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ON THE CONTACT PROBLEM FOR A SMOOTH PUNCH IN PIEZOELECTROELASTICITY

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The problem of electroelasticity for piezoelectric materials is considered. For axially symmetric states, three potentials are introduced, which determine displacements, electric potential, stresses, components of the electric field vector and electric displacements in the piezoelectric body. These fundamental solutions are utilized to solve a smooth contact problem. Exact solutions are obtained for elastic and electric fields in the contact problem. The numerical results are presented graphically to show the influence of applied mechanical and electrical loading on the analyzed quantities and to clarify the effect of anisotropy of piezoelectric materials. It is also shown that the influence of anisotropy of the materials on these fields is significant.

Key words: piezoelectric medium, transverse isotropy, potential theory method, circular punch

1. Introduction

Mechanical durability and reliability of piezoelectric materials offer important considerations in the design of "smart" structures and devices. Actually, over a hundred piezoelectric materials or composites are known. Piezoelectric materials, particularly piezoelectric ceramics, have been widely used for applications such as sensors, filters, ultrasonic generators and actuators. The piezoelectric composite materials have also been used for hydrophone application and transducers for medical imaging. The readers interested in this problem are referred to the state of the art survey by Rao and Sunar (1994).

The body of literature concerning the mechanics of piezoelectric materials is enormous. We referred to a few fundamental works (Cady, 1946; Berlincourt *et al.*, 1964; Tiersten, 1969; Parton and Kudryatvsev, 1988).

In particular, the contact problem of electroelasticity is very interesting from the point of view of application, since the contact is the direct way of transmission of loading from one element to another. Fan *et al.* (1996) considered the two-dimensional contact problem of a piezoelectric half-plane. These authors by means of Stroh's formalism formulated the nonslip and slip conditions of contact on the half-plane. The three-dimensional contact problem for piezoelectric materials was solved by Chen (2000), who used Fabrikant's potentials (1989) and the solution which was found by Ding *et al.* (1996). The solutions related to elliptical contact problems and piezo-electro and magneto-electro elastic bodies have been recently obtained in papers by Ding *et al.* (1999) and Hou *et al.* (2003).

In this paper, three potential functions are introduced to simplify the basic equations for piezoelectric materials with transversely isotropic electrical and mechanical properties. Using the operator theory, we derive a general solution that is expressed in terms of the three potentials. These functions satisfy differential equations of the second order and are quasi-harmonic functions. Making use of these fundamental solutions, the punch problem is investigated. The integral equations are derived from the corresponding mixed boundary-value problems of a half space. The exact solutions are obtained. The formulae in a closed form, describing elastic and electric fields in piezoelectric materials, are obtained. Also relationships between the force, electric charge, indentation depth of the punch and the potential on the boundary are derived. These relationships are presented graphically.

2. Basic equations and their fundamental solution

As our point of departure, we take partial differential equations of equilibrium of linear elasticity for a transversely isotropic piezoelectric material

$$
\begin{aligned}
&c_{11}\mathcal{B}_1 u_r + c_{44}D^2 u_r + (c_{13}+c_{44})D\frac{\partial u_z}{\partial r} + (e_{31}+e_{15})\frac{\partial \phi}{\partial r} = 0\\
&c_{44}\mathcal{B}_0 u_z + c_{33}D^2 u_z + (c_{13}+c_{44})D\frac{\partial [r u_r]}{r\partial r} + e_{15}\mathcal{B}_0\phi + e_{33}D^2\phi = 0 \qquad (2.1)\\
&(e_{31}+e_{15})D\frac{\partial [r u_r]}{r\partial r} + e_{15}\mathcal{B}_0 u_z + e_{33}D^2 u_z - \varepsilon_{11}\mathcal{B}_0\phi - \varepsilon_{33}D^2\phi = 0
\end{aligned}
$$

where the following differential operators have been introduced

$$\mathcal{B}_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k}{r^2} \qquad k = 0,1 \qquad D = \frac{\partial}{\partial z} \tag{2.2}$$

In above equations: u_r and u_z are components of the displacement vector in the radial and axial directions of the cylindrical coordinate system (r, θ, z), ϕ is the electric potential and c_{ij}, e_{kl}, ε_{kl} stand for the elastic, piezoelectric and dielectric constants, respectively. The problem considered is axially symmetric in which $u_\theta \equiv 0$ and physical quantities are independent on the θ-coordinate.

