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# STABILITY OF COMPOSITE PLATES WITH NON-UNIFORM DISTRIBUTION OF CONSTITUENTS[1]

Bohdan Michalak

*Department of Structural Mechanics, Łódź University of Technology*
*e-mail: bmichala@p.lodz.pl*

This contribution deals with stability of certain composite plates with a deterministic material structure which is not periodic but can be approximately regarded as periodic in small regions of a plate. The formulation of an approximate mathematical model of these plates, based on a tolerance averaging method, was discussed in Woźniak and Wierzbicki (2000), where the plates under consideration were referred to as heteroperiodic.

*Key words:* plate, modelling, non-periodic structure, stability

## 1. Introduction

The main objects of considerations in the paper are thin composite annular plates made of two families of elastic beams with axes intersecting under the right angle. A homogeneous elastic matrix fulfils regions situated between the beams (Fig. 1).

Buckling of annular homogeneous plates was investigated, for example, by Waszczyszyn (1976). Eigenvalues of circular plates resting on elastic foundations were determined by Gomuliński (1967). Woźniak and Zieliński (1967) investigated some stability problems of circular perforated plates.

The aim of this contribution is to propose and apply a mathematical model of heteroperiodic plates. In order to apply the general modelling procedure given in Woźniak, Wierzbicki (2000) we have to solve a whole family of the periodic variational cell problems, where every such problem is related to a

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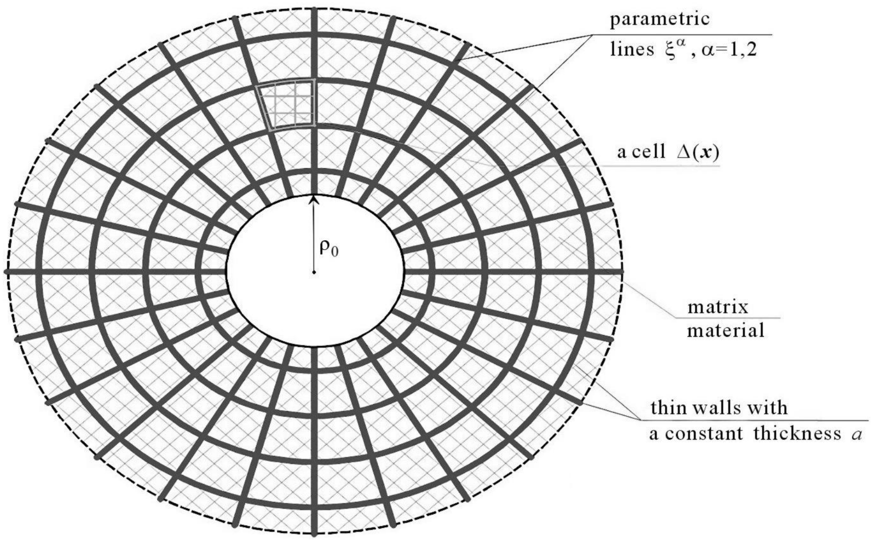


Fig. 1. A scheme of the analysed plate

small region in which the plate, with a sufficient tolerance, can be treated as periodic.

In this contribution, a certain approximate solution to the periodic cell problems for the composite plates under consideration are proposed. These solutions are based on some heuristic assumptions and lead to a system of equations with functional but slowly-varying coefficients for the averaged displacement vector field. The derived equations are dependent on the microstructure size in contrast to the equations obtained by the method of nonuniform homogenization, Bensoussan *et al.* (1978). Following Woźniak and Wierzbicki (2000) we can observe that the mathematical modelling of media which are periodic and related to a certain curvilinear coordinate system, see Lewiński and Telega (2000), is not able to describe composite plates under consideration with a constant cross section of the beams.

## 2. Preliminaries

Introduce a polar coordinate system in a physical space denoted by $O\xi^1\xi^2\xi^3$. Throughout the paper the indices $\alpha, \beta, \ldots$ run over 1, 2 and a vertical line before the subscripts stands for the covariant derivative in the polar coordinate system. The summation convention holds for all aforementioned indices. Setting $\boldsymbol{x} \equiv (\xi^1, \xi^2)$ and $z \equiv \xi^3$ it is assumed that the undeformed plate occupies the region $\Omega \equiv \{(\boldsymbol{x}, z): \ -h/2 \leqslant z \leqslant h/2, \ \ \boldsymbol{x} \in \Pi\}$, where $\Pi$ is the plate midplane and $h$ is the plate thickness. The orthogonal Cartesian coordinate system $Oy_1y_2$, with the vector basis $\boldsymbol{e}_\alpha$ $(\alpha = 1, 2)$, is a local coordinate system in an arbitrary cell $\Delta(\boldsymbol{x})$ (Fig. 2).

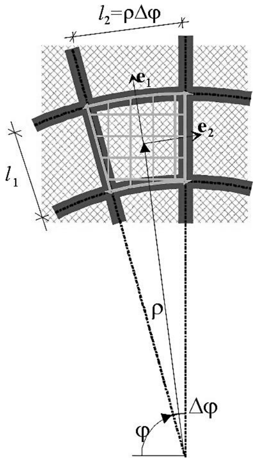


Fig. 2. An arbitrary cell $\Delta(\boldsymbol{x})$ of the plate

The considerations are based on the well-known second order non-linear theory for thin plates (Woźniak *et al.*, 2001):
— strain-displacement relations

$$\varepsilon_{\alpha\beta} = u_{(\alpha|\beta)} \qquad \kappa_{\alpha\beta} = -w_{|\alpha\beta} \tag{2.1}$$

— constitutive equations

$$n^{\alpha\beta} = DH^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} \qquad m^{\alpha\beta} = BH^{\alpha\beta\gamma\delta}\kappa_{\gamma\delta} \tag{2.2}$$

where

$$H^{\alpha\beta\gamma\delta} = \frac{1}{2}[g^{\alpha\delta}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\delta} + \nu(\epsilon^{\alpha\gamma}\epsilon^{\beta\delta} + \epsilon^{\alpha\delta}\epsilon^{\beta\gamma})]$$

$$D \equiv \frac{Eh}{1-\nu^2} \qquad B \equiv \frac{Eh^3}{12(1-\nu^2)}$$

— equilibrium equations

$$n^{\alpha\beta}_{|\alpha} + p^\beta = 0 \qquad m^{\alpha\beta}_{|\alpha\beta} + (n^{\alpha\beta}w_{|\beta})_{|\alpha} + p = 0 \tag{2.3}$$

