Internal resonances in nonlinear vibrations of a continuous rod with microstructure
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Abstract: Nonlinear longitudinal vibrations of a periodically heterogeneous rod are considered. Geometrical nonlinearity is described by the Cauchy–Green strain tensor. Physical nonlinearity is modelled expressing the energy of deformation as a series expansion in powers of the strains. The governing macroscopic dynamical equation is obtained by the higher-order asymptotic homogenization method. An asymptotic solution is developed by the method of multiple time scales. The effects of internal resonances and modes coupling are predicted. The specific objective of the paper is to analyse how the presence of the microstructure influences on the processes of mode interactions. It is shown that depending on a scaling relation between the amplitude of the vibrations and the size of the unit cell different scenarios of the modes coupling can be realised.

1. Introduction

The effect of internal resonances may arise in nonlinear multi-degree of freedom systems, when natural frequencies of the modes become commensurable with each other. Then, the presence of nonlinearity induces a coupling between different modes even in zero-order approximation. Complicated modal interactions occur, which may result in a self-generation of higher-order modes. In such a case, truncation to the modes having non-zero initial energy (which is usually applied studying vibrations of continuous structures) will not be valid and all resonant modes should be taken into account simultaneously.

The nonlinear phenomena of modes coupling and internal resonances have been intensively studied for homogeneous structures [1–3]. Meantime, the nonlinear dynamic behaviour of heterogeneous solids was considered significantly less. Several studies of nonlinear vibrations of composite structures were presented in [4–6]. However, many authors have focused on laminated plates and shells that include a small number of layers (usually only a few), so the influence of a microstructure was not investigated thoroughly. Only very recently, vibrations of a heterogeneous rod embedded in a nonlinear elastic medium were considered in [7].

In this paper, natural longitudinal vibrations of an elastic periodically heterogeneous rod are studied. Geometrical and physical nonlinearity of the problem is taken into account. We wish to predict how the presence of the microstructure affects the processes of internal resonances and mode interactions. The governing macroscopic dynamical equation was obtained earlier with the help of the asymptotic homogenization method [8]. The method of multiple time scales [1–3] is applied for the analysis of nonlinear dynamical behaviour of the rod.

The paper is organized as follows. In Section 2, the input problem is formulated. In Section 3, the perturbation procedure for a homogeneous rod is introduced. In Section 4, the influence of the microstructure is analysed. Conclusion remarks are presented in Section 5.

2. Input problem

We consider a periodically heterogeneous composite rod consisting of alternating layers of two different components $\Omega^{(1)}$ and $\Omega^{(2)}$ with a perfect bonding at the interface $\partial\Omega$ (figure 1). Natural longitudinal vibrations in the direction x are studied.

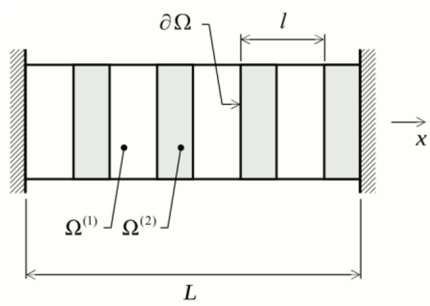


Figure 1. Heterogeneous rod under consideration.

Geometrical nonlinearity appears due to nonlinear relations between the elastic strains and the gradients of displacements and is described by the Cauchy–Green strain tensor [9]. Physical nonlinearity displays a deviation of the stress–strain relations from the proportional Hooke's law. It is modelled representing the energy of deformation as a series expansion in powers of invariants of the strain tensor and taking into account the higher-order terms. Such expansion is usually referred to as the Murnaghan elastic potential [10]. In our previous paper [8], we have obtained a macroscopic dynamical equation that describes nonlinear vibrations of heterogeneous layered structures. For the problem under consideration it can be written as follows:

$$E_1 \frac{\partial^2 u}{\partial x^2} + E_2 \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + l^2 E_3 \frac{\partial^4 u}{\partial x^4} = \rho \frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where u is the displacement; l is the size of the unit cell; E_1, E_2, E_3 are the effective elastic coefficients; ρ is the effective mass density. In the l.h.s. of equation (1), the first term is associated with a linear elastic response of a homogeneous solid. The second term accounts for nonlinear effects (both physical and geometrical). The third term describes the influence of the microstructure.

