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Comparison of the Aluminium Versus Steel Telecommunication Towers in Stochastic Finite Element Method Eigenvibrations Analysis

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The main aim of this paper is to make a comparison of the eigenfrequencies of the high telecommunication towers alternatively manufactured using the stainless steel and the aluminium components. It is provided each time assuming that the Young modulus of the applied material is the Gaussian input random variable and using the generalized stochastic perturbation method using the global version of the Response Function Method. Up to the fourth order probabilistic moments and characteristics are computed in the three dimensional Finite Element Method model of the tower composed from the continuous linear elastic edge beams spanned by the large number of the linear elastic bars. A computational part of the work is made using the hybrid usage of the computer algebra system MAPLE and the FEM engineering package ROBOT used widely in the civil engineering practice.

Keywords: Stochastic dynamics, Stochastic Finite Element Method, response function method, stochastic perturbation technique

1. Introduction

The telecommunication towers undoubtedly belong to the class of the lightweight structures exhibited to the stochastic influence of the wind blow and since that, their reliability needs to be evaluated with respect to the strength, to maximum deflections and rotations as well as to the eigenfrequencies. The uncertainty in the structural response of the towers and masts in general follows the quasi–periodic and temporary ice covers increasing both mass and effective surfaces of the structural elements, temperature fluctuations leading to the significant thermal stresses

not necessarily uniformly influencing the entire structure (concerning dominating southern exposure to the sun heating). The separate role in overall uncertainty magnitude play the geometrical imperfections in the elements connections (especially in welds) and the elements themselves since material defects following productions stage and composite character of the micro– and nanostructure. Because aluminium has decisively more micro–compounds in its total volume, randomization of the basic material properties is even more justified than for stainless steels and, finally, a random dispersion of those properties needs to be significantly larger.

It is known from the engineering practice that the telecommunication structures (towers, masts and antennas) designing and manufacturing is still relatively new and the very modern area for the engineers and scientists because the development of the mobile phones is still in progress and may demand the brand new extensions in the nearest future. Therefore, an optimization of the relevant supporting structures' shape and the materials' design is still an ongoing development – an example is alternative usage of the aluminium and steel based supporting towers.

On the other hand, the engineering reliability analysis is still being developed, concerning at least the demands of the Eurocodes and, at the same time, acquisition of various stochastic methods in engineering practice [1, 8–11]. It is widely known, that the reliability measured with some indices must be computed not only for the load capacity and maximum deflections of some structures but also for their vibrations and fatigue under dynamic loads, which needs further, more advanced stochastic computer methods.

These are the main reasons to investigate the matter of comparison of the aluminium and steel manufactured telecommunication structures in the presence of uncertainty in material properties of the structural components. We study random fluctuations of the eigenvibrations modes for the same towers made of steel and, than of aluminium to discuss in this context their reliability issues and we assume in this case that the Young modulus of both materials is a truncated Gaussian random variable with the given expected value and the coefficient of variation being an extra input parameter to this analysis. The generalized stochastic perturbation technique is employed to achieve this goal since the expected time savings (with respect to the Monte–Carlo technique) and a determination of up to the fourth order probabilistic moments and coefficients (in addition to the other stochastic methods).

Computational part is provided using the Finite Element Method engineering system ROBOT, where all the eigenfrequencies are determined with respect to the initially modified Young moduli of both towers. Further computations of the response functions, their automatic differentiation, the probabilistic moments and the coefficients as well as their visualization are provided with the use of computer algebra system MAPLE. This analysis is planned to be extended towards a full verification of the static and dynamic reliability, buckling fragility as well as including of the fatigue and ageing phenomena into the overall structure mathematical and computational models.

2. Variational formulation

Let us consider the following linear elasto–dynamic problem consisting of [6]

- the equations of motion

$$\mathbf{D}^{\mathbf{T}}\sigma + \hat{\mathbf{f}} = \rho\ddot{\mathbf{u}}, \quad \mathbf{x} \in \Omega \quad , \quad \tau \in [t_0, \infty) \tag{1}$$

- the constitutive equations

$$\sigma = \mathbf{C}\varepsilon, \quad \mathbf{x} \in \Omega \quad , \quad \tau \in [t_0, \infty) \tag{2}$$

- the geometric equations

$$\varepsilon = \mathbf{D}\mathbf{u}, \quad \mathbf{x} \in \Omega, \quad \tau \in [t_0, \infty) \tag{3}$$

- the displacement boundary conditions

$$\mathbf{u} = \hat{\mathbf{u}}, \quad \mathbf{x} \in \partial\Omega_u, \quad \tau \in [t_0, \infty) \tag{4}$$

- the stress boundary conditions

$$\mathbf{N}\sigma = \hat{\mathbf{t}}, \quad \mathbf{x} \in \partial\Omega_\sigma, \quad \tau \in [t_0, \infty) \tag{5}$$

- the initial conditions

$$\mathbf{u} = \hat{\mathbf{u}}^{\mathbf{0}}, \quad \dot{\mathbf{u}} = \hat{\dot{\mathbf{u}}}^{\mathbf{0}}, \quad \tau = t_0 \tag{6}$$

It is assumed that all the state functions appearing in this system are sufficiently smooth functions of the independent variables $\mathbf{x}$ and τ. Let us consider the variation $\mathbf{u}(\mathbf{x}, \tau)$ in some time moment $\tau = t$ denoted by $\delta\mathbf{u}(\mathbf{x}, \tau)$. Using the above equations one can show that

$$-\int_{\Omega} (\mathbf{D}^{\mathbf{T}}\sigma + \hat{\mathbf{f}} - \rho\ddot{\mathbf{u}})^{\mathbf{T}}\delta\mathbf{u}d\Omega + \int_{\partial\Omega_\sigma} (\mathbf{N}\sigma - \hat{\mathbf{t}})^{\mathbf{T}}\delta\mathbf{u}d(\partial\Omega) = 0 \tag{7}$$

Assuming further that the displacement function $\mathbf{u}(\mathbf{x},\mathbf{t})$ has known values at the initial moment $\mathbf{u}(\mathbf{x}, t_1) = 0$ and at the end of the process $\mathbf{u}(\mathbf{x}, t_2) = 0$, so that the variations of this function also equal 0 at those time moments

$$\delta\mathbf{u}(\mathbf{x}, t_1) = 0, \quad \delta\mathbf{u}(\mathbf{x}, t_2) = 0 \tag{8}$$

Integrating by parts with respect to the variables $\mathbf{x}$ and τ we can obtain that

$$\int_{\mathbf{t_1}}^{\mathbf{t_2}} [\delta\mathbf{T} - \int_{\Omega} \sigma^{\mathbf{T}}\delta\varepsilon d\Omega + \int_{\Omega} \hat{\mathbf{f}}^{\mathbf{T}}\delta\mathbf{u}d\Omega + \int_{\partial\Omega} \hat{\mathbf{t}}^{\mathbf{T}}\delta\mathbf{u}d(\partial\Omega)]d\tau = 0 \tag{9}$$

where the kinetic energy of the region Ω is defined as

$$T = \frac{1}{2}\int \rho\dot{\mathbf{u}}^{\mathbf{T}}\dot{\mathbf{u}}d\Omega \tag{10}$$