The displacement vector field of the plate midplane is denoted by

$$\boldsymbol{u}(\xi^\alpha,t) = u^\beta(\xi^\alpha,t)\boldsymbol{g}_\beta + w(\xi^\alpha,t)\boldsymbol{g}_3 \qquad \xi^\alpha \in \Pi \tag{2.4}$$

and the external surface loading by

$$\boldsymbol{p}(\xi^\alpha,t) = p^\beta(\xi^\alpha,t)\boldsymbol{g}_\beta + p(\xi^\alpha,t)\boldsymbol{g}_3 \qquad \xi^\alpha \in \Pi \tag{2.5}$$

Setting the external surface loading $p^\beta = p = 0$, we obtain equilibrium equations (2.3) in the form

$$m^{\alpha\beta}_{|\alpha\beta} + n^{\alpha\beta}w_{|\beta\alpha} = 0 \tag{2.6}$$

This direct description leads to plate equations with highly-oscillating coefficients, which are too complicated to be used in the analysis of stability problems and numerical calculations.

## 3. Modelling procedure

By a heteroperiodic plate we shall mean a microheterogeneous plate which in subregions of $\Pi$, much smaller than $\Pi$, can be approximately regarded as periodic. The characteristic feature of every periodic plate is that there exists a representative cell $\Delta$. The edge length dimensions of the cell $\Delta$ are equal to the periods of the heterogeneous material structure of this plate. Now we define

$$\Delta(\boldsymbol{x}) := \Big\{\boldsymbol{y} = \boldsymbol{x} + \eta^\alpha \boldsymbol{l}_\alpha(\boldsymbol{x}), \quad \eta \in \Big(-\frac{1}{2},\frac{1}{2}\Big)\Big\} \qquad \boldsymbol{x} \in \Pi_\Delta \tag{3.1}$$

where $l_\alpha = |\boldsymbol{l}_\alpha|$ are the cell length dimensions, $\Pi_\Delta := \{\boldsymbol{x} \in \Pi : \ \Delta(\boldsymbol{x}) \subset \Pi\}$. Denoting by $l(\boldsymbol{x})$ the diameter of $\Delta(\boldsymbol{x})$ and define $l = \sup l(\boldsymbol{x})$ as *a meso-structure parameter*, we assume that $l$ is sufficiently small compared to the smallest characteristic length dimension $L_\Pi$ of $\Pi$ ($l \ll L_\Pi$) and sufficiently large compared to the plate thickness $h$ ($h \ll l$) (Fig. 2). In this case, every $\Delta(\boldsymbol{x})$ defined by Eq. (3.1) will be called a cell with the center at $\boldsymbol{x}$.

Now we assume that a certain cell distribution $\Delta(\cdot)$ has been assigned to $\Pi$. The averaging formula can be now generalized to the form

$$\langle\varphi\rangle(\boldsymbol{x}) = \frac{1}{|\Delta(\boldsymbol{x})|}\int\limits_{\Delta(\boldsymbol{x})} \varphi(\boldsymbol{y})\,dy \qquad \boldsymbol{x} \in \Pi_\Delta \tag{3.2}$$

In order to derive an averaged mathematical model for the plate under consideration we will adapt *the tolerance averaging method* developed by Woźniak and Wierzbicki (2000). In the framework of the method for periodic plates, we introduce the concept of *a slowly varying* and *periodic-like function* for the tolerance system $T = (F, \varepsilon(\cdot))$. The continuous function $\varPhi(\cdot) \in F$, defined on the periodic plate region $\varPi$, will be called *slowly varying* if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \overline{\varPi} \qquad \|\boldsymbol{x} - \boldsymbol{y}\| \leqslant l \ \Rightarrow \ |\varPhi(\boldsymbol{x}) - \varPhi(\boldsymbol{y})| \leqslant \varepsilon_{\varPhi} \tag{3.3}$$

The continuous function $f(\cdot)$ defined on $\varPi$ will be called a *periodic-like function* if for every $\boldsymbol{x} \in \varPi_{\varDelta}$ there exists a $\varDelta$-periodic function $f_x(\cdot)$ such that for every $\boldsymbol{y} \in \varPi_{\varDelta}$

$$\|\boldsymbol{x} - \boldsymbol{y}\| \leqslant l \ \Rightarrow \ |f(\boldsymbol{x}) - f_x(\boldsymbol{y})| \leqslant \varepsilon_f \tag{3.4}$$

We shall write $\varPhi(\cdot) \in SV_{\varDelta}(T)$ if $\varPhi(\cdot)$ and all its derivatives are slowly-varying functions, and $f(\cdot) \in PL_{\varDelta}(T)$ if $f(\cdot)$ and all its derivatives are periodic-like functions. The periodic-like function $f(\cdot)$ will be called an *oscillating periodic-like function* if the condition $\langle cf\rangle(\boldsymbol{x}) \cong 0$ holds for every $\boldsymbol{x} \in \varPi_{\varDelta}$, where $c(\cdot)$ is a positive value $\varDelta$-periodic function.

Now definitions (3.3), (3.4) can be generalized, and after interpreting the symbol $\varDelta$ as a cell distribution $\varDelta(\cdot)$, the definition of *slowly varying* and *periodic-like functions* will be given by

$$\begin{aligned} \varPhi(\cdot) \in SV_{\varDelta}(T) \ &\Leftrightarrow \ \{\forall \boldsymbol{x} \in \varPi_{\varDelta}: \quad \varPhi_{|P(\boldsymbol{x})}(\cdot) \in SV_{\varDelta(\boldsymbol{x})}(T)\} \\ f(\cdot) \in PL_{\varDelta}(T) \ &\Leftrightarrow \ \{\forall \boldsymbol{x} \in \varPi_{\varDelta}: \quad f_{|P(\boldsymbol{x})}(\cdot) \in PL_{\varDelta(\boldsymbol{x})}(T)\} \end{aligned} \tag{3.5}$$

for a certain region $P(\boldsymbol{x})$ such that $\overline{\varDelta}(\boldsymbol{x}) \subset P(\boldsymbol{x}) \subset \varPi$; the symbol $f_{|P(\boldsymbol{x})}(\cdot)$ denotes here the restriction on the function $f(\cdot)$ to $P(\boldsymbol{x})$.

