ASYMPTOTIC AND TOLERANCE 2D-MODELLING IN ELASTODYNAMICS OF CERTAIN THIN-WALLED STRUCTURES

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The object of analysis is a plane structures reinforced by system of periodically distributed thin parallel ribs (Fig.1). The aim of contribution is to derive a 2D-macroscopic mathematical model describing elastodynamics behaviour of this structure. The considerations are based on those summarized in monographs (Woźniak, Michalak, Jędrysiak, eds. 2008). Some applications of the tolerance averaging technique for the modelling of various dynamic and stability problems for elastic microheterogeneous structures are given in series of paper by: Baron (2003), Jędrysiak and Michalak (2011), Michalak and Wirowski (2011), Nagórko and Woźniak (2002), Wągrowska and Woźniak (1996), Wierzbicki and Woźniak (2000).

1. FORMULATION OF THE MODELLING PROBLEM

Introduce the orthogonal Cartesian coordinate system $Ox^1x^2x^3$ in the physical space occupied by a plate under consideration. Let $\Xi = (0, L_1) \times (0, L_2)$ be the midplane (the symmetry plane) of the structure. It is assumed that a thickness of the plate h is small compared to the minimum length dimension of the midplane of the plate, $h \ll \min(L_1, L_2)$. At the same time the thickness h is supposed to be small compared to the width of the stiffened ribs H, $h \ll H$ (Fig.2).

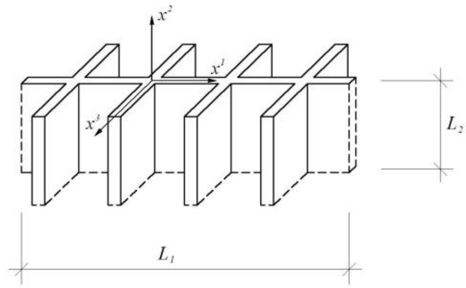

Fig. 1. A fragment of a plate structure with periodic system of stiffeners

Subsequently it will be assumed that number n of the ribs is very large, $1/n \ll 1$, and the maximum distance l between ribs is very small when compared to L_1. Hence $l = L_1/n$ will be treted as a microstructure length parameter. At the same time, the thickness h of the plate is supposed to be small compared to the microstructure length parameter l, $h \ll l$.

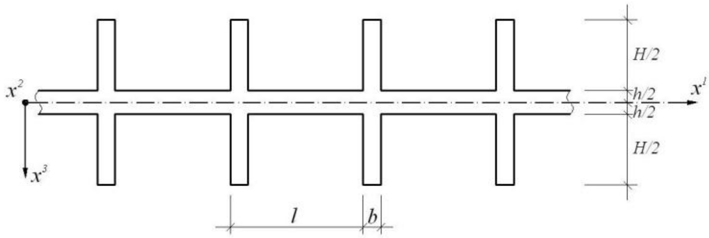

Fig. 2. A fragment of a cross-section of the stiffened plate structure

The aim of this contribution is to formulate 2D macroscopic models of dynamics behaviour of the plate under consideration. There models will be referred to as asymptotic and tolerance, respectively. By the 2-dimensional macroscopic model we shall understand mathematical model governed by averaged equations of motion with smooth coefficients and unknowns functions dependent on coordinates x^1 and x^2.

Throughout the paper, indices $i,k,l\ldots$ run over $1,2,3$, indices $\alpha,\beta,\gamma,\ldots$run over $1,2$ and t stand for the time coordinate. Subsequently we shall use denotations $x = x^1$, $\partial_1 = \partial/\partial x^1$, $\partial_2 = \partial/\partial x^2$. The summation convention holds all aforementioned sub-and superscripts.

2. PRELIMINARES

The considerations are based on the well-known equations for the plane stress state in the plate. It is assumed that the undeformed midplane of the plate occupies region $\Xi = (0,L_1)\times(0,L_2)$. Denoting by l distance between the ribs of the plate-structures, every Δ_i, where $x_i = l/2 + (i-1)l$, $i = 1,2,\ldots,n,$, $(1/n \ll 1)$, will be referred to the cell in Ξ with centre at x_i (Fig.3). Let $\overline{\Omega} = \bigcup\Delta_i \times [0,L_2]$ will be region in the physical space occupied by plate-structure and $\mathrm{int}(\bigcup\Delta_i)$ -cross section of Ω by every $x^2 \in (0,L_2)$ -plane. Let subcells $\Delta_i^{\ P}$, $\Delta_i^{\ S}$, $\Delta_i^{\ SP}$ will be parts of every cell $\Delta_i(x)$; belonging to plate, ribs-stiffeners and part belonging both to plate and stiffeners, respectively.