We apply Hankel's transforms of the first order for equation $(2.1)_1$ and the zero order for equations $(2.1)_{2,3}$, namely

$$\begin{aligned} \widehat{u}_r(\xi, z) &= \mathcal{H}_1[u_r(r,z); r \to \xi] \equiv \int\limits_0^\infty u_r(r,z) r J_1(r\xi)\,dr \\ \Big\{\widehat{u}_z(\xi,z), \widehat{\phi}(\xi,z)\Big\} &= \mathcal{H}_0[u_z(r,z), \phi(r,z); r \to \xi] \equiv \\ &\equiv \int\limits_0^\infty \Big\{u_z(r,z), \phi(r,z)\Big\} r J_0(r\xi)\,d\xi \end{aligned} \tag{2.3}$$

where $J_1(\xi)$ and $J_0(r\xi)$ are the Bessel functions of the first kind and order one or zero, respectively, and ξ is the transform parameter. We use the properties of Hankel's transforms

$$\begin{aligned} &\mathcal{H}_\nu[B_\nu f(r,z); r \to \xi] = -\xi^2 \widehat{f}_\nu(\xi, z) \\ &\mathcal{H}_1\Big[\frac{\partial f(r,z)}{\partial r}; r \to \xi\Big] = -\xi \widehat{f}_0(\xi, z) \\ &\mathcal{H}_0\Big[\frac{\partial [r f(r,z)]}{r \partial r}; r \to \xi\Big] = \xi \widehat{f}_1(\xi, z) \end{aligned} \tag{2.4}$$

where the index $\nu = 0$ or $\nu = 1$ denotes the transforms of the zero or first order, respectively.

From partial differential equations of equilibrium (2.1) three-coupled ordinary differential equations are then obtained, which may be written in the form

$$\mathbf{D} \begin{bmatrix} \widehat{u}_r \\ \widehat{u}_z \\ \widehat{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.5}$$

where $\mathbf{D}$ is the following operator matrix

$$\mathbf{D} = \begin{bmatrix} -c_{11}\xi^2 + c_{44}D^2 & -\xi(c_{13} + c_{44})D & -\xi(e_{31} + e_{15})D \\ \xi(c_{13} + c_{44})D & -c_{44}\xi^2 + c_{33}D^2 & -e_{15}\xi^2 + e_{33}D^2 \\ \xi(e_{31} + e_{15})D & -e_{15}\xi^2 + e_{33}D^2 & \varepsilon_{11}\xi^2 - \varepsilon_{33}D^2 \end{bmatrix} \tag{2.6}$$

We have

$$|\mathbf{D}| = -a_0(D^2 - \lambda_1^2\xi^2)(D^2 - \lambda_2^2\xi^2)(D^2 - \lambda_3^2\xi^2) \tag{2.7}$$

where λ_i^2 $(i = 1,2,3)$ are the roots of the following cubic algebraic equation in λ_i^2

$$a_0\lambda^6 + b_0\lambda^4 + c_0\lambda^2 + d_0 = 0 \tag{2.8}$$

with the coefficients defined by equations

$$\begin{aligned}
a_0 &= c_{44}(c_{33}\varepsilon_{33} + e_{33}^2)\\
b_0 &= (e_{31} + e_{15})[2c_{13}e_{33} - c_{33}(e_{31} + e_{15})] + 2c_{44}e_{33}e_{31} - c_{11}e_{33}^2 + \\
&\quad - \varepsilon_{11}c_{33}c_{44} - \varepsilon_{33}c^2\\
c_0 &= 2e_{15}[c_{11}e_{33} - c_{13}(e_{31} + e_{15})] + c_{44}e_{31}^2 + \varepsilon_{33}c_{11}c_{44} + \varepsilon_{11}c^2\\
d_0 &= -c_{11}(c_{44}\varepsilon_{11} + e_{15}^2)\\
c^2 &= c_{11}c_{33} - c_{13}(c_{13} + 2c_{44})
\end{aligned} \tag{2.9}$$