Let $f(\cdot)$ be an integrable function defined on $\varPi$ such that $\langle f\rangle(\cdot)$ is a slowly varying function, $\langle f\rangle \in SV_{\varDelta(\boldsymbol{x})}(T)$. We assume that averaged values $\langle f\rangle(\boldsymbol{x})$, $\boldsymbol{x} \in \varPi_{\varDelta}$ have to be calculated with some tolerance determined by a certain tolerance parameter $\varepsilon_{\langle f\rangle}$. The function $f(\cdot)$ will be called a $\varDelta$-heteroperiodic function if for every $\boldsymbol{x} \in \varPi_{\varDelta}$ there exists a $\varDelta(\boldsymbol{x})$-periodic function $f_x(\cdot)$ such that

$$\forall \boldsymbol{x} \in \varPi_{\varDelta} \qquad \langle |f - f_x|\rangle(\boldsymbol{x}) \leqslant \varepsilon_{\langle f\rangle} \tag{3.6}$$

A heterogeneous plate will be called heteroperiodic if all material properties of this plate can be described by heteroperiodic functions. Otherwise, by a heteroperiodic plate we mean a plate which in small regions (small neighbourhoods of $\varDelta(\boldsymbol{x})$) can be approximately regarded as a periodic one.

## 4. Averaged description

The tolerance averaging applied to the plate under consideration is based on two additional modelling assumptions. The assumption of *macromodelling* states that every cell $\Delta(\boldsymbol{x})$ (Fig. 2), within a certain tolerance, can be treated as nondiscernible with a rectangular cell shown in Fig. 3.

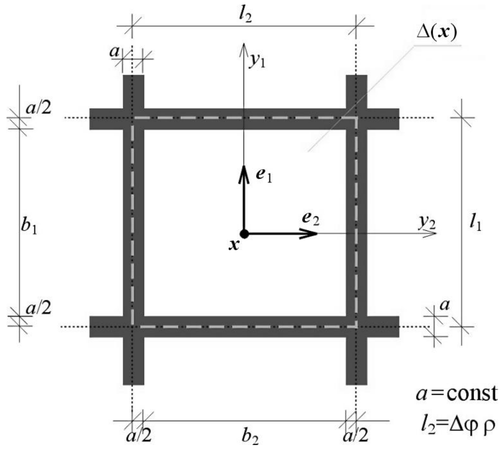


Fig. 3. The rectangular cell

The *conformability* assumption states that the deflection $w(\cdot)$ of the plate midplane is in a small region $P(\boldsymbol{x})$ (Eq. (3.5)) a periodic-like function, $w(\cdot) \in PL_{\Delta(\boldsymbol{x})}(T)$, that means the deflection is conformable to the plate structure. This condition may be violated only near the boundary of the plate. Bearing in mind the lemmas of the tolerance averaging method (see Woźniak and Wierzbicki, 2000), the conformability assumption can be represented by the decomposition

$$w(\xi^\beta,t) = w^0(\xi^\beta,t) + \widetilde{w}(\xi^\beta,t) \tag{4.1}$$

where $w^0 = \langle w\rangle$, $w^0(\cdot) \in SV_{\Delta(\boldsymbol{x})}(T)$, $\widetilde{w}(\cdot) \in PL^1_{\Delta(\boldsymbol{x})}(T)$ is called the deflection disturbance and satisfy the condition $\langle\widetilde{w}\rangle \cong 0$.

Substituting the right-hand side of Eq. (4.1) into equilibrium equation (2.6) and applying the tolerance averaging, we arrive at the equation

$$\Big[\langle BH^{\alpha\beta\gamma\delta}\rangle(\xi^\tau)w^0_{|\gamma\delta}(\xi^\tau,t)\Big]_{|\alpha\beta} + \Big[\langle BH^{\alpha\beta\gamma\delta}\widetilde{w}_{|\gamma\delta}\rangle(\xi^\tau,t)\Big]_{|\alpha\beta} - N^{\alpha\beta}w^0_{|\beta\alpha}(\xi^\tau,t) = 0 \tag{4.2}$$

where $N^{\alpha\beta} = \langle n^{\alpha\beta}\rangle$. According to the conformability assumption, we have to assume that the forces in the plate midplane are determined by the periodic-like function $n^{\alpha\beta}(\cdot) \in PL_{\Delta(\boldsymbol{x})}(T)$.

Hence, these forces can be represented by the decomposition

$$n^{\alpha\beta}(\xi^\beta, t) = N^{\alpha\beta}(\xi^\beta, t) + \widetilde{n}^{\alpha\beta}(\xi^\beta, t) \tag{4.3}$$

where $N^{\alpha\beta}(\cdot) \in SV_{\Delta(\boldsymbol{x})}(T)$, and $\widetilde{n}^{\alpha\beta}(\cdot) \in PL^1_{\Delta(\boldsymbol{x})}(T)$ is a fluctuating part of forces $n^{\alpha\beta}(\cdot)$, such that $\langle \widetilde{n}^{\alpha\beta} \rangle \cong 0$.

Multiplying Eq. (2.6) by an arbitrary $\Delta(\boldsymbol{x})$-periodic test function $\delta w$, such that $\langle \delta w \rangle = 0$, averaging this equation over $\Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Delta$, and using the tolerance averaging formulae (see Woźniak and Wierzbicki, 2000), we obtain a periodic problem on the cell $\Delta(\boldsymbol{x})$ for the $\Delta(\boldsymbol{x})$-periodic function $\widetilde{w}_x(\cdot)$, given by the following variational condition

$$\begin{aligned} &\langle \delta w_{|\alpha\beta} BH^{\alpha\beta\gamma\delta} \widetilde{w}_{x|\gamma\delta} \rangle(\xi^\tau, t) + \langle \delta w_{|\beta} n^{\alpha\beta} \widetilde{w}_{x_{|\alpha}} \rangle(\xi^\tau, t) = \\ &= -\langle \delta w_{|\alpha\beta} BH^{\alpha\beta\gamma\delta} \rangle(\xi^\tau) w^0_{|\gamma\delta}(\xi^\tau, t) \end{aligned} \tag{4.4}$$

which has to hold for every test function $\delta w$.