We also notice that

$$\delta\varepsilon = \mathbf{D}\delta\mathbf{u}, \quad \mathbf{x} \in \Omega \quad , \quad \tau \in [t_0, \infty) \tag{11}$$

Next, we introduce the assumption that the mass forces $\hat{\mathbf{f}}$ and the surface loadings $\hat{\mathbf{t}}$ are independent from the displacement vector $\mathbf{u}$, which means that the external loadings do not follow the changes in the domain initial configuration. Therefore, equation (9) can be modified to the following statement:

$$\delta \int_{t_1}^{t_2} (T - J_p)\, d\tau = 0 \tag{12}$$

where J_p means the potential energy stored in the entire domain Ω

$$J_{\mathbf{p}} = U - \int_{\Omega} \hat{\mathbf{f}}^{\mathbf{T}}\mathbf{u} d\Omega - \int_{\partial\Omega_\sigma} \hat{\mathbf{t}}^{\mathbf{T}}\mathbf{u} d(\partial\Omega) = 0 \tag{13}$$

whereas the variation is determined with respect to the displacement function and U is the elastic strain energy given by the formula

$$U = \tfrac{1}{2} \int_{\Omega} \varepsilon^{\mathbf{T}}\mathbf{C}\varepsilon d\Omega \tag{14}$$

It is well known that the equation (12) represents the Hamilton principle widely used in structural dynamics in conjunction with the Finite Element Method approach.

3. Computational implementation

Let us consider a discretization of the displacement field $\mathbf{u}(\mathbf{x}, \tau)$using the following forms:

$$\mathbf{u}^{\alpha}_{3x1}(\mathbf{x},\tau) \cong \varphi_{3xN_{(e)}}(\mathbf{x})\mathbf{q}^{\alpha}_{N_{(e)}x1}(\tau), \quad \mathbf{u}^{\alpha}_{3x1}(\mathbf{x},\tau) \cong \mathbf{\Phi}_{3xN}(\mathbf{x})\mathbf{r}^{\alpha}_{Nx1}(\tau) \tag{15}$$

where $\mathbf{q}$ is a vector of the generalized coordinates for the considered finite element, $\mathbf{r}$ is a vector for the generalized coordinates of the entire discretized system, $N_{(e)}$ is the total number of the eth finite element degrees of freedom; N is the total number of degrees of freedom in the structure model. The generalized coordinates vector for the entire structure model is composed from the finite element degrees of freedom and the transformation matrix as

$$\mathbf{r}^{\alpha}_{Nx1} = a_{NxN_{(e)}}\mathbf{q}^{\alpha}_{N_{(e)}x1} \tag{16}$$

φ and $\mathbf{\Phi}$ are the corresponding shape function matrices (local and global). Contrary to the classical formulations of both FEM and the perturbation-based Stochastic Finite Element Method [2,3,6], we introduce here the additional index α=1,...,M to distinguish between various solutions of the elastodynamic problem necessary to build up the response function (around the mean value of the input random parameter).

The strain tensor can be expressed as

$$\varepsilon^{\alpha}_{6x1}(\mathbf{x},\tau) = B_{6xN(e)}(\mathbf{x})\mathbf{q}^{\alpha}_{N(e)x1}(\tau) = \tilde{B}_{6xN}(\mathbf{x})\mathbf{r}^{\alpha}_{Nx1}(\tau) \tag{17}$$

The discretized version of the Hamilton's principle is obtained as

$$\delta \int_{t_1}^{t_2} \left(\tfrac{1}{2} \sum_{e=1}^{E} \mathbf{q}^{\alpha T} \mathbf{m}^{\alpha}_{N_{(e)}xN_{(e)}} \mathbf{q}^{\alpha} - \tfrac{1}{2} \sum_{e=1}^{E} \mathbf{q}^{\alpha T} \mathbf{k}^{\alpha}_{N_{(e)}xN_{(e)}} \mathbf{q}^{\alpha} + \sum_{e=1}^{E} \boldsymbol{Q}^{\alpha T}_{N_{(e)}} \mathbf{q}^{\alpha} \right) d\tau = 0 \tag{18}$$

and hence

$$\delta \int_{t_1}^{t_2} \left(\tfrac{1}{2} \dot{\mathbf{r}}^{\alpha T} \mathbf{M}^{\alpha} \dot{\mathbf{r}}^{\alpha} - \tfrac{1}{2} \mathbf{r}^{\alpha T} \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} + \mathbf{R}^{\alpha T} \mathbf{r}^{\alpha} \right) d\tau = 0 \tag{19}$$

The global mass matrix is defined as

$$\mathbf{M}^{\alpha}_{NxN} = \int_{\Omega} \rho^{\alpha}(\mathbf{x}) \tilde{\mathbf{B}}^{T}_{Nx6}(\mathbf{x}) \tilde{\mathbf{B}}_{6xN}(\mathbf{x}) d\Omega \tag{20}$$

so that all partial derivatives of it with respect to random Young modulus equal to 0; the global stiffness matrix equals to

$$\mathbf{K}^{\alpha}_{NxN} = \int_{\Omega_{(e)}} \tilde{\mathbf{B}}^{T}_{Nx6} \mathbf{C}^{\alpha}_{6x6} \tilde{\mathbf{B}}_{6xN} d\Omega \tag{21}$$

and since 3D bar and beam elements are used in further computations (as the linearly dependent on Young modulus), only the first partial derivatives differ from 0. Henceforth, equation (19) can be rewritten with those substitutions as

$$\dot{\mathbf{r}}^{\alpha \mathbf{T}} \mathbf{M}^{\alpha} \delta \mathbf{r} - \int_{t_1}^{t_2} (\ddot{\mathbf{r}}^{\alpha \mathbf{T}} \mathbf{M}^{\alpha} + \mathbf{r}^{\alpha \mathbf{T}} \mathbf{K}^{\alpha} - \mathbf{R}^{\alpha \mathbf{T}}) \delta \mathbf{r} \, d\tau = 0 \tag{22}$$