For the effective coefficients, explicit analytical expressions were derived [8]. The coefficients E_1, E_3 are always positive. E_2 is negative for the most industrial materials, but it will be positive in the case of a physically linear solid. The typical magnitudes of the elastic coefficients are as follows: $|E_2|/E_1 \sim 10$, $E_3/E_1 \sim 10^{-2}$.

Equation (1) presents an asymptotic approximation of the original problem. It is valid only if the size of the microstructure l is smaller than the macroscopic size L of the entire body, $l/L < 1$. It was shown [11] that a good accuracy is achieved for $l/L < 0.4$, i.e. $l^2/L^2 \leq 10^{-1}$.

Let us introduce non-dimensional variables $\bar{x} = x(\pi/L)$, $\bar{t} = t(\pi/L)\sqrt{E_1/\rho}$, $\bar{u} = u/A$, where A is the amplitude of the vibrations. For the simplicity, after the substitution we drop the over bars. Then, equation (1) reads

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \eta \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 u}{\partial t^2}, \tag{2}$$

where $\varepsilon = \pi(E_2/E_1)(A/L)$, $\eta = \pi^2(E_3/E_1)(l^2/L^2)$. Let us note that η is always positive, whereas the sign of ε depends on the properties of the material: $\varepsilon < 0$ if $E_3 < 0$ (soft nonlinearity) and $\varepsilon > 0$ if $E_3 > 0$ (hard nonlinearity). The variables η and ε may be considered as natural small parameters characterizing, accordingly, the rate of heterogeneity and the rate of nonlinearity.

Let us consider the case of clamped-clamped edges. The boundary and the initial conditions are:

$$u(0,t) = u(\pi,t) = 0, \quad u(x,0) = U_0(x), \quad \partial u(x,0)/\partial t = U_1(x). \tag{3}$$

It should be noted that equation (2) includes the fourth-order spatial derivative and, consequently, additional boundary conditions are required. This is a typical difficulty that arises when higher-order models, derived originally for infinite media, are applied to bounded domains. It has been shown [12] that general solutions of the higher-order models combine contributions of long-wave solutions associated with the macroscopic problem and short-wave solutions localised in the vicinity of boundaries. The latter are induced particularly by the presence of higher-order derivative terms. The short-wave solutions describe extraneous boundary layers that have no physical sense. Therefore, additional boundary conditions for equation (2) should be formulated in such a way to eliminate the effect of short-wave boundary layers. This principle yields [12]:

$$\partial^2 u(0,t)/\partial x^2 = \partial^2 u(\pi,t)/\partial x^2 = 0. \tag{4}$$

3. Vibrations of a homogeneous rod

The behaviour of the nonlinear problem (2)–(4) depends on the scaling relation between the small parameters η and ε, which, in its turn, is determined by the size l of the microstructure and by the amplitude A of the vibrations. If the size of the microstructure is considerably small, one can estimate $\eta \sim \varepsilon^2$. In such a case, $l^2/L^2 \sim 10^{-3}$ for $A/L \sim 10^{-4}$ and $l^2/L^2 \sim 10^{-1}$ for $A/L \sim 10^{-3}$. Up to $O(\varepsilon^2)$ approximation, the influence of the microstructure can be neglected, so equation (2) reads:

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + O(\varepsilon^2) = \frac{\partial^2 u}{\partial t^2} . \tag{5}$$