The approximate solution to the above variational cell problem will be assumed in the form

$$\widetilde{w}_x(\boldsymbol{y}, t) \cong h^\alpha(\boldsymbol{y}) V_\alpha(\boldsymbol{x}, t) \tag{4.5}$$

where $\boldsymbol{y} \in \Delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi_\Delta$; $h^\alpha(\cdot)$ are postulated $\Delta(\boldsymbol{x})$-periodic functions such that $\langle h^\alpha \rangle = 0$, and $V_\alpha(\cdot, t)$ are new unknowns which are assumed to be slowly varying functions, $V_\alpha(\cdot) \in SV_{\Delta(\boldsymbol{x})}(T)$. The functions $h^\alpha(\cdot)$, called shape functions, depend on the mesostructure parameter $l$ such that $l^{-1}h^\alpha(\cdot) \in O(l)$, $lh^\alpha_{|\gamma\beta}(\boldsymbol{x}) \in O(l)$, $\max|h^\alpha(\boldsymbol{y})| \leqslant l^2$, $\boldsymbol{y} \in \Delta(\boldsymbol{x})$.

Substituting the right-hand sides of Eq. (4.5) into (4.2) and (4.4) and setting $\delta w = h^\alpha(\boldsymbol{y})$ in (4.4) on the basis of the tolerance averaging relations, we finally arrive at the governing equations for the considered plates

$$\begin{aligned} &\Big[\langle BH^{\alpha\beta\gamma\delta}\rangle(\xi^\tau) w^0_{|\gamma\delta}(\cdot, t)\Big]_{|\alpha\beta} + \Big[\langle BH^{\alpha\beta\gamma\delta} h^\mu_{|\gamma\delta}\rangle(\xi^\tau) V_\mu(\cdot, t)\Big]_{|\alpha\beta} - N^{\alpha\beta} w^0_{|\alpha\beta} = 0 \\ &\langle BH^{\alpha\beta\gamma\delta} h^\mu_{|\alpha\beta}\rangle(\xi^\lambda) w^0_{|\gamma\delta}(\cdot, t) + \langle BH^{\alpha\beta\gamma\delta} h^\mu_{|\alpha\beta} h^\tau_{|\gamma\delta}\rangle(\xi^\lambda) V_\tau(\cdot, t) + \\ &+\underline{N^{\alpha\beta}\langle h^\mu_{|\beta} h^\tau_{|\alpha}\rangle V_\tau} = 0 \end{aligned} \tag{4.6}$$

where the underlined term depends on the mesostructure parameter $l$. In Eq. $(4.6)_2$ we have assumed that the fluctuating part $\widetilde{n}^{\alpha\beta}(\cdot)$ of the forces $n^{\alpha\beta}(\cdot)$ is very small compared to their averaging part $N^{\alpha\beta}(\cdot)$, and hence $\langle h^\mu_{|\beta} n^{\alpha\beta} h^\tau_{|\alpha}\rangle \cong N^{\alpha\beta}\langle h^\mu_{|\beta} h^\tau_{|\alpha}\rangle$.

Taking into account Eq. (4.5), the plate deflection can be approximated by means of the formula

$$w(\xi^\beta,t) \cong w^0(\xi^\beta,t) + h^\alpha(\boldsymbol{y})V_\alpha(\xi^\beta,t) \tag{4.7}$$

The presented model has a physical sense when the basic unknowns $w^0(\xi^\beta,t)$, $V_\alpha(\xi^\beta,t)$ are $\varDelta(\boldsymbol{x})$-slowly varying functions, $w^0(\cdot) \in SV_{\varDelta(\boldsymbol{x})}(T)$, $V_\alpha(\cdot) \in SV_{\varDelta(\boldsymbol{x})}(T)$.

The characteristic features of the derived *length-scale model* are:

- The model takes into account the effect of the cell size on the stability of the considered plate.
- The governing equations have averaged coefficients that are slowly varying functions.

The simplified model of the stability of plates with non-uniform distribution of constituents can be derived from the length-scale model, Eq. (4.6), by passing to the limit $l \to 0$, i.e. by neglecting the parameter $l$, which is placed in the underlined term. Hence, we arrive at *the local model* governed by

$$\begin{gathered}\Big[\langle BH^{\alpha\beta\gamma\delta}\rangle(\xi^\tau)w^0_{|\gamma\delta}(\cdot,t)\Big]_{|\alpha\beta} + \Big[\langle BH^{\alpha\beta\gamma\delta}h^\mu_{|\gamma\delta}\rangle(\xi^\tau)V_\mu(\cdot,t)\Big]_{|\alpha\beta} - N^{\alpha\beta}w^0_{|\alpha\beta} = 0\\ \langle BH^{\alpha\beta\gamma\delta}h^\mu_{|\alpha\beta}\rangle(\xi^\lambda)w^0_{|\gamma\delta}(\cdot,t) + \langle BH^{\alpha\beta\gamma\delta}h^\mu_{|\alpha\beta}h^\tau_{|\gamma\delta}\rangle(\xi^\lambda)V_\tau(\cdot,t) = 0\end{gathered} \tag{4.8}$$

This model can be treated as a certain homogenized model, in which through the tolerance averaging method one can calculate an approximate value of the averaged stiffnesses modulus.