Considering the assumptions that

$$\delta \mathbf{r}(t_1) = 0, \quad \delta \mathbf{r}(t_2) = 0 \tag{23}$$

we finally obtain the dynamic equilibrium system

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha} \tag{24}$$

which represents the equations of motion of the discretized system. When we complete this equation with the component $\mathbf{C}^{\alpha}_{NxN} \mathbf{r}^{\alpha}_{Nx1}$ getting

$$\mathbf{M}^{\alpha} \ddot{\mathbf{r}}^{\alpha} + \mathbf{C}^{\varepsilon} \dot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha} \mathbf{r}^{\alpha} = \mathbf{R}^{\alpha} \tag{25}$$

then we decompose the damping matrix as

$$\mathbf{C}^{\alpha} = \alpha_0 \mathbf{M}^{\alpha} + \alpha_1 \mathbf{K}^{\alpha} \tag{26}$$

where the coefficients α_0 and α_1 are determined using the specific eigenfunctions for this problem, so that

$$\mathbf{M}^{\alpha}\ddot{\mathbf{r}}^{\alpha} + \alpha_0\mathbf{M}^{\alpha}\dot{\mathbf{r}}^{\alpha} + \alpha_1\mathbf{K}^{\alpha}\dot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha}\mathbf{r}^{\alpha} = \mathbf{R}^{\alpha} \tag{27}$$

where no summation over the doubled indices α is applied here. As it is known [2], the case of the undamped free vibrations leads to the following algebraic system:

$$\mathbf{M}^{\alpha}\ddot{\mathbf{r}}^{\alpha} + \mathbf{K}^{\alpha}\mathbf{r}^{\alpha} = \mathbf{0} \tag{28}$$

and the solution $\mathbf{r}^{\alpha} = \mathbf{A}^{\alpha}\sin\omega_{\alpha}t$ leads to the relation

$$-\mathbf{M}^{\alpha}\mathbf{A}^{\alpha}\omega_{\alpha}^2\sin\omega_{\alpha}t + \mathbf{K}^{\alpha}\mathbf{A}^{\alpha}\sin\omega_{\alpha}t = \mathbf{0} \tag{29}$$

so that for $\sin\omega_{\alpha}t \neq \mathbf{0}$ and $\mathbf{A}^{\alpha} \neq \mathbf{0}$ there holds

$$-\mathbf{M}^{\alpha}\omega_{\alpha}^2 + \mathbf{K}^{\alpha} = \mathbf{0} \tag{30}$$

4. The stochastic perturbation–based approach in the eigenproblems

4.1. *The stochastic Taylor expansion with random coefficients*

Let us introduce the random variable $b \equiv b(\omega)$ and its probability density function as $p(b)$. Then, the expected values and the mth central probabilistic moment are defined as

$$E[b] \equiv b^0 = \int_{-\infty}^{+\infty} bp(b)\,db, \quad \mu_m(b) = \int_{-\infty}^{+\infty} (b - E[b])^m p(b)\,db \tag{31}$$

The basic idea of the stochastic perturbation approach is to expand all the input variables and the state functions via Taylor series about their spatial expectations using some small parameter $\varepsilon > 0$. In case of random quantity $e = e(b)$, the following expression is employed [4,7]:

$$e = e^0 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\varepsilon^n \tfrac{\partial^n e}{\partial b^n}(\Delta b)^n \tag{32}$$

where

$$\varepsilon\Delta b = \varepsilon\left(b - b^0\right) \tag{33}$$

is the first variation of b about b^0. Symbol $(.)^0$ represents the function value $(.)$ taken at the expectation b^0, while $(.)^{,b}$,$(.)^{,bb}$ denote the first and the second partial derivatives with respect to b evaluated at b^0, respectively. Let us analyze further the expected values of any state function $f(b)$ defined analogously to the formula (3) by its expansion via Taylor series with a given small parameter ε as follows:

$$E[f(b);b] = \int_{-\infty}^{+\infty} f(b)p(b)\,db = \int_{-\infty}^{+\infty} \left(f^0 + \sum_{n=1}^{\infty} \tfrac{1}{n!}\varepsilon^n f^{(n)}\Delta b^n\right) p(b)\,db \tag{34}$$

Let us remind that this power expansion is valid only if the state function is analytic in ε and the series converge and, therefore, any criteria of convergence should include the magnitude of the perturbation parameter; perturbation parameter is taken as equal to 1 in numerous practical computations. From the numerical point of view, the expansion provided by the formula (32) is carried out for the summation over the finite number of components. Now, let us focus on an analytical derivation of the probabilistic moments for the structural response function. It is easy to prove that the general 6^{th} order expansion results in the formula

$$E\left[f\left(b\right)\right] = f^{0}(b) + \varepsilon\Delta b\frac{\partial f}{\partial b} + \tfrac{1}{2}\varepsilon^{2}\mu_{2}\left(b\right)\tfrac{\partial^{2} f}{\partial b^{2}} + \tfrac{1}{3!}\varepsilon^{3}\mu_{3}(b)\tfrac{\partial^{3} f}{\partial b^{3}}$$
$$+\tfrac{1}{4!}\varepsilon^{4}\mu_{4}(b)\tfrac{\partial^{4} f}{\partial b^{4}} + \tfrac{1}{5!}\varepsilon^{5}\mu_{5}(b)\tfrac{\partial^{5} f}{\partial b^{5}} + \tfrac{1}{6!}\varepsilon^{6}\mu_{6}(b)\tfrac{\partial^{6} f}{\partial b^{6}} \tag{35}$$

where for Gaussian variables the even components need to be dropped off. Thanks to such an extension of the random output, any desired efficiency of the expected values as well as higher probabilistic moments can be achieved by an appropriate choice of the distribution parameters. Similar considerations lead to the 4th order expressions for a variance; there holds

$$Var\left(f(b)\right) = \varepsilon^{2}\mu_{2}\left(b\right)\frac{\partial f}{\partial b}\frac{\partial f}{\partial b} + \varepsilon^{4}\mu_{4}\left(b\right)\left(\tfrac{1}{4}\tfrac{\partial^{2} f}{\partial b^{2}}\tfrac{\partial^{2} f}{\partial b^{2}} + \tfrac{2}{3!}\tfrac{\partial f}{\partial b}\tfrac{\partial^{3} f}{\partial b^{3}}\right)$$
$$+\varepsilon^{6}\mu_{6}(b)\left(\left(\tfrac{1}{3!}\right)^{2}\frac{\partial^{3} f}{\partial b^{3}}\frac{\partial^{3} f}{\partial b^{3}} + \tfrac{1}{4!}\tfrac{\partial^{4} f}{\partial b^{4}}\tfrac{\partial^{2} f}{\partial b^{2}} + \tfrac{2}{5!}\tfrac{\partial^{5} f}{\partial b^{5}}\tfrac{\partial f}{\partial b}\right) \tag{36}$$

The third order probabilistic moments are derived including the lowest orders only as

$$\mu_{3}\left(f\left(b\right)\right) = \int_{-\infty}^{+\infty}\left(f\left(b\right) - E[f(b)]\right)^{3} p\left(b\right) db$$
$$= \int_{-\infty}^{+\infty}\left(f^{0} + \varepsilon\frac{\partial f}{\partial b}\Delta b + ... - E[f(b)]\right)^{3} p\left(b\right) db \tag{37}$$
$$= \int_{-\infty}^{+\infty}\left(\varepsilon\frac{\partial f}{\partial b}\Delta b + \tfrac{1}{2}\varepsilon^{2}\tfrac{\partial^{2} f}{\partial b^{2}}\Delta b\Delta b + ...\right)^{3} p\left(b\right) db$$
$$\cong \tfrac{3}{2}\varepsilon^{4}\mu_{4}(b)\left(\tfrac{\partial f}{\partial b}\right)^{2}\tfrac{\partial^{2} f}{\partial b^{2}} + \tfrac{1}{8}\varepsilon^{6}\mu_{6}(b)\left(\tfrac{\partial^{2} f}{\partial b^{2}}\right)^{3}$$