The model equations for the dynamic behaviour of the plate-structure under consideration will be obtained for plane-stress state in the plate.

Subcells $\Delta_i^{\ P}$. Plane stress $n^{33} = 0$, hence

$$n^{11} = \frac{hE}{1-\nu^2}(e_{11} + \nu e_{22})$$
$$n^{22} = \frac{hE}{1-\nu^2}(e_{22} + \nu e_{11}), \qquad (1)$$
$$n^{12} = \frac{hE}{1+\nu}e_{12}$$

where $e_{\alpha\beta}$ strain tensors.

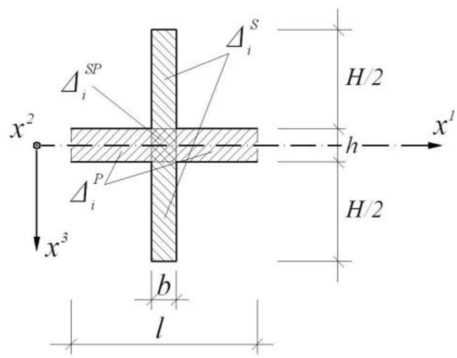

Fig. 3. The basic cell of the stiffened plate structure

Subcells Δ_i^{SP}. In this region of the structure we consider 3D-stress state

$$n^{11} = h(\lambda + 2\mu)e_{11} + h\lambda(e_{22} + e_{33})$$
$$n^{22} = h(\lambda + 2\mu)e_{22} + h\lambda(e_{11} + e_{33})$$
$$n^{33} = h(\lambda + 2\mu)e_{33} + h\lambda(e_{11} + e_{22}), \qquad (2)$$
$$n^{12} = \frac{hE}{1+\nu}e_{12}$$

where λ, μ are Lame's constants.

Subcells Δ_i^S. Plane stress $n^{11} = 0$, hence

$$n^{22} = \frac{hE}{1-\nu^2}(e_{22} + \nu e_{33})$$
$$n^{33} = \frac{hE}{1-\nu^2}(e_{33} + \nu e_{22}), \qquad (3)$$
$$n^{12} = \frac{hE}{1+\nu}e_{12}$$

From condition of continuity on interfaces $\overline{\Delta}_i^S \cap \overline{\Delta}_i^{SP}$

$$h(\lambda+2\mu)e_{33}+h\lambda(e_{11}+e_{22})=\frac{hE}{1-\nu^2}(e_{33}+\nu e_{22}), \tag{4}$$

we derive

$$e_{33}=[\frac{E}{1-\nu^2}e_{22}-\frac{h}{H}\lambda(e_{11}+e_{22})]/[\frac{h}{H}(\lambda+2\mu)-\frac{E}{1-\nu^2}] \tag{5}$$

and bearing in mind that $h \ll H$ we shall assume approximation that h/H is negligibly small in formula (4) and then formula (4) take the form $e_{33}=-\nu e_{22}$. Hence we assume conditions; $e_{33}=-\nu e_{22}$ in subcell Δ_i^{S} and $e_{33}=0$ in subcell Δ_i^{SP}.

Averaging formulae (2), (3) in Ξ_S over $(-(h+H)/2,(h+H)/2)$ we have

$$\begin{aligned} N^{11} &= h(\lambda+2\mu)e_{11}+h\lambda e_{22} \\ N^{22} &= [HE+h(\lambda+2\mu)]e_{22}+h\lambda e_{11}, \\ N^{12} &= \frac{hE}{1+\nu}e_{12} \end{aligned} \tag{6}$$

and in Ξ_P over $(-h,h)$

$$\begin{aligned} n^{11} &= \frac{hE}{1-\nu^2}(e_{11}+\nu e_{22}) \\ n^{22} &= \frac{hE}{1-\nu^2}(e_{22}+\nu e_{11}), \\ n^{12} &= \frac{hE}{1+\nu}e_{12} \end{aligned} \tag{7}$$

we derive constitutive equations for 2-dimensional model of the heterogeneous structure under consideration.