## 5. Applications

We shall investigate the linear stability of plates for polar-symmetric buckling. Assume that the matrix and walls of a plate are made of two different isotropic homogeneous materials. The bending stiffness of the walls is denoted by $B_1$ and that of the matrix by $B_2 = \alpha_1 B_1$, Poisson's ratio respectively by $\nu_1$ and $\nu_2 = \alpha_2\nu_1$. Moreover, the loadings $p$ are neglected. On the leading assumption, the physical components of shape functions, for the cell shown in Fig. 3, will be taken as

$$\begin{aligned} h^{\langle 1\rangle}(\boldsymbol{y}) &= h^1(\boldsymbol{y}) = s_1(y_1)\Big[1 - \Big(\frac{2y_2}{b_2}\Big)^2\Big]\\ h^{\langle 2\rangle}(\boldsymbol{y}) &= \rho h^2(\boldsymbol{y}) = s_2(y_2)\Big[1 - \Big(\frac{2y_1}{b_1}\Big)^2\Big]\end{aligned} \tag{5.1}$$

where

$$
s_1(y_1) = \begin{cases} a^2\Big[\dfrac{4}{a^2}\Big(y_1 - \dfrac{1}{2}l_1\Big)^2 - 1 - \dfrac{2}{3}\dfrac{l_1 - 2a}{l_1}\Big] & y_1 \in \Big\langle \dfrac{1}{2}b_1, \dfrac{1}{2}l_1 \Big\rangle \\ a^2\Big[-\dfrac{4}{(l_1 - a)^2}(y_1)^2 + 1 - \dfrac{2}{3}\dfrac{l_1 - 2a}{l_1}\Big] & y_1 \in \Big\langle -\dfrac{1}{2}b_1, \dfrac{1}{2}b_1 \Big\rangle \\ a^2\Big[\frac{4}{a^2}\Big(y_1 + \frac{1}{2}l_1\Big)^2 - 1 - \frac{2}{3}\frac{l_1-2a}{l_1}\Big] & y_1 \in \Big\langle -\dfrac{1}{2}l_1, -\dfrac{1}{2}b_1 \Big\rangle \end{cases}
$$

$$
s_2(y_2) = \begin{cases} a^2\Big[\dfrac{4}{a^2}\Big(y_2 - \dfrac{1}{2}\Delta\varphi\rho\Big)^2 - 1 - \dfrac{2}{3}\dfrac{\Delta\varphi\rho - 2a}{\Delta\varphi\rho}\Big] & y_2 \in \Big\langle \dfrac{1}{2}b_2, \dfrac{1}{2}l_2 \Big\rangle \\ a^2\Big[-\dfrac{4}{(\Delta\varphi\rho - a)^2}(y_2)^2 + 1 - \dfrac{2}{3}\dfrac{\Delta\varphi\rho - 2a}{\Delta\varphi\rho}\Big] & y_2 \in \Big\langle -\dfrac{1}{2}b_2, \dfrac{1}{2}b_2 \Big\rangle \\ a^2\Big[\dfrac{4}{a^2}\Big(y_2 + \dfrac{1}{2}\Delta\varphi\rho\Big)^2 - 1 - \dfrac{2}{3}\dfrac{\Delta\varphi\rho - 2a}{\Delta\varphi\rho}\Big] & y_2 \in \Big\langle -\dfrac{1}{2}l_2, -\dfrac{1}{2}b_2 \Big\rangle \end{cases}
\tag{5.2}
$$

### 5.1. Governing equations for the length-scale model

Using Eq. (4.6) with shape functions given by Eq. (5.1), (5.2), we obtain a system of governing equations for polar-symmetric buckling. These equations, describing the buckling of the plate in the framework of the length-scale model, take the form

$$
\begin{aligned}
&(\langle BH^{11\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta})_{,11} + \Big(\frac{2}{\rho}\langle BH^{11\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta}\Big)_{,1} - 2\langle BH^{22\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta} + \\
&-\rho(\langle BH^{22\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta})_{,1} + (\langle BH^{11\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1)_{,11} + (\langle BH^{11\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2)_{,11} + \\
&+\frac{2}{\rho}(\langle BH^{11\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1)_{,1} + \frac{2}{\rho}(\langle BH^{11\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2)_{,1} - 2\langle BH^{22\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1 + \\
&-2\langle BH^{22\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2 - \rho(\langle BH^{22\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1)_{,1} - \rho(\langle BH^{22\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2)_{,1} + \\
&-N^{11}w^0_{,11} - N^{22}w^0_{,1} = 0 \\[1ex]
&[C^{11}(\rho) + N^{11}\langle(h^1_{|1})^2\rangle + N^{22}\langle(h^1_{|2})^2\rangle]V_1 + \\
&+[C^{12}(\rho) + N^{11}\langle h^1_{|1}h^2_{|1}\rangle + N^{22}\langle h^1_{|2}h^2_{|2}\rangle]V_2 + B^{111}(\rho)w^0_{,11} + B^{221}(\rho)w^0_{,1} = 0 \\[1ex]
&[C^{21}(\rho) + N^{11}\langle h^1_{|1}h^2_{|1}\rangle + N^{22}\langle h^1_{|2}h^2_{|2}\rangle]V_1 + \\
&+[C^{22}(\rho) + N^{11}\langle(h^2_{|1})^2\rangle + N^{22}\langle(h^2_{|2})^2\rangle]V_2 + B^{211}(\rho)w^0_{,11} + B^{222}(\rho)w^0_{,1} = 0
\end{aligned}
\tag{5.3}
$$

where the following denotations have been introduced

$$
\begin{aligned}
&B^{111}(\rho) = \langle BH^{1111}h^1_{|11}\rangle + \langle BH^{1122}h^1_{|22}\rangle \\
&B^{211}(\rho) = \langle BH^{1111}h^2_{|11}\rangle + \langle BH^{2211}h^2_{|22}\rangle \\
&B^{221}(\rho) = \langle BH^{1122}h^1_{|11}\rangle + \langle BH^{2222}h^1_{|22}\rangle \\
&B^{222}(\rho) = \langle BH^{1122}h^2_{|11}\rangle + \langle BH^{2222}h^2_{|22}\rangle \\
&C^{11}(\rho) = \langle BH^{1111}h^1_{|11}h^1_{|11}\rangle + 2\langle BH^{1122}h^1_{|11}h^1_{|22}\rangle + \\
&\qquad +4\langle BH^{1212}h^1_{|12}h^1_{|12}\rangle + \langle BH^{2222}h^1_{|22}h^1_{|22}\rangle \\
&C^{12}(\rho) = C^{21}(\rho) = \langle BH^{1111}h^1_{|11}h^2_{|11}\rangle + \langle BH^{1122}h^1_{|11}h^2_{|22}\rangle + \\
&\qquad +4\langle BH^{1212}h^1_{|12}h^2_{|12}\rangle + \langle BH^{2211}h^1_{|22}h^2_{|11}\rangle + \langle BH^{2222}h^1_{|22}h^2_{|22}\rangle \\
&C^{22}(\rho) = \langle BH^{1111}h^2_{|11}h^2_{|11}\rangle + 2\langle BH^{1122}h^2_{|11}h^2_{|22}\rangle + \\
&\qquad +4\langle BH^{1212}h^2_{|12}h^2_{|12}\rangle + \langle BH^{2222}h^2_{|22}h^2_{|22}\rangle
\end{aligned}
\tag{5.4}
$$