By virtue of the operator theory, we obtain the following general solution to equations (2.5)

$$\begin{aligned}
\widehat{u}_r(\xi,z) &= A_{i1}\widehat{F}(\xi,z)\\
\widehat{u}_z(\xi,z) &= A_{i2}\widehat{F}(\xi,z)\\
\widehat{\phi}(\xi,z) &= A_{i3}\widehat{F}(\xi,z)
\end{aligned} \tag{2.10}$$

where A_{ij} are the algebraic cominors of the matrix operator and $\widehat{F}(\xi,z)$ is the zero order Hankel's transform of the general solution $F(r,z)$, which satisfies the equations, respectively

$$\begin{aligned}
&|\mathbf{D}|\widehat{F}(\xi,z) = 0\\
&(D^2 + \lambda_1^2\varDelta)(D^2 + \lambda_2^2\varDelta)(D^2 + \lambda_3^2\varDelta)F(r,z) = 0
\end{aligned} \tag{2.11}$$

Here, $\varDelta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r$ is the Laplacian and $D^2 = \partial^2/\partial z^2$.

Taking $i = 3$ and writing down the expressions for A_{3j}, we obtain

$$\begin{aligned}
\widehat{u}_r(\xi,z) &= (a_1D^2 + b_1\xi^2)\xi D\widehat{F}(\xi,z)\\
\widehat{u}_z(\xi,z) &= -(a_2D^4 + b_2\xi^2D^2 + c_2\xi^4)\widehat{F}(\xi,z)\\
\widehat{\phi}(\xi,z) &= (a_3D^4 + b_3\xi^2D^2 + c_3\xi^4)\widehat{F}(\xi,z)
\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}
a_1 &= c_{33}(e_{31} + e_{15}) - (c_{13} + c_{44})e_{33} & b_1 &= c_{13}e_{15} - c_{44}e_{31}\\
a_2 &= c_{44}e_{33} & b_2 &= (c_{13} + c_{44})e_{31} + c_{13}e_{15} - c_{11}e_{33}\\
c_2 &= c_{11}e_{15} & a_3 &= c_{44}c_{33}\\
b_3 &= c_{13}^2 + 2c_{13}c_{44} - c_{11}c_{33} & c_3 &= c_{11}c_{44}
\end{aligned} \tag{2.13}$$

Note that in equations (2.12), $\widehat{u}_r(\xi,z)$ is the first order Hankel's transform of the displacement $u_r(r,z)$, while $\widehat{u}_z(\xi,z)$ and $\widehat{\phi}(\xi,z)$ are the zero order Hankel's transforms of the displacement $u_z(r,z)$ and electric potential , $\phi(r,z)$, as well as $\widehat{F}(\xi,z)$ and $F(r,z)$.

Applying the inverse Hankel's transforms to equations (2.12), the oryginal solution for the displacements and electric potential are obtained as follows

$$\begin{aligned}
u_r(r,z) &= -(a_1D^2 - b_1\varDelta)\frac{\partial^2 F(r,z)}{\partial r\partial z}\\
u_z(r,z) &= -(a_2D^4 - b_2\varDelta D^2 + c_2\varDelta^2)F(r,z)\\
\phi(r,z) &= (a_3D^4 - b_3\varDelta D^2 + c_3\varDelta^2)F(r,z)
\end{aligned} \tag{2.14}$$

Using the generalized Almansi's theorem, the function $F(r,z)$, which satisfies equation $(2.11)_2$, can be expressed in terms of three quasi-harmonic functions

$$F = \begin{cases} F_1 + F_2 + F_3 & \text{for distinct } \lambda_i \\ F_1 + F_2 + zF_3 & \text{for } \lambda_1 \neq \lambda_2 = \lambda_3 \\ F_1 + zF_2 + z^2F_3 & \text{for } \lambda_1 = \lambda_2 = \lambda_3 \end{cases} \tag{2.15}$$

where $F_i(r,z)$ satisfies, respectively

$$\Big(\varDelta + \frac{1}{\lambda_i^2}D^2\Big)F_i(r,z) = 0 \qquad\qquad i = 1,2,3 \tag{2.16}$$

For the sake of simplicity, we proceed to consider the case of distinct roots here and after. On the other hand, the special case of multiple roots can be obtained from the general solution by appropriately limited calculation.