Let us represent the displacement u as a Fourier-sine expansion:

$$u(x,t) = q_1(t)\sin(x) + q_2(t)\sin(2x) + q_3(t)\sin(3x)\ldots . \tag{6}$$

Substituting (6) into (5), we obtain:

$$\frac{d^2 q_1}{dt^2} + \omega_1^2 q_1 + \varepsilon\left(q_1 q_2 + 3q_2 q_3 + \ldots\right) = 0 , \tag{7}$$

$$\frac{d^2 q_2}{dt^2} + \omega_2^2 q_2 + \varepsilon\left(\frac{1}{2}q_1^2 + 3q_1 q_3 + \ldots\right) = 0 ,$$

$$\frac{d^2 q_3}{dt^2} + \omega_3^2 q_3 + \varepsilon\left(3q_1 q_2 + \ldots\right) = 0 , \quad \ldots ;$$

where ω_n is the frequency in the linear case, $\omega_n = n$; n is the number of the mode, $n = 1,2,3,\ldots$.

Let us introduce different time scales $t_0 = t$, $t_1 = \varepsilon t$ and represent q_n as an asymptotic expansion in powers of ε:

$$q_n(t) = q_{n0}(t_0,t_1) + \varepsilon q_{n1}(t_0,t_1) + O(\varepsilon^2) . \tag{8}$$

We note that $d^2/dt^2 = \partial^2/\partial t_0^2 + 2\varepsilon\partial^2/(\partial t_0 \partial t_1) + O(\varepsilon^2)$. Next we substitute expressions (8) into equations (7) and collect the coefficients at equal powers of ε.

In $O(\varepsilon^0)$ approximation we obtain

$$q_{n0} = a_n(t_1)\cos(\omega_n t_0) + b_n(t_1)\sin(\omega_n t_0) , \tag{9}$$

where $a_n(0) = (2/\pi)\int_0^\pi U_0(x)\sin(nx)dx$, $b_n(0) = \left[2/(\pi\omega_n)\right]\int_0^\pi U_1(x)\sin(nx)dx$.

In $O\left(\varepsilon^1\right)$ approximation equations (7) for give:

$$\frac{\partial^2 q_{11}}{\partial t_0^2} + \omega_1^2 q_{11} = -2\frac{\partial^2 q_{10}}{\partial t_0 \partial t_1} - q_{10}q_{20} - 3q_{20}q_{30} + \dots, \tag{10}$$

$$\frac{\partial^2 q_{21}}{\partial t_0^2} + \omega_2^2 q_{21} = -2\frac{\partial^2 q_{20}}{\partial t_0 \partial t_1} - \frac{1}{2}q_{10}^2 - 3q_{10}q_{30} + \dots,$$

$$\frac{\partial^2 q_{31}}{\partial t_0^2} + \omega_3^2 q_{31} = -2\frac{\partial^2 q_{30}}{\partial t_0 \partial t_1} - 3q_{10}q_{20} + \dots, \quad \dots;$$

A straightforward integration of system (10) will lead to the appearance of secular terms in the expressions for q_{n1}. Secular terms grow without a bound in time, which is inconsistent with the physical properties of the conservative system under consideration. In order to eliminate secular terms, the coefficients of $\cos(\omega_n t_0)$ and $\sin(\omega_n t_0)$ in the r.h.s. of equations (10) must be equal to zero. Substituting expressions (9) into equations (10) and fulfilling the aforementioned condition, we obtain a system of equations for a_n and b_n, which gives a possibility to investigate the interactions between different modes.

We note that in the problem under consideration an infinite number of modes can be involved into the resonant interactions. In this paper, we consider only two leading modes and examine in detail the internal resonance between the modes 1 and 2. Coupling between higher-order modes can be investigated in a similar way.