Eliminating the internal variables

$$
\begin{aligned}
V_1 &= \underline{A^{11}}w^0{}_{,11} + \underline{A^1}\rho w^0{}_{,1} = \frac{B^{211}K_1 - B^{111}K_2}{K_3K_2 - K_1^2}w^0{}_{,11} + \frac{B^{222}K_1 - B^{221}K_2}{K_3K_2 - K_1^2}\rho w^0{}_{,1} \\
V_2 &= \underline{A^{22}}w^0{}_{,11} + \underline{A^2}\rho w^0{}_{,1} = \frac{B^{111}K_1 - B^{211}K_3}{K_3K_2 - K_1^2}w^0{}_{,11} + \frac{B^{221}K_1 - B^{222}K_3}{K_3K_2 - K_1^2}\rho w^0{}_{,1}
\end{aligned}
\tag{5.5}
$$

where

$$
\begin{aligned}
K_1 &= C^{12}(\rho) + N^{11}\langle h^1_{|1}h^2_{|1}\rangle + N^{22}\langle h^1_{|2}h^2_{|2}\rangle \\
K_2 &= C^{22}(\rho) + N^{11}\langle (h^2_{|1})^2\rangle + N^{22}\langle (h^2_{|2})^2\rangle \\
K_3 &= C^{11}(\rho) + N^{11}\langle (h^1_{|1})^2\rangle + N^{22}\langle (h^1_{|2})^2\rangle
\end{aligned}
$$

we obtain the equilibrium equation in the form

$$
\begin{aligned}
&(\underline{C_1(\rho, N^{\alpha\beta})}w^0{}_{,11})_{,11} + \underline{C_2(\rho, N^{\alpha\beta})}w^0{}_{,11} + (\rho\underline{C_2(\rho, N^{\alpha\beta})}w^0{}_{,11})_{,1} + \\
&+(\rho\underline{C_3(\rho, N^{\alpha\beta})}w^0{}_{,1})_{,11} + \rho\underline{C_4(\rho, N^{\alpha\beta})}w^0{}_{,1} + (\rho^2\underline{C_4(\rho, N^{\alpha\beta})}w^0{}_{,1})_{,1} + \\
&-N^{11}w^0{}_{,11} - N^{22}\rho w^0{}_{,1} = 0
\end{aligned}
\tag{5.6}
$$

where

$$
\begin{aligned}
\underline{C_1(\rho, N^{\alpha\beta})} &= \langle BH^{1111}\rangle + B^{111}\underline{A^{11}} + B^{211}\underline{A^{22}} \\
\underline{C_2(\rho, N^{\alpha\beta})} &= \frac{2}{\rho^2}\langle BH^{1111}\rangle - \langle BH^{2211}\rangle + \Big(\frac{2}{\rho^2}B^{111} - B^{221}\Big)\underline{A^{11}} + \\
&+ \Big(\frac{2}{\rho^2}B^{211} - B^{222}\Big)\underline{A^{22}} \\
\underline{C_3(\rho, N^{\alpha\beta})} &= \langle BH^{1122}\rangle + B^{111}\underline{A^{1}} + B^{211}\underline{A^{2}} \\
\underline{C_4(\rho, N^{\alpha\beta})} &= \frac{2}{\rho^2}\langle BH^{1122}\rangle - \langle BH^{2222}\rangle + \Big(\frac{2}{\rho^2}B^{111} - B^{221}\Big)\underline{A^{1}} + \\
&+ \Big(\frac{2}{\rho^2}B^{211} - B^{222}\Big)\underline{A^{2}}
\end{aligned}
\tag{5.7}
$$

### 5.2. Governing equations for the local model

Now we consider buckling of a plate in the framework of the local model. This model can be derived directly from the length-scale model Eqs (5.3)-(5.7) by passing $l \to 0$, i.e. by neglecting terms with the mesostructure parameter $l$. Hence, we arrive at equilibrium equations

$$
\begin{aligned}
&(\langle BH^{11\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta})_{,11} + \Big(\frac{2}{\rho}\langle BH^{11\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta}\Big)_{,1} - 2\langle BH^{22\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta} + \\
&-\rho(\langle BH^{22\gamma\delta}\rangle(\rho)w^0_{|\gamma\delta})_{,1} + (\langle BH^{11\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1)_{,11} + (\langle BH^{11\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2)_{,11} + \\
&+\frac{2}{\rho}(\langle BH^{11\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1)_{,1} + \frac{2}{\rho}(\langle BH^{11\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2)_{,1} - 2\langle BH^{22\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1 + \\
&-2\langle BH^{22\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2 - \rho(\langle BH^{22\gamma\delta}h^1_{|\gamma\delta}\rangle(\rho)V_1)_{,1} - \rho(\langle BH^{22\gamma\delta}h^2_{|\gamma\delta}\rangle(\rho)V_2)_{,1} + \\
&-N^{11}w^0_{,11} - N^{22}w^0_{,1} = 0 \\
&C^{11}(\rho)V_1 + C^{12}(\rho)V_2 + B^{111}(\rho)w^0_{,11} + B^{221}(\rho)w^0_{,1} = 0 \\
&C^{21}(\rho)V_1 + C^{22}(\rho)V_2 + B^{211}(\rho)w^0_{,11} + B^{222}(\rho)w^0_{,1} = 0
\end{aligned}
\tag{5.8}
$$

with the denotations given by Eq. (5.4).