Finally, the fourth probabilistic moment is approximated with the first few perturbation terms as

$$\mu_4(f(b)) = \int_{-\infty}^{+\infty} (f(b) - E[f(b)])^4 \, p(b)\, db$$

$$= \int_{-\infty}^{+\infty} \left(f^0 + \varepsilon \frac{\partial f}{\partial b} \Delta b + \ldots - E[f(b)] \right)^4 p(b)\, db \tag{38}$$

$$= \varepsilon^4 \mu_4(b) \left(\frac{\partial f}{\partial b} \right)^4 + \tfrac{3}{2} \varepsilon^6 \mu_6(b) \left(\frac{\partial f}{\partial b} \frac{\partial^2 f}{\partial b^2} \right)^2 + \tfrac{1}{16} \varepsilon^8 \mu_8(b) \left(\frac{\partial^2 f}{\partial b^2} \right)^4$$

Let us mention that it is necessary to multiply in each of these equations by the relevant order probabilistic moments of the input random variables to get the algebraic form convenient for any symbolic computations. Therefore, this method in its generalized form is convenient for all the random distributions, where the above mentioned moments may be analytically derived (or at least computed for a specific combination of those distributions parameters). Finally, one may recover the kurtosis and the skewness after their well–known definitions as

$$\kappa(f(b)) = \frac{\mu_4(f(b))}{\sigma^4(f(b))} - 3, \quad \beta(f(b)) = \frac{\mu_3(f(b))}{\sigma^3(f(b))} \tag{39}$$

and, independently, the reliability index for the particular eigenfrequencies as

$$R(f(b)) = \frac{E\left(\hat{f} - f_\alpha\right)}{\sigma\left(\hat{f} - f_\alpha\right)} \tag{40}$$

where the pair $\left(\hat{f}; f_\alpha\right)$ denotes the induced frequency of the vibrations and the additional eigenfrequency. The relevant civil engineering codes state that this difference cannot be smaller than 25% of the eigenfrequency, so that Eqn. (40) may serve for the straightforward estimation of the reliability for the structures subjected to the dynamic excitations.

4.2. *Eigenfrequencies determination via the response function method*

As shown during derivation of equations for the generalized perturbation based approach, one of the most complicated issues is a reliable numerical determination of up to nth order partial derivatives of the structural response function with respect to the randomized parameter. It is possible to determine this function first by multiple solutions of the boundary value problem around the expectation of the random parameter to complete this task. The response function for each eigenvalue is built up from uniform symmetric discretization in the neighborhood of this expectation, with equidistant intervals. A set of classical deterministic re–computations of the all the components of the eigenvalues vector leads to the final formation of the responses function for all ω_α. That is why we consider further a problem of the

unknown response function approximation by the following polynomial of n-1 order [5]:

$$\omega_\alpha = A_1^{(\alpha)} b^{n-1} + A_2^{(\alpha)} b^{n-2} + ... + A_n^{(\alpha)} b^0 \tag{41}$$

having the values of this function determined computationally for n different arguments. With this representation, the algebraic system of equations is formed

$$\begin{cases} A_1^{(\alpha)} b_1^{n-1} + A_2^{(\alpha)} b_1^{n-2} + ... + A_n^{(\alpha)} b_1^0 = \omega_{\alpha(1)} \\ A_1^{(\alpha)} b_2^{n-1} + A_2^{(\alpha)} b_2^{n-2} + ... + A_n^{(\alpha)} b_2^0 = \omega_{\alpha(2)} \\ ... \\ A_1^{(\alpha)} b_n^{n-1} + A_2^{(\alpha)} b_n^{n-2} + ... + A_n^{(\alpha)} b_n^0 = \omega_{\alpha(n)} \end{cases} \tag{42}$$

where the coefficients $\omega_{\alpha(i)}$ for i=1,...,n denote the approximated function values in ascending order of the arguments bi. Therefore, the following algebraic system of equations is formed to determine the polynomial coefficients $A_i^{(\alpha)}$:

$$\begin{bmatrix} b_1^{n-1} & b_1^{n-2} & ... & b_1^0 \\ b_2^{n-1} & b_2^{n-2} & & b_2^0 \\ ... & ... & & ... \\ b_n^{n-1} & b_n^{n-2} & ... & b_n^0 \end{bmatrix} \begin{Bmatrix} A_1^{(\alpha)} \\ A_2^{(\alpha)} \\ ... \\ A_n^{(\alpha)} \end{Bmatrix} = \begin{Bmatrix} \omega_{\alpha(1)} \\ \omega_{\alpha(2)} \\ ... \\ \omega_{\alpha(n)} \end{Bmatrix} \tag{43}$$

The crucial point in this method is a proper determination of the set of input parameters $\{b_1^0, ..., b_n^0\}$ inserted into this equation. This determination is started with a choice of the computational domain $[b - \Delta b, b + \Delta b]$, where $2\Delta b = 0.05b$. Then, this domain is subdivided into the set of equidistant n–1 intervals with the length $\Delta b_{(m,m+1)} = \frac{2\Delta b}{n-1}$ for any m=1,..,n-1. So that assuming that $b_0 = b - \Delta b$ it is obtained that $b_m = b - \Delta b + m\frac{2\Delta b}{n-1}$. Let us note that since this linear system of equations is non–symmetric, its solution cannot be done by the integration with the FEM solver, and some separate numerical procedure based on the Gauss–Jordan elimination scheme must be employed. The unique solution for this system makes it possible to calculate up to the nth order ordinary derivatives of the homogenized elasticity tensor with respect to the parameter b at the given b_0 as 1^{st} order derivative

$$\frac{\partial \omega_\alpha}{\partial b} = (n-1) A_1^{(\alpha)} b^{n-2} + (n-2) A_2^{(\alpha)} b^{n-3} + ... + A_{n-1}^{(\alpha)} \tag{44}$$

2^{nd} order derivative

$$\frac{\partial^2 \omega_\alpha}{\partial b^2} = (n-1)(n-2) A_1^{(\alpha)} b^{n-3} + (n-2)(n-3) A_2^{(\alpha)} b^{n-4} + ... + A_{n-2}^{(\alpha)} \tag{45}$$

kth order derivative

$$\frac{\partial^k \omega_\alpha}{\partial b^k} = \prod_{i=1}^{k} (n-i) A_1^{(\alpha)} b^{n-k} + \prod_{i=2}^{k} (n-i) A_2^{(\alpha)} b^{n-(k+1)} + ... + A_{n-k}^{(\alpha)} \tag{46}$$

Providing that the response function of the structural eigenvalue has a single independent argument, that is, the input random variable of the problem, it is possible to employ the stochastic perturbation technique based on the Taylor representation to compute up to the mth order probabilistic moments $\mu_m(\omega_\alpha)$. It is clear from the derivation above that to complete the mth order approximation we need to solve the initial deterministic problem m times, with its number of degrees of freedom and a single system of algebraic equations mxm, to find a single response function. Including the formulas above for the derivatives of the response function in a definition of the probabilistic moments, one can determine the expectations, variances as well as any order random characteristics of the structural response.