For the further analysis, it is convenient to introduce polar coordinates as follows: $a_n = r_n\cos(\varphi_n)$, $b_n = r_n\sin(\varphi_n)$, where r_n is the amplitude and φ_n is the phase. After routine transformations, the condition of the elimination of secular terms gives:

$$\frac{dr_1}{dt_1} = \frac{1}{4}r_1r_2\sin(\varphi_2 - 2\varphi_1), \tag{11}$$

$$r_1\frac{d\varphi_1}{dt_1} = -\frac{1}{4}r_1r_2\cos(\varphi_2 - 2\varphi_1), \tag{12}$$

$$\frac{dr_2}{dt_1} = -\frac{1}{16}r_1^2\sin(\varphi_2 - 2\varphi_1), \tag{13}$$

$$r_2\frac{d\varphi_2}{dt_1} = -\frac{1}{16}r_1^2\cos(\varphi_2 - 2\varphi_1). \tag{14}$$

A simple analysis shows that equations (11)–(14) allow vibrations by a single mode 2 ($r_2 \neq 0$, $r_1 = 0$). In this case the amplitude is constant in time, $r_2(t_1) = r_2(0)$. This is true up to the order $O(\varepsilon)$ for the time $t \leq O(\varepsilon^{-1})$. If we start with zero initial energy in the 1st mode, there will be no energy present up to the order $O(\varepsilon)$ on the timescale $O(\varepsilon^{-1})$. On the other hand, vibrations by a single mode 1 are not possible, because system (11)–(14) does not hold for $r_1 \neq 0$, $r_2 = 0$. If there is initial energy present in the mode 1, energy transfers occur between the modes 1 and 2. Thereby, the modes 1 and 2 are coupled in $O(\varepsilon^0)$ approximation. This effect is called the internal resonance.

Multiplying equation (11) with r_1 and equation (13) with r_2, adding both equations and performing the integration, one obtains: $r_1^2 + 4r_2^2 = E^2$. This formula represents the energy conservation law, where E is the constant of integration having a physical sense of the full energy of the vibrations. Since in the input non-dimensional equation (2) the displacement u has been normalized to the amplitude A of the vibrations, without loss of generality we let $E = 1$. Then, equations (11)–(14) can be written as follows:

$$\frac{dr_2}{dt_1} = \frac{4r_2^2 - 1}{16}\sin(\psi), \tag{15}$$

$$\frac{d\psi}{dt_1} = \frac{12r_2^2 - 1}{16r_2}\cos(\psi);$$

where $\psi = \varphi_2 - 2\varphi_1$, $0 \le r_2 \le 1/2$.

We performed a numerical integration of system (15) using the Runge-Kutta fourth-order method; the obtained results are presented in (r_2, ϕ) phase plane in figure 2. The solution is 2π-periodic in ψ; the parts of the phase diagram at $r_2 > 0$ and at $r_2 < 0$ are symmetric with respect to the line $r_2 = 0$. The critical points are located at $r_2 = \pm\sqrt{3}/6$, $\psi = \pm\pi m$ (centres) and at $r_2 = \pm 1/2$, $\psi = \pi/2 \pm \pi m$ (saddles); $m = 0,1,2,...$. We can observe that the system oscillates around an equilibrium state with a periodic energy transfer between the modes 1 and 2.

4. Influence of the microstructure

As the size of the microstructure increases, the parameters η and ε become the same order of magnitude: $\eta \sim \varepsilon$. In such a case, $l^2/L^2 \sim 10^{-1}$ for $A/L \sim 10^{-4}$. The presence of the microstructure provides a kind of detuning effect for the phenomenon of internal resonance. Let us introduce the detuning parameter γ of the order $O(1)$ as follows: $\gamma = \eta/\varepsilon = \pi(E_3/E_2)(l^2/L^2)(L/A)$. The input dynamical equation (2) takes the form:

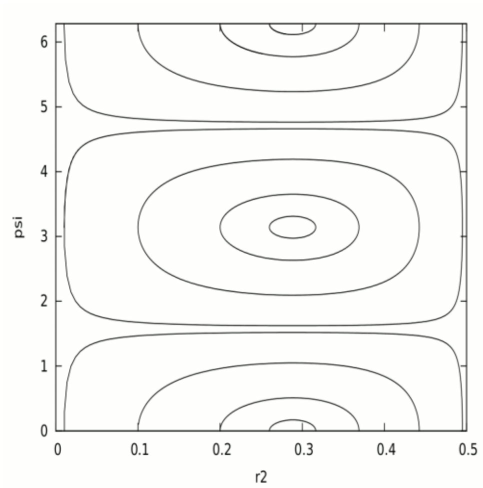


Figure 2. Phase plane in the case of a homogeneous rod.

$$\frac{\partial^2 u}{\partial x^2} + \varepsilon \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \varepsilon\gamma \frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 u}{\partial t^2} .$$

Following the asymptotic procedure presented in Section 3, we obtain an additional contribution in system (15), which now reads:

$$\frac{dr_2}{dt_1} = \frac{4r_2^2 - 1}{16}\sin(\psi), \tag{16}$$

$$\frac{d\psi}{dt_1} = \frac{12r_2^2 - 1}{16r_2}\cos(\psi) + 3\gamma .$$

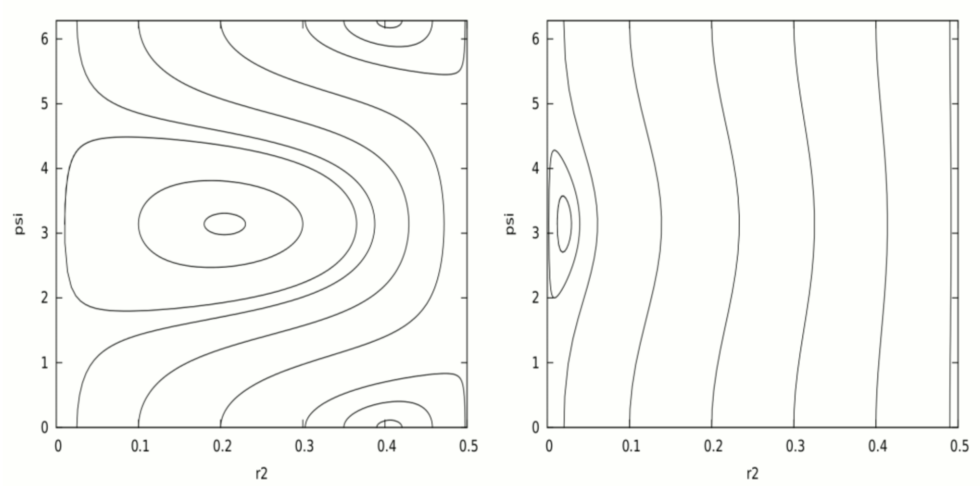
Let us examine the solution in the domain $0 \le r_2 \le 1/2$, $0 \le \psi \le 2\pi$, because for other values of r_2 and ψ it continues periodically.

In the case of soft nonlinearity, $\gamma < 0$, examples of the phase plane are shown in figure 3. As $|\gamma|$ increases, one centre moves left along the line $\psi = \pi$, whereas two centres move right along the lines $\psi = 0$, $\psi = 2\pi$. One saddle moves up and the other saddle moves down along the line $r_2 = 1/2$. For $\gamma = -1/12$, centres and saddles coincide at the points $r_2 = 1/2$, $\psi = 0$ and $r_2 = 1/2$, $\psi = 2\pi$ and then disappear. The only one centre remains and, with the further increase in $|\gamma|$, it continues moving left along the line $\psi = \pi$. The area of the periodic energy transfers between the modes 1 and 2 narrows.