Eliminating the internal variables

$$
\begin{aligned}
V_1 &= A^{11}w^0,_{11} + A^1\rho w^0,_1 = \\
&= \frac{B^{211}C^{12} - B^{111}C^{22}}{C^{11}C^{22} - (C^{12})^2} w^0,_{11} + \frac{B^{222}C^{12} - B^{221}C^{22}}{C^{11}C^{22} - (C^{12})^2} \rho w^0,_1 \\
V_2 &= A^{22}w^0,_{11} + A^2\rho w^0,_1 = \\
&= \frac{B^{111}C^{12} - B^{211}C^{11}}{C^{11}C^{22} - (C^{12})^2} w^0,_{11} + \frac{B^{221}C^{12} - B^{222}C^{11}}{C^{11}C^{22} - (C^{12})^2} \rho w^0,_1
\end{aligned}
\tag{5.9}
$$

we obtain the equilibrium equation in the form similar to Eq. (5.6)

$$
\begin{aligned}
&(C_1(\rho)w^0,_{11}),_{11} + C_2(\rho)w^0,_{11} + (\rho C_2(\rho)w^0,_{11}),_1 + (\rho C_3(\rho)w^0,_1),_{11} + \\
&+\rho C_4(\rho)w^0,_1 + (\rho^2 C_4(\rho)w^0,_1),_1 - N^{11}w^0,_{11} - N^{22}\rho w^0,_1 = 0
\end{aligned}
\tag{5.10}
$$

where

$$
\begin{aligned}
C_1(\rho) &= \langle BH^{1111}\rangle + B^{111}A^{11} + B^{211}A^{22} \\
C_2(\rho) &= \frac{2}{\rho^2}\langle BH^{1111}\rangle - \langle BH^{2211}\rangle + \Big(\frac{2}{\rho^2}B^{111} - B^{221}\Big)A^{11} + \\
&+ \Big(\frac{2}{\rho^2}B^{211} - B^{222}\Big)A^{22} \\
C_3(\rho) &= \langle BH^{1122}\rangle + B^{111}A^{1} + B^{211}A^{2} \\
C_4(\rho) &= \frac{2}{\rho^2}\langle BH^{1122}\rangle - \langle BH^{2222}\rangle + \Big(\frac{2}{\rho^2}B^{111} - B^{221}\Big)A^{1} + \\
&+ \Big(\frac{2}{\rho^2}B^{211} - B^{222}\Big)A^{2}
\end{aligned}
\tag{5.11}
$$

### 5.3. Illustrative example

Now we will investigate a special case of polar- symmetrical buckling of an annular plate. Assume that the cell length $l_1 = \Delta\varphi\rho$, Poisson's ratio $\nu_1 = \nu_2 = 0$ and the beam thickness $a = ml_1$. Hence, all averaged plate stiffenesses are constant, and for the local model equilibrium equation (5.10) have the form

$$
\widetilde{C}_1 w^0_{,1111} + \frac{2}{\rho}\widetilde{C}_1 w^0_{,111} - \frac{1}{\rho^2}\widetilde{C}_1 w^0_{,11} + \frac{1}{\rho^3}\widetilde{C}_1 w^0_{,1} - N_\rho w^0_{,11} - \frac{1}{\rho}N_\varphi w^0_{,1} = 0 \tag{5.12}
$$

where

$$
\begin{aligned}
&\widetilde{C}_1(\rho) = \langle B\widetilde{H}^{1111}\rangle + \widetilde{B}^{111}\widetilde{A}^{11} + \widetilde{B}^{211}\widetilde{A}^{22} \\
&\widetilde{A}^{11} = A^{11} \qquad\qquad \widetilde{A}^{22} = \frac{1}{\rho}A^{22} \\
&\widetilde{B}^{111} = B^{111} \qquad\qquad \widetilde{B}^{211} = \rho B^{211} \\
&\langle B\widetilde{H}^{1111}\rangle = B_1[m + m(1-m) + \alpha_1(1-m)^2]
\end{aligned}
\tag{5.13}
$$

In Eq. (5.13), $B_1$ denotes the bending stiffeness of the beams, and $\alpha_1 = E_{matrix}/E_{beams}$.

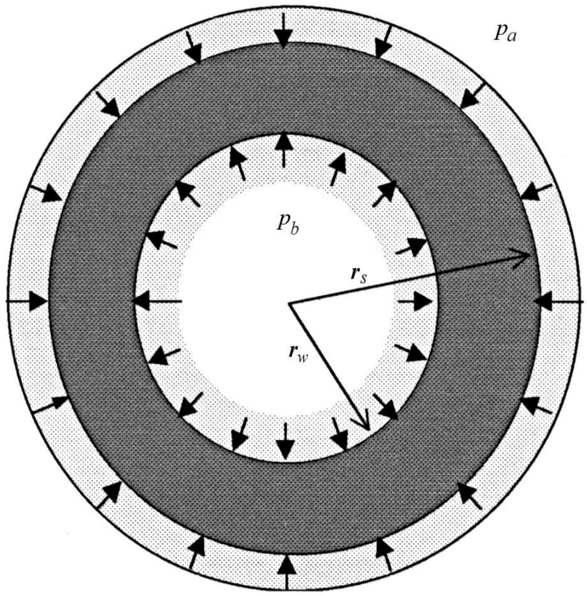


Fig. 4. An annular plate subjected to constant compressive forces

We will investigate the stability of the annular plate subjected to constant compressive forces distributed along the edges of the plate (Fig. 4). Bearing in mind that the tensile forces $N_\rho$ and $N_\varphi$ are averaged parts of the middle surface forces $n^{\alpha\beta}$, from the equilibrium equations for membrane forces in the midplane one gets, for $p_a = p_b$, the following condition $N_\rho = N_\varphi = N$. In this case, equilibrium equation (5.12) can be assumed in the form

$$
L[Lw^0(x)] - \gamma Lw^0(x) = 0 \tag{5.14}
$$

where, adopting a new dimensionless independent variable $x = \rho/r_z$ ($r_z$ is the external radius of the annular plate)

$$L = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \qquad\qquad \gamma = \frac{N(r_z)^2}{\widetilde{C}_1} \tag{5.15}$$

Fourth order differential equations (5.14) can be replaced by two independent second order Bessel's differential equations. The solution to these equations will be obtained as

$$w^0(x) = D_1 + D_2 \ln x + D_3 J_0(\lambda x) + D_4 Y_0(\lambda x) \tag{5.16}$$

where $\lambda = \sqrt{-\gamma}$ and $J_0(\lambda x)$, $Y_0(\lambda x)$ are Bessel's functions.