5. Numerical illustrations

Computational analysis has been provided on the example of the steel telecommunication tower with the height equal to 42.0 meters discretized with 396 two–noded linear elastic beams and bars having common 165 nodal points and presented schematically in Fig. 1. The cross–sectional areas in both towers are (1) exactly the same and (2) optimally designed to fulfill the strength criteria concerning the aluminium usage, which automatically means some overdesigning load capacity effect on the steel version of this tower. Further computational experiments will deal with the optimally chosen cross–sections for the steel and aluminium made profiles, where the expected values for the steel members of the Young modulus is taken as $E[E_s] = 210\ GPa$ and we have for aluminium $E[E_a] = 75\ GPa$. The geometrical data for the specific cross–sections are contained in Tab. 1.

Segment no	**Tower legs**	**Tower braces**
1	RO Ø 60,3x3,6 mm	RO Ø 22,0x2,0 mm
2	RO Ø 60,3x3,6 mm	RO Ø 25,0x3,0 mm
3	RO Ø 76,1x5,0 mm	RO Ø 38,0x4,0 mm
4	RO Ø 88,9x6,3 mm	C 30x30x3 mm
5	RO Ø 88,9x6,3 mm	C 30x30x3 mm
6	RO Ø 114,3x6,3 mm	C 40x40x3 mm
7	RO Ø 114,3x6,3 mm	C 40x40x3 mm

The results of computational modeling are presented in Figs 2–7, where we have in turn: the expected values, standard deviations, kurtosis, skewness as well as the reliability index for both aluminium (left graphs) and steel structures (right graphs). They are all shown with respect to the coefficient of variations of the Young modulus for the tower basic designed material; this coefficient belongs to the interval [0.00,0.20], which is relatively wide interval considering maximum 10% random dispersion accompanying most of experimental tests in this case. Further, it is important that all those moments and coefficients have been determined using the sixth order perturbation approach.

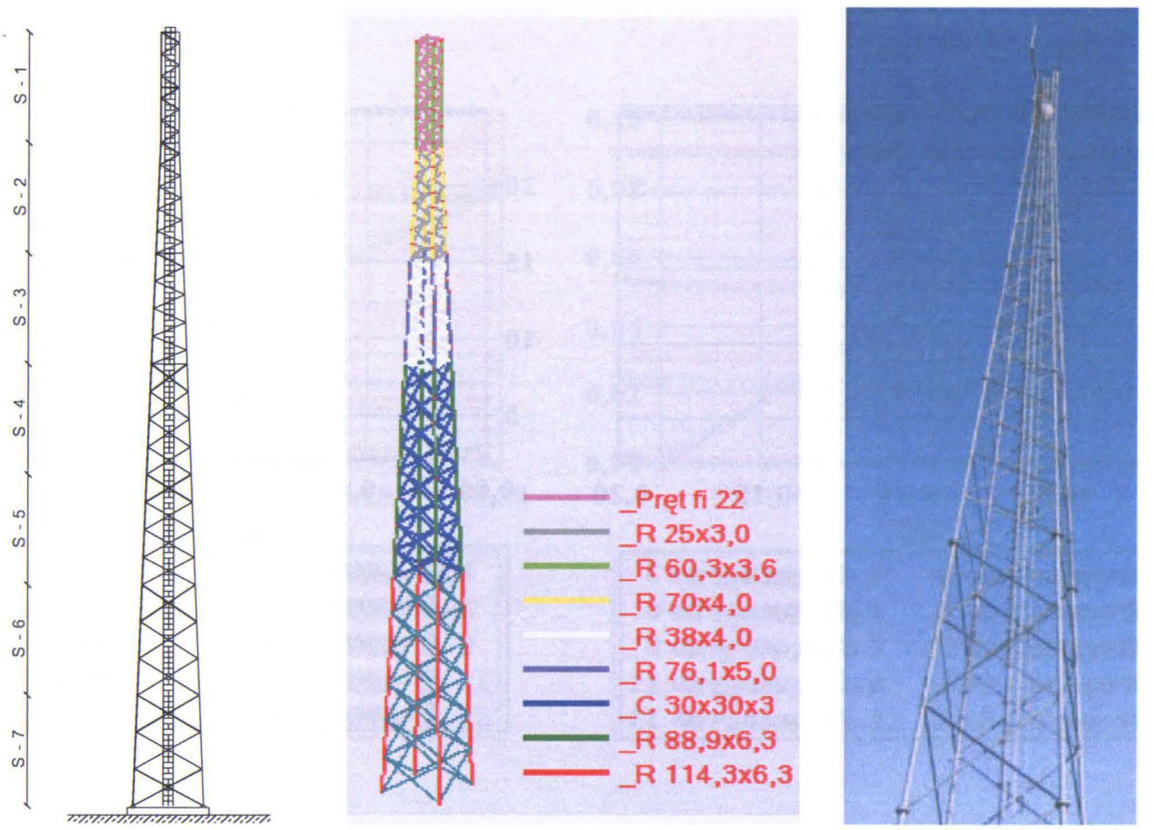

Figure 1 Static scheme and photo of the telecommunication tower

As it is apparent from both graphs in Fig. 2, the expected values for the eigenfrequencies from 1^{st} to 13^{th} are clearly very similar to each other for both materials. It follows directly the analytical results for the eigenfrequencies of the elastic beams, where, independently from the boundary conditions of the simple beam, those frequencies are proportional to the Young modulus and intertia moment, while inversely proportional to the mass of the element (into the square root). Let us recall here that the ratio of the Young moduli for aluminium and steel is very similar to the ratio of their unit masses, so that this conclusion seems to be very well justified. Let us note that the intermediate eigenfrequencies differ from each other here, so that it cannot be concluded precisely that they are always almost equal (see 11^{th} and 12^{th} eigenfrequencies, for instance). The sensitivity of the expected values with respect to the input coefficient of variation α is, however somewhat different – steel structure exhibits no such a sensitivity, while in the case of aluminium tower – the expectations show no sensitivity until α=0.15, while for larger values – may increase or decrease as well.

Further, we analyze the standard deviations, necessary in the reliability index, as well as the output coefficients of variation, also for particular eigenfrequencies. It is quite clear that the model is linear in the probabilistic sense, because the ratio of output and input coefficients of variation remains constant. We notice also the apparent damping since 0.10–0.12 is obtained for both materials from 0.20 as the input value of the parameter α and it also follows the analytical result recalled above.

