory involve periodic highly oscillating and noncontinuous coefficients. Hence, in most cases direct application of these equations to analyze engineering problems in periodic shells is very complicated, particularly from the computational viewpoint. That is why the aim of this contribution was to propose a new nonasymptotic model of stability problems for uniperiodically, densely stiffened cylindrical shells, which has constant coefficients in the direction of periodicity and hence can be applied as a proper analytical tool for investigations of stability problems in the shell under considerations. Moreover, the proposed model takes into account the effect of periodicity cell size on the global shell stability (*the length-scale effect*), which is neglected in the known homogenized models derived by asymptotic methods.

In order to derive the model equations, the *tolerance averaging procedure* given by Woźniak and Wierzbicki (2000) has been applied to governing equations of the Kirchhoff-Love second-order shell theory for thin linear-elastic cylindrical shells, i.e. to Eqs. (2.2)-(2.4). The proposed averaged model called *the tolerance model of stability problems for uniperiodically densely stiffened cylindrical shells* is represented by a system of partial differential equations (4.8), (4.9) with coefficients which are constant in the direction of periodicity. The basic unknowns are: the *macrodisplacements* U_α, W and the *fluctuation variables* Q_α^A, V^A, $A = 1, 2, \ldots, N$, which have to be *locally slowly-varying* functions with respect to the cell and certain tolerance system. This requirement imposes certain restrictions on the class of problems described by the model under consideration. In order to obtain the governing equations, the *shape functions* h^A, g^A, $A = 1, 2, \ldots, N$, should be derived from the periodic finite element method discretization of the cell or obtained as solutions to periodic eigenvalue problem on the cell given by equation (4.5).

In contrast with the homogenized (asymptotic) models, the proposed one makes it possible to describe the effect of a periodicity cell size on the critical forces (the length-scale effect). From the illustrative example it follows that this effect plays an important role in stability problems and cannot be neglected.

In the framework of the non-asymptotic model proposed in this contribution, not only the fundamental, lower, but also the additional, higher critical forces can be determined and analyzed.

Problems related to various applications of the proposed Eqs. (4.7)-(4.9) to stability of uniperiodically, densely stiffened cylindrical shells and determination of the mode-shape functions from periodic eigenvalue problem given by (4.5) are reserved for a separate paper.

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Stateczność cienkich periodycznie gęsto użebrowanych powłok walcowych

Streszczenie

W pracy zaproponowano nowy uśredniony nieasymptotyczny model służący do analizy stateczności cienkich liniowo-sprężystych powłok walcowych, periodycznie, gęsto użebrowanych w jednym kierunku stycznym do powierzchni środkowej. Przy wyprowadzaniu równań modelu wykorzystano znaną metodę tolerancyjnego uśredniania, zaproponowaną przez Woźniaka i Wierzbickiego (2000). Zastosowanie tej metody do znanych równań teorii powłok Kirchhoffa-Love'a doprowadziło do modelu reprezentowanego przez równania różniczkowe cząstkowe o stałych współczynnikach w kierunku periodyczności, zależnych od długości okresu periodyczności. Oznacza to, że proponowany model, w przeciwieństwie do znanych modeli zhomogenizowanych, umożliwia badanie wpływu wielkości komórki periodyczności na wartości sił krytycznych w powłoce walcowej (wpływ ten zwany jest efektem skali). Wyprowadzony model porównano z modelem bez efektu skali i pokazano, że wpływ długości okresu periodyczności odgrywa znaczącą rolę w zagadnieniach stateczności periodycznie, gęsto użebrowanych powłok walcowych.

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