In the case of an annular plate clamped along the circumference, the boundary conditions have the form

$$\begin{aligned} &w^0(x=\eta) = 0 \qquad\qquad && \frac{dw^0(x=\eta)}{d\rho} = 0 \\ &w^0(x=1) = 0 && \frac{dw^0(x=1)}{d\rho} = 0 \end{aligned} \tag{5.17}$$

where $\eta = r_w/r_z$ ($r_z$ – external and $r_w$ – internal radius of the annular plate). Substituting Eq. (5.16) into (5.17), we obtain the condition

$$\begin{vmatrix} J_0(\eta\lambda) & Y_0(\eta\lambda) & \ln\eta & 1 \\ -\eta\lambda J_1(\eta\lambda) & -\eta\lambda Y_1(\eta\lambda) & 1 & 0 \\ J_0(\lambda) & Y_0(\lambda) & 0 & 1 \\ -\lambda J_1(\lambda) & -\lambda Y_1(\lambda) & 1 & 0 \end{vmatrix} = 0 \tag{5.18}$$

from which we calculate the critical value of the coefficient $\lambda_{cr}$ and the critical compressive force

$$N_{cr} = \frac{(\lambda_{cr})^2 \widetilde{C}_1}{(r_z)^2} \tag{5.19}$$

Introducing notations $N_{cr} = s_{cr}B_1/(r_z)^2$, where $B_1 = E_{beams}h^3/12(1-\nu^2)$, we derive diagrams of the parameter $s_{cr}$ versus the ratio $n = r_w/r_z$. On the diagram in Fig. 5 one can observe the smallest value of the critical parameter $s_{cr}$ versus the ratio $n$ for the ratio of the matrix and beams Young moduli $\alpha_1 = E_{matrix}/E_{beams} = 0.5$, where the ratio $a/l_1$ was used as a parameter. The diagram presenting the parameter $s_{cr}$ for $n = a/l_1 = 1.0$ shows the parameter corresponding to the critical force for a homogeneous plate made of the same material as that of the beams, while the diagram for $n = a/l_1 = 0$ shows the critical parameter for a homogeneous plate made of the matrix material. Figure 6 shows the critical parameter $s_{cr}$ for $n = a/l_1 = 0.1$, where the ratio $\alpha_1 = E_{matrix}/E_{beams}$ is used as a parameter.

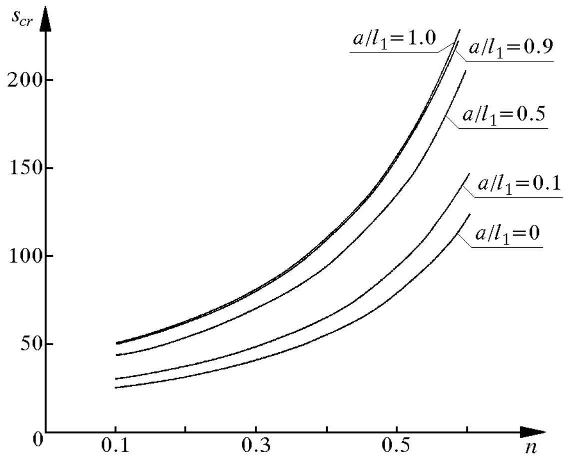


Fig. 5. The smallest value of the parameter $s_{cr}$ of critical forces $N$ versus the ratio $n = r_w/r_z$. The ratio $a/l_1$ is used as a parameter. It is assumed that $\alpha_1 = E_{matrix}/E_{walls} = 0.5$

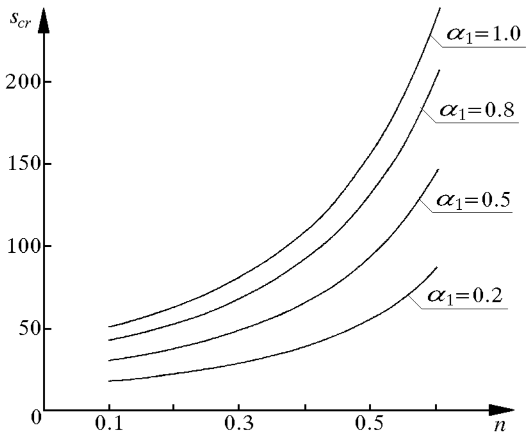


Fig. 6. The smallest value of the parameter $s_{cr}$ of the critical forces $N$ versus the ratio $n = r_w/r_z$. The ratio $\alpha_1 = E_{matrix}/E_{walls}$ is used as a parameter. It is assumed that $a/l_1 = 0.1$

## 6. Conclusions

In this paper, the tolerance averaging method, developed by Woźniak and Wierzbicki (2000) for heteroperiodic solids, is adopted to the analysis of stability of composite plates with non-uniform distribution of constituents. From the above considerations it follows that the tolerance averaging method can be successfully applied to the formulation of the averaging model of the linear stability of such plates.

The modelling approach is different from the known homogenization methods and leads to a model in which governing equations depend on the microstructure size. It has to be mentioned that the results obtained in this contribution cannot be derived by using the homogenization method related to solids which are periodic with respect to a certain curvilinear parametrization, see Lewiński and Telega (2000).

It can be seen that the above modelling approach leads from equations, which have highly oscillating coefficients, to a system of equations with nonconstant but slowly varying coefficients. A solution to these equations can be obtained by applying known typical numerical procedures.

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### Stateczność płyt kompozytowych z niejednorodnym rozkładem składników


Streszczenie

Celem pracy jest sformułowanie i zbadanie uśrednionego modelu opisującego stateczność płyty kompozytowej z niejednorodnym rozkładem składników. Rozpatrywana płyta ma określoną budowę, która nie jest periodyczna, ale która w małym

obszarze rozpatrywanej płyty może być w przybliżeniu traktowana jako periodyczna. Przedmiotem analizy jest kolista płyta kompozytowa zbudowana z dwóch rodzajów sprężystych prętów, których osie są prostopadłe. Obszar pomiędzy prętami wypełnia jednorodny sprężysty materiał matrycy. Sformułowanie przybliżonego modelu matematycznego bazuje na koncepcji uśredniania tolerancyjnego przedstawionej w pracy Woźniaka i Wierzbickiego (2000), gdzie ciało tego rodzaju nazwane jest ciałem heteroperiodycznym.

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