2-dimensional model of the plate structures with the periodic distribution of ribs.

Let displacement of the midplane of the plate will be denoted by $w_\alpha(x^\beta,t)$, external forces by $p_\alpha(x^\beta,t)$ and by $\tilde{\rho}$ the mass density averaged over the plate thickness related to the midplane.

In the framework of the linear theory for plane-stress state we obtain:

- equations of motion

$$\partial_\alpha \tilde{N}^{\alpha\beta}+p^\beta-\tilde{\rho}\,\ddot{w}^\beta=0, \tag{8}$$

where

$$\tilde{\rho} = \begin{cases} \rho h & \text{in}\,\Xi_P \\ \rho(h+H) & \text{in}\,\Xi_S \end{cases}, \tag{9}$$

$$\tilde{N}^{\alpha\beta} = \begin{cases} n^{\alpha\beta} & \text{in}\,\Xi_P \\ N^{\alpha\beta} & \text{in}\,\Xi_S \end{cases}, \tag{10}$$

- constitutive equations, which we shall write in the form

$$\tilde{N}^{\alpha\beta} = D^{\alpha\beta\gamma\delta} e_{\gamma\delta}, \tag{11}$$

where

$$\begin{aligned} \tilde{N}^{11} &= D^{1111} e_{11} + D^{1122} e_{22}, \\ \tilde{N}^{22} &= D^{2211} e_{11} + D^{2222} e_{22}, \\ \tilde{N}^{12} &= D^{1212} e_{12}. \end{aligned} \tag{12}$$

It can be seen that the coefficients in above equations are discontinuous and highly oscillating. These equations are to complicated to be used in the engineering analysis and will be used as starting point in the tolerance modelling procedure.

3. TOLERANCE MODELLING

In order to derive averaged model equations we applied tolerance averaging approach. This technique based on the concept of tolerance and indiscrenibility relations and on the definition of slowly-varying functions. The general modelling procedures of this technique are given in books (Woźniak et al. 2008, 2010).

The fundamental concept of the modelling technique is the averaging an arbitrary integrable function $f(\cdot)$ over the cell Δ_i

$$< f >= \frac{1}{\Delta_i} \int_{\Delta_i} f(y)\,dy \qquad y \in \Delta_i \ . \tag{13}$$

The first assumption in the tolerance modelling is micro-macro decomposition of the displacement field

$$w_\alpha(x^\beta, t) = u_\alpha(x^\beta, t) + g(x^1) V_\alpha(x^\beta, t) \tag{14}$$

for $x^\alpha \in \Xi$ and $t \in (t_0, t_1)$.

The modelling assumption states that $u_\alpha(\cdot)$, $V_\alpha(\cdot)$ are slowly-varying functions with respect to the argument $x^1 \in (0, L_1)$. Functions $u_\alpha(\cdot, x^2, t) \in SV_\delta^1(\Xi, \Delta)$,

$V_\alpha(\cdot, x^2, t) \in SV^1_\delta(\Xi, \Delta)$ are the basic unknowns of the tolerance model. Function $g(x^1)$ is known, dependent on the microstructure length parameter l, fluctuation shape function.

Let $\tilde{g}(\cdot)$, $\partial_1\tilde{g}(\cdot)$ stand for periodic approximation of $g(\cdot)$, $\partial_1 g(\cdot)$ in Δ, respectively. Due to the fact that $w_\alpha(\cdot, x^2, t)$ are tolerance periodic functions, it can be observe that the periodic approximation of $w_{\alpha h}(\cdot, x^2, t)$ and $\partial_\beta w_{\alpha h}(\cdot, x^2, t)$ in $\Delta(x^1)$, $x^1 \in \Xi$ have the form

$$
\begin{aligned}
&w_{\alpha h}(y, x^2, t) = u_\alpha(x^\beta, t) + g(y) V_\alpha(x^\beta, t),\\
&\partial_\beta w_{\alpha h}(y, x^2, t) = \partial_\beta u_\alpha(x^\gamma, t) + \partial_1 g(y) V_\alpha(x^\gamma, t) + g(y)\partial_2 V_\alpha(x^\gamma, t),\\
&\dot{w}_{\alpha h}(y, x^2, t) = \dot{u}_\alpha(x^\beta, t) + g(y)\dot{V}_\alpha(x^\beta, t),
\end{aligned}
\tag{15}
$$

for every $x^1 \in \Xi$, almost every $y \in \Delta(x^1)$ and every $t \in (t_0, t_1)$.