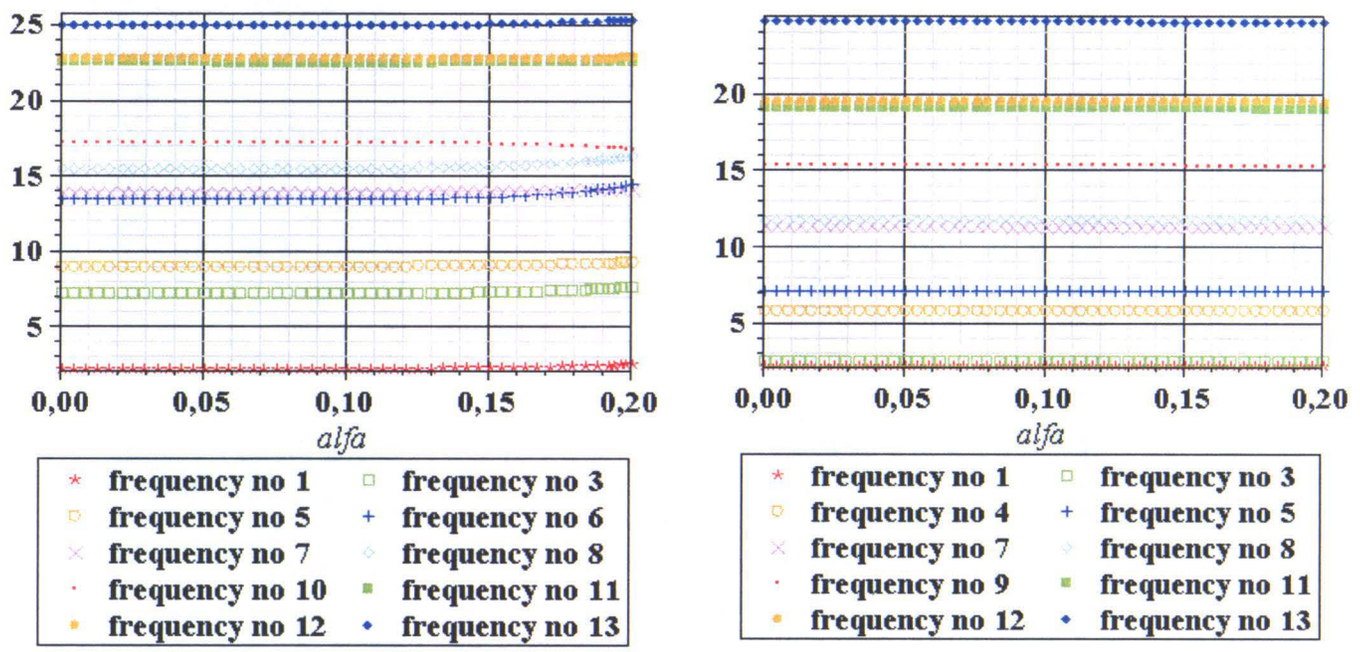

Figure 2 Expected values of the eigenvalues for the aluminium and steel towers

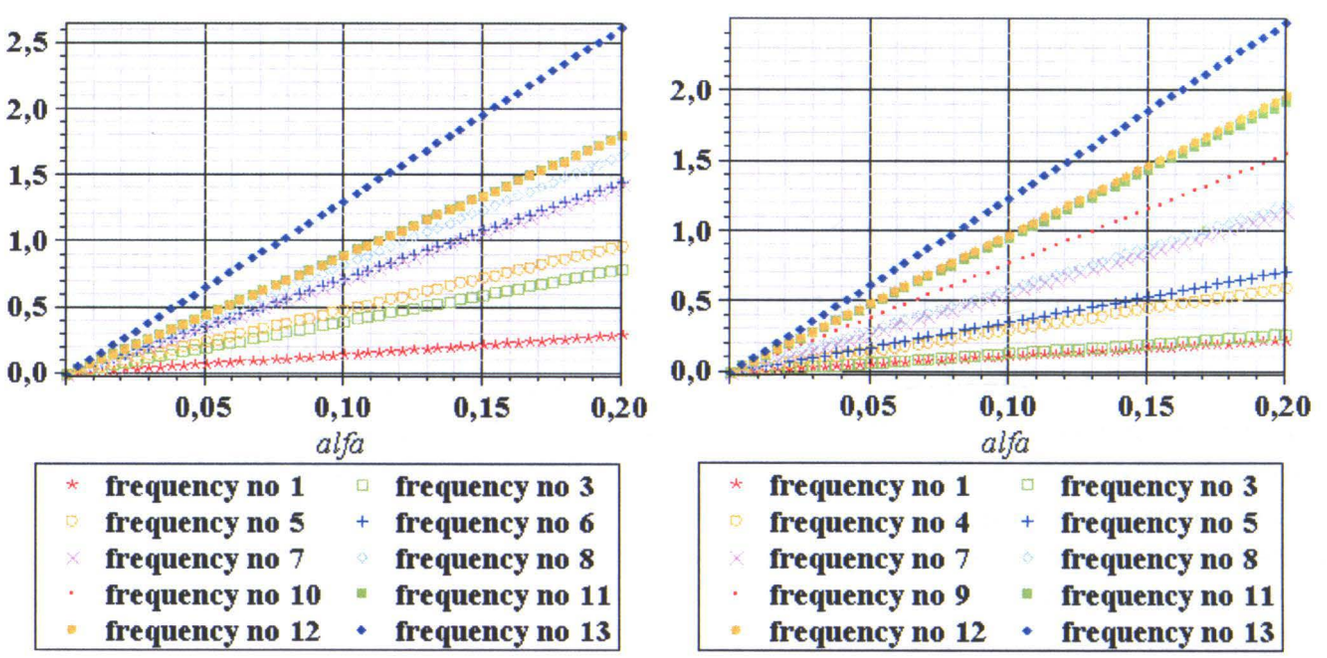

Figure 3 Standard deviations of the eigenvalues for the aluminium and steel towers