In the case of hard nonlinearity, $\gamma > 0$, the behaviour of the system is illustrated in figure 4. As γ increases, one centre moves right along the line $\psi = \pi$ and two centres move left along the lines

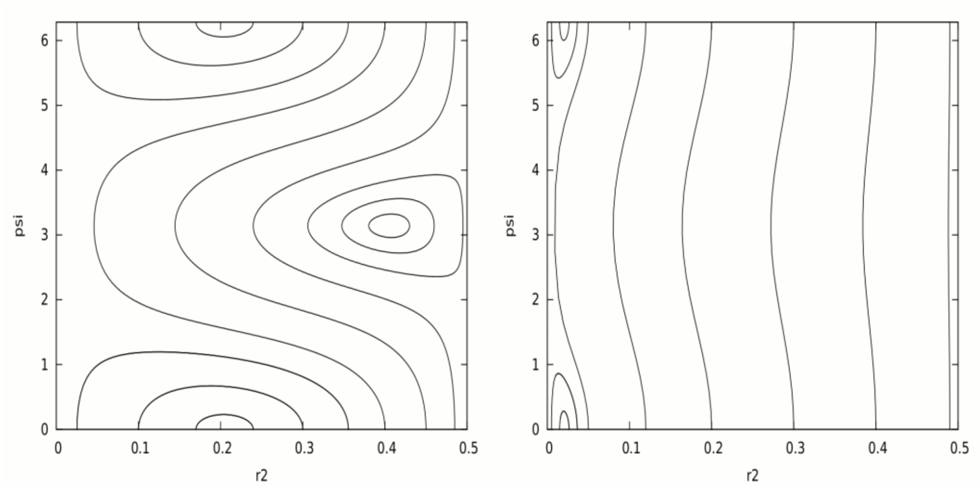
$\psi = 0$, $\psi = 2\pi$. For $\gamma = 1/12$, one centre and two saddles coincide at the point $r_2 = 1/2$, $\psi = \pi$ and disappear. With the further increase in γ , two centres continues moving left along the lines $\psi = 0$, $\psi = 2\pi$ and the intensity of the energy transfers between the modes decreases.

For the both soft and hard nonlinearity, the increase of the size of the microstructure suppresses the energy exchange between the modes, so the effect of internal resonance becomes negligible.



$\gamma = -0.05$ $\gamma = -1$

Figure 3. Phase plane in the case of soft nonlinearity.



$\gamma = 0.05$ $\gamma = 1$

Figure 4. Phase plane in the case of hard nonlinearity.

5. Conclusions

Natural vibrations of a periodically heterogeneous rod are considered with an account for geometrical and physical nonlinearity. The governing dynamical equation was obtained earlier by the method of higher-order asymptotic homogenization. In this paper, we present the asymptotic analysis of the problem with the help of the method of multiple time scales.

If the size of the microstructure is relatively small in comparison to the amplitude of the vibrations, the effect of internal resonance takes place. It results in periodic energy transfers between different modes and in a modulation of their amplitudes. The resonant modes are coupled in $O(\varepsilon^0)$ approximation, so the truncation to the modes having non-zero initial energy is not possible. We studied in details the internal resonance between the leading modes 1 and 2, which is of primary importance for the engineering practice. The behaviour of the system was analysed in the phase plane using the Runge-Kutta fourth-order method and numerical results were presented.

If the size of the microstructure increases, the intensity of the energy transfers between different modes decreases and the effect of internal resonance is suppressed

The results presented in the paper can be applied to facilitate the development of new efficient methods of non-destructive testing. Measuring the characteristics of nonlinear vibrations at different amplitudes allows us to receive precise information about the internal structure of heterogeneous solids. This is sometimes that may be not possible within a linear framework.

Changing properties of the microstructure (e.g., using piezoelectric effects or saturation/desaturation of porous media) make it possible to tune the macroscopic dynamic response of nonlinear structures. This can be useful for a design of new active control devices in various branches of engineering.

Finally, we remark that the effect of internal resonance may be applied for the purposes of vibration damping. Nonlinear coupling between the vibrating modes may help to transfer mechanical energy from low- to high-order modes and, therefore, to decrease essentially the amplitude of the vibrations.

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