The modelling assumption states that if in every cell $\Delta(x^1)$, $x^1 \in \Xi$ will be define residual forces

$$
r^\beta = \partial_\alpha \tilde{N}^{\alpha\beta} + p^\beta - \tilde{\rho}\,\ddot{w}^\beta \tag{16}
$$

then the following orthogonality conditions holds

$$
< r^\beta >_T (x^1) = 0, \qquad < g\, r^\beta >_T (x^1) = 0 \tag{17}
$$

where operator $< \cdot >_T (x^1)$ stands for tolerance averaging over the cell $\Delta(x^1)$.

Substituting the right-hand side of formula (13) into equations (16) and bearing in mind orthogonality conditions (17), we obtain the following system of equations of motion

$$
\begin{aligned}
&\partial_\alpha(< D^{\alpha\beta\gamma\delta} > \partial_\gamma u_\delta + < D^{\alpha\beta 1\delta}\partial_1 g > V_\delta + < D^{\alpha\beta 2\delta} g > \partial_2 V_\delta) + < p^\beta > -\\
&< \tilde{\rho} > \ddot{u}^\beta - < \tilde{\rho} g > \ddot{V}^\beta = 0\\
&\partial_2(< D^{2\beta\gamma\delta} g > \partial_\gamma u_\delta + < g\, D^{2\beta 1\delta}\partial_1 g > V_\delta + < g\, D^{2\beta 2\delta} g > \partial_2 V_\delta) - < D^{1\beta\gamma\delta}\partial_1 g > \partial_\gamma u_\delta -\\
&< \partial_1 g\, D^{1\beta 1\delta}\partial_1 g > V_\delta - < \partial_1 g\, D^{1\beta 2\delta} g > \partial_2 V_\delta + < p^\beta g > - < \tilde{\rho}\, g > \ddot{u}^\beta - < g\,\tilde{\rho} g > \ddot{V}^\beta = 0
\end{aligned}
\tag{18}
$$

The above results represent the system equations for averaged displacements $u_\alpha(x^\beta, t)$, and displacements fluctuation amplitudes $V_\alpha(x^\beta, t)$. These equations together with micro-macro decomposition of displacement fields (14) and physical condition that solutions have to be slowly-varying functions with respect to the argument $x^1 \in (0, L_1)$, constitute the tolerance model of structural plate under consideration.

3. ASYMPTOTIC MODEL

For asymptotic modelling procedure we retain only the concept of highly oscillating function. We shall not deal with the concept of the tolerance periodic function as well as slowly-varying function. Using the asymptotic procedure we introduce parameter $\varepsilon = 1/n$, $n = 1, 2, \ldots$. Let εl, εh, εH and εb are a scaled dimensions of the cell Δ A scaled cell is defined by $\Delta_\varepsilon \equiv (-\varepsilon l/2, \varepsilon l/2)$ and $\Delta_\varepsilon(x) = x + \Delta_\varepsilon$ is a scaled cell with a centre at $x \in \overline{\Xi}$.

The mass density $\tilde{\rho}(\cdot)$ and tensor of elastic moduli $D^{\alpha\beta\delta}(\cdot)$ are assumed to be highly oscillating and discontinuous functions, $\tilde{\rho}(\cdot)$, $D^{\alpha\beta\gamma\delta}(\cdot) \in HO^0_\delta(\Xi, \Delta)$, for almost every $x \in \overline{\Xi}$. If $\tilde{\rho}(\cdot)$, $D^{\alpha\beta\delta}(\cdot) \in HO^0_\delta(\Xi, \Delta)$ then for every $x \in \overline{\Xi}$ there exist functions $\tilde{\rho}(y, x^2)$, $D^{\alpha\beta\gamma\delta}(y, x^2)$ which are periodic approximation of functions $\tilde{\rho}(\cdot)$, $D^{\alpha\beta\gamma\delta}(\cdot)$, respectively.

The fundamental assumption of the asymptotic modelling is that we introduce decomposition of displacement as family of fields

$$w_{\alpha\varepsilon}(y, x^2, t) = u_\alpha(y, t) + \varepsilon \tilde{g}\left(\frac{y}{\varepsilon}\right) V_\alpha(y, t), \qquad y \in \Delta(x),\ t \in (t_0, t_1) \tag{19}$$

where $\tilde{g}(\cdot, x^2)$ are periodic approximation of highly oscillating functions $g(\cdot) \in HO^1_\delta(\Xi, \Delta)$. From formula (19) we obtain

$$\begin{aligned} &\partial_\beta w_{\alpha\varepsilon}(y, x^2, t) = \partial_\beta u_\alpha(y, x^2, t) + \partial_1 \tilde{g}\left(\frac{y}{\varepsilon}\right) V_\alpha(y, x^2, t) + \varepsilon \tilde{g}\left(\frac{y}{\varepsilon}\right) \partial_2 V_\alpha(y, x^2, t), \\ &\dot{w}_{\alpha\varepsilon}(y, x^2, t) = \dot{u}_\alpha(y, x^2, t) + \varepsilon \tilde{g}\left(\frac{y}{\varepsilon}\right) \dot{V}_\alpha(y, x^2, t), \end{aligned} \tag{20}$$

Bearing in mind that by means of property of the mean value, Jikov et al. (1994), function $\tilde{g}(y/\varepsilon, x^2)$, $y \in \Delta_\varepsilon(x)$, is weakly bounded and has under $\varepsilon \to 0$ weak limit. Under limit passage $\varepsilon \to 0$ for $y \in \Delta_\varepsilon(x)$ we obtain

$$\begin{aligned} &u_\alpha(y, x^2, t) = u_\alpha(x^\beta, t) + O(\varepsilon), && \partial_\beta u_\alpha(y, x^2, t) = \partial_\beta u_\alpha(x^\gamma, t) + O(\varepsilon), \\ &V_\alpha(y, x^2, t) = V_\alpha(x^\beta, t) + O(\varepsilon), && \partial_\beta V_\alpha(y, x^2, t) = \partial_\beta V_\alpha(x^\gamma, t) + O(\varepsilon), \\ &\dot{u}_\alpha(y, x^2, t) = \dot{u}_\alpha(x^\beta, t) + O(\varepsilon), && \dot{V}_\alpha(y, x^2, t) = \dot{V}_\alpha(x^\beta, t) + O(\varepsilon). \end{aligned} \tag{21}$$

By means of (21) we rewrite formulae (19) and (20) in the form

$$
\begin{aligned}
&w_{\alpha\varepsilon}(y, x^2, t) = u_\alpha(x^\beta, t) + O(\varepsilon),\\
&\partial_\beta w_{\alpha\varepsilon}(y, x^2, t) = \partial_\beta u_\alpha(x^\beta, t) + \partial_1 \tilde{g}(\frac{y}{\varepsilon}) V_\alpha(x^\beta, t) + O(\varepsilon).
\end{aligned} \tag{22}
$$

Using formulae (22) for orthogonality conditions (17) we obtain equations

$$
\begin{aligned}
&\partial_\alpha (< D^{\alpha\beta\gamma\delta} > \partial_\gamma u_\delta + < D^{\alpha\beta 1\delta} \partial_1 g > V_\delta) + < p^\beta > - < \tilde{\rho} > \ddot{u}^\beta = 0\\
&< D^{1\beta\gamma\delta} \partial_1 g > \partial_\gamma u_\delta + < \partial_1 g\, D^{1\beta 1\delta} \partial_1 g > V_\delta = 0
\end{aligned} \tag{23}
$$

Eliminating V_δ from equations (23)

$$
V_\delta = -\frac{< D^{1\beta\gamma\delta} \partial_1 g >}{< \partial_1 g\, D^{1\beta 1\delta} \partial_1 g >} \partial_\gamma u_\delta\,, \tag{24}
$$

we arrive the following equation of motion for the averaged displacements of the plate midplane $u_\alpha(x^\beta, t)$

$$
\partial_\alpha (< D^{\alpha\beta\gamma\delta} > - \frac{< D^{\alpha\beta 1\tau} \partial_1 g >< D^{1\mu\gamma\delta} \partial_1 g >}{< \partial_1 g\, D^{1\mu 1\tau} \partial_1 g >}) \partial_\gamma u_\delta + < p^\beta > - < \tilde{\rho} > \ddot{u}^\beta = 0\,. \tag{25}
$$

Equations (24), (25) represent the asymptotic model of the structural plate under consideration.

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