Using

$$\varDelta F_i = -\frac{1}{\lambda_i^2}D^2F_i \tag{2.17}$$

and summing in equations (2.14), we obtain

$$\begin{aligned}
u_r(r,z) &= -\sum_{i=1}^{3}\alpha_{i1}\frac{\partial^4 F_i}{\partial r\partial z^3} \qquad\qquad u_z(r,z) = -\sum_{i=1}^{3}\alpha_{i2}\frac{\partial^4 F_i}{\partial z^4}\\
\phi(r,z) &= \sum_{i=1}^{3}\alpha_{i3}\frac{\partial^4 F_i}{\partial z^4}
\end{aligned} \tag{2.18}$$

The coefficients α_{ij} are

$$\alpha_{ij} = a_j + \frac{b_j}{\lambda_i^2} + \frac{c_j}{\lambda_i^4} \tag{2.19}$$

where a_j, b_j and c_j are defined by equations (2.13) and $c_1 \equiv 0$.

It is now assumed that

$$\alpha_{i2}\frac{\partial^3 F_i(r,z)}{\partial z^3} = -\frac{1}{\lambda_i}\varphi_i(r,z) \tag{2.20}$$

then equations (2.18) can be further simplified to

$$\begin{aligned} u_r(r,z) &= \sum_{i=1}^{3} a_{i1}\lambda_i\frac{\partial\varphi_i}{\partial r} \qquad\qquad u_z(r,z) = \sum_{i=1}^{3}\frac{1}{\lambda_i}\frac{\partial\varphi_i}{\partial z} \\ \phi(r,z) &= -\sum_{i=1}^{3}\frac{a_{i3}}{\lambda_i}\frac{\partial\varphi_i}{\partial z} \end{aligned} \tag{2.21}$$

where

$$a_{i1} = \frac{\alpha_{i1}}{\alpha_{i2}}\frac{1}{\lambda_i^2} = \frac{a_1\lambda_i^2 + b_1}{a_2\lambda_i^4 + b_2\lambda_i^2 + c_2} \qquad\qquad a_{i3} = \frac{\alpha_{i3}}{\alpha_{i2}} = \frac{a_3\lambda_i^4 + b_3\lambda_i^2 + c_3}{a_2\lambda_i^4 + b_2\lambda_i^2 + c_2} \tag{2.22}$$

and

$$\Big(\varDelta + \frac{1}{\lambda_i^2}\frac{\partial^2}{\partial z^2}\Big)\varphi_i(r,z) = 0 \tag{2.23}$$

It can be verified that

$$a_{i3} = \frac{c_{13}+c_{44}}{e_{31}+e_{15}} - \frac{c_{11}-c_{44}\lambda_i^2}{e_{31}+e_{15}}a_{i1} = \frac{-e_{15}+e_{33}\lambda_i^2}{\varepsilon_{11}-\varepsilon_{33}\lambda_i^2} - \frac{(e_{31}+e_{15})\lambda_i^2}{\varepsilon_{11}-\varepsilon_{33}\lambda_i^2}a_{i1} \tag{2.24}$$

The relationships between stress, displacement and electric potential for a transversely isotropic piezoelectric medium, in the case of axial symmetry, are

$$\begin{aligned} \sigma_{rr} &= c_{11}\frac{\partial u_r}{\partial r} + c_{12}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial\phi}{\partial z} \\ \sigma_{\theta\theta} &= c_{12}\frac{\partial u_r}{\partial r} + c_{11}\frac{u_r}{r} + c_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial\phi}{\partial z} \\ \sigma_{zz} &= c_{13}\frac{\partial u_r}{\partial r} + c_{13}\frac{u_r}{r} + c_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial\phi}{\partial z} \\ \sigma_{zr} &= c_{44}\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big) + e_{15}\frac{\partial\phi}{\partial r} \end{aligned} \tag{2.25}$$

Substituting equations (2.21) into equations (2.25), we obtain

$$\begin{aligned} \sigma_{rr} &= -\sum_{i=1}^{3}\frac{a_{i4}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial z^2} - (c_{11}-c_{12})\frac{u_r}{r} \qquad\qquad \sigma_{zz} = \sum_