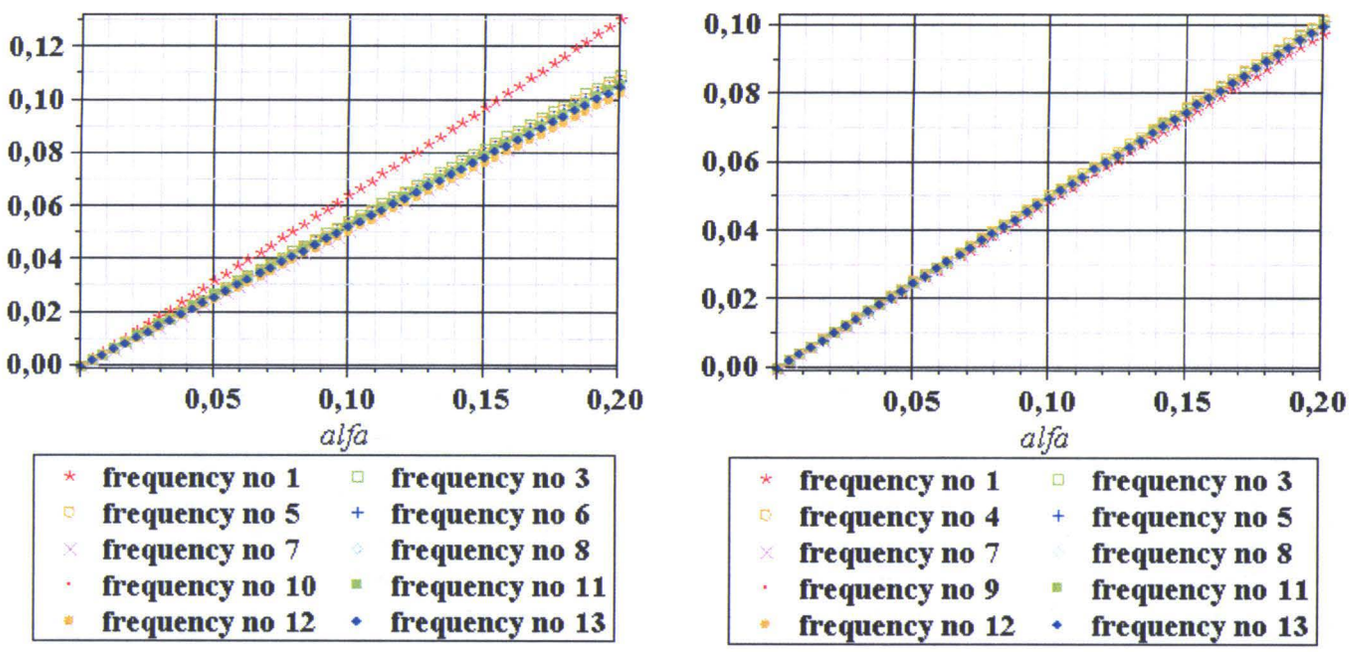

Figure 4 Coefficients of variation of the eigenvalues for the aluminium and steel towers

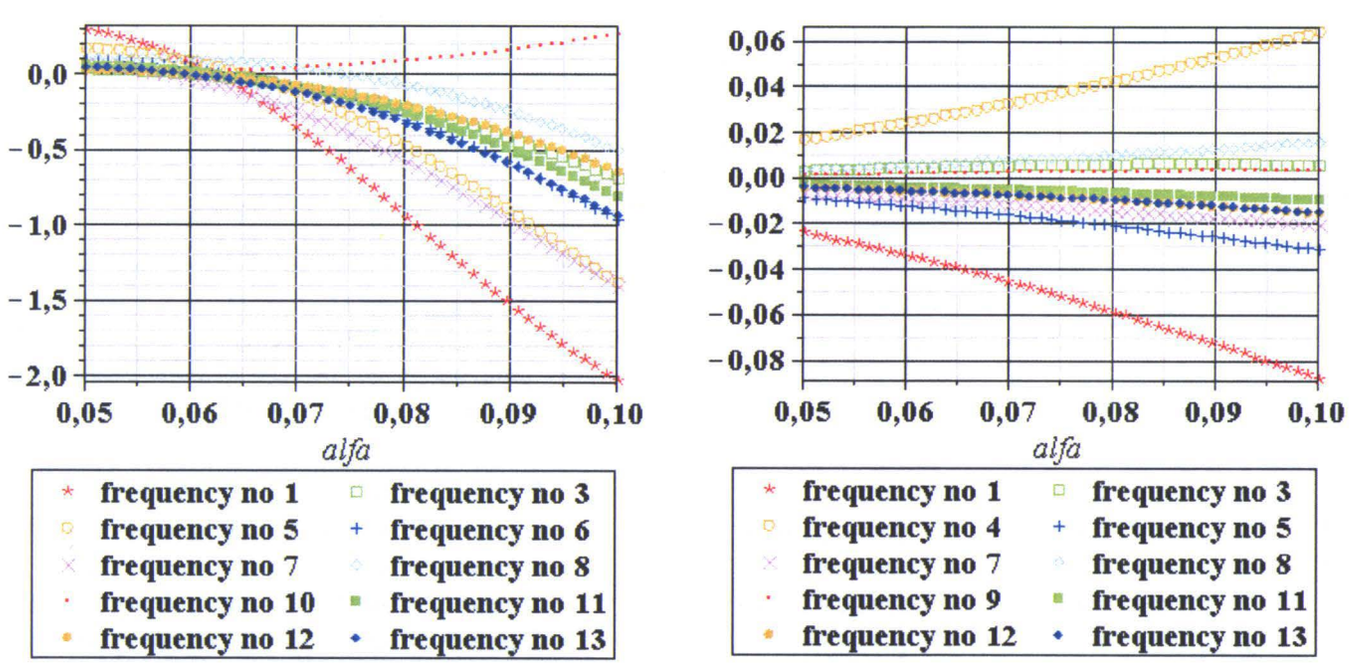

Figure 5 Kurtosis of the eigenvalues for the aluminium and steel towers

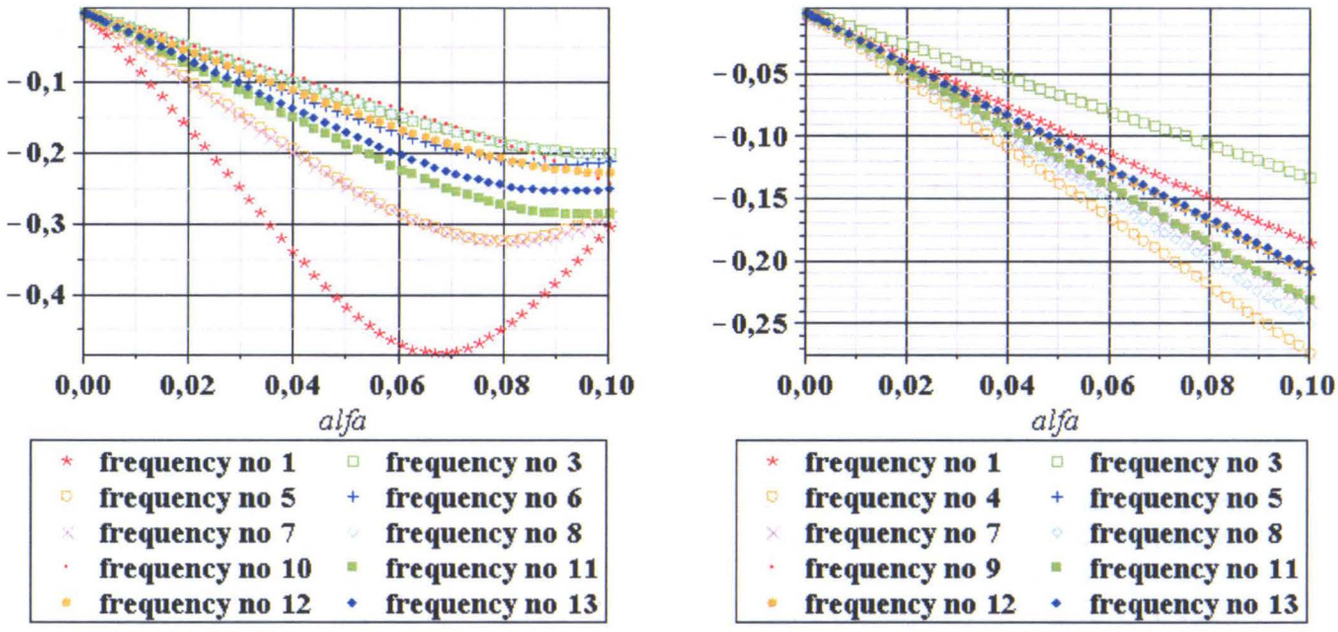

Figure 6 Skewness of the eigenvalues for the aluminium and steel towers

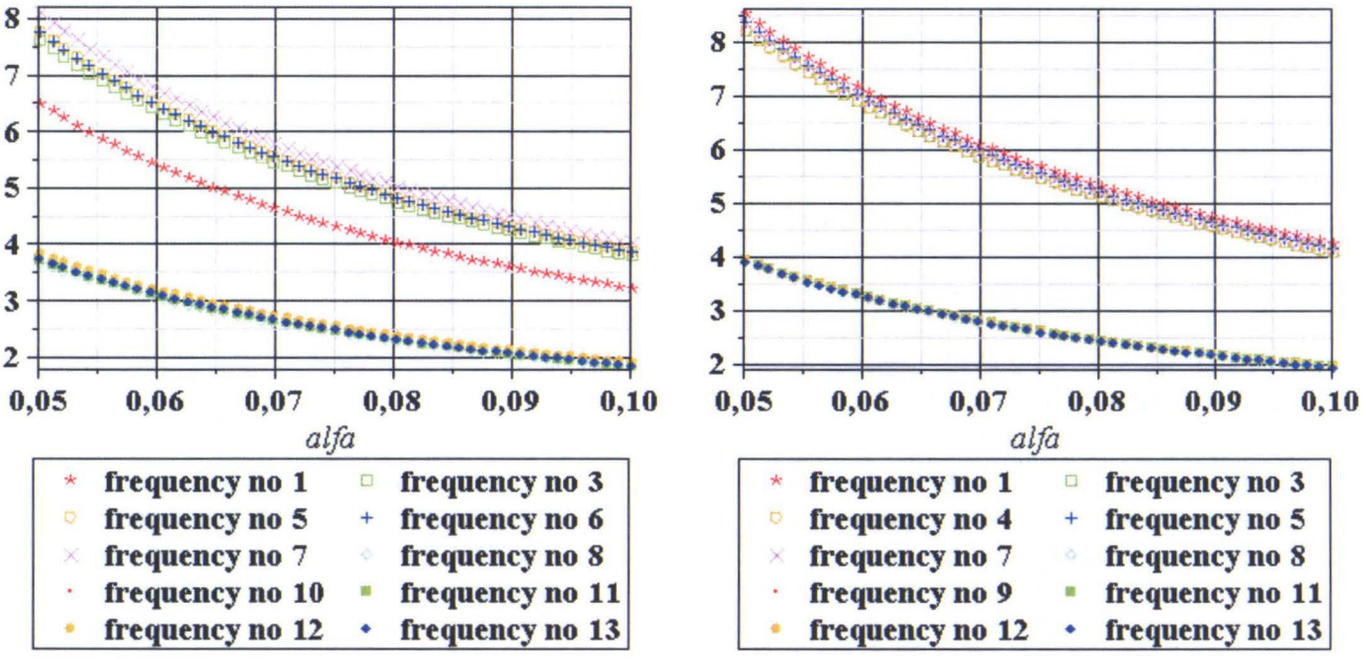

Figure 7 Reliability index in eigenvibrations analysis of the aluminium and steel towers

Higher damping is noticed for the steel tower, where additionally the results for particular eigenfrequencies are less dispersed than for the aluminium structures.

Kurtosis given in Fig. 5, however, is apparently different for both materials – they both start from 0 for the input coefficient of variation close to 0 to some mostly negative values for the aluminium and relatively small positive as well as negative values in the case of steel (closer to 0). Contrary to the previous moments and coefficients, the fourth order quantities are computed with lower accuracy lost with larger values of α, so that the results are restricted to the 10% input random dispersion; this is also the case of skewness as the result of the third order approximations (see Fig. 6). These skewnesses exhibit similar properties as the kurtosis – in the sense that they have smaller absolute values for steel tower than for the aluminium one. Steel structure shows a linear interrelation between output skewness and input coefficient of variation of the Young modulus. This is absolutely not the case of aluminium tower eigenfrequencies, where this interrelation does not seem to be linear, while the minimum values apparently differ from 0. Trying to generalize those results one may notice that Gaussian Young modulus of the tower result in the eigenfrequencies being almost Gaussian, when the tower is made of steel, whereas the final distributions of aluminium eigenfrequencies are more distant from the Gaussian one (due to negative skewness and kurtosis).

Finally, we study the variations of the reliability index as the function of the input uncertainty for α belonging to the interval [0.0, 0.10]. The results obtained for both materials are quite similar – the larger input coefficient of variation, the smaller final reliability index value. It is known from the Eurocode 0 regulations, that the unconditional structural safety is preserved in the case of reliability index larger than about 4.5. Fig. 7 shows clearly that the safety margin for both structures is rather small, because this limit value is reached for the eigenfrequencies lower or equal to 10^{th} at the input coefficient $\alpha = 0.07$ (for aluminium) and for $\alpha = 0.07$ (in the case of steel). This structure is, however, never safe in the view of higher eigenfrequencies, because for the entire variability of input coefficient of variation the final reliability index is equal or smaller than 4. It is seen that the structure safely designed according to the strength and deflections condition not necessarily exhibit full safety in the view of eigenvibrations analysis. One needs to remember also that usually input coefficient of variation increases together with the exploitation time, so that the graphs, provided may be directly interpreted during full stochastic reliability analysis, after a sensible calibration of time versus input random dispersion level.

6. Concluding remarks

The main result of the analyses presented in this paper is that the eigenfrequencies expectations computed for aluminium and steel towers are very close to each other, which follows almost identical interrelations of the Young moduli and densities of both materials. Both materials exhibit probabilistic damping in free vibrations analysis decreasing almost twice the input uncertainty level. The probabilistic distributions for all eigenfrequencies in the steel tower are essentially closer to the Gaussian origin than for the aluminium tower, where both skewness and kurtosis show clearly negative values. The reliability analysis is also straightforward pro-

cedure with the Stochastic Finite Element Method perturbation–based technique implemented provided that the direct difference in–between induced frequency of vibrations and the eigenfrequency is declared in percents with respect to this last quantity. Otherwise, of course, full stochastic forced vibrations analysis is necessary, which needs further extensive developments of the SFEM procedures. There is no doubt that the computer algebra system plays the crucial role in the computational strategy – one may try to use this hybrid strategy with the response function method in addition to the other probability density functions, especially for the lognormal variables, where all central moments of any order have additional analytical forms. Otherwise, some further numerical techniques must be employed to recover those moments for the needs of specific input random variables configuration. The structural open research problems may be for instance the SFEM analysis of stochastic earthquake vibrations applied at the foundations of such towers, significantly influencing stochastic reliability of those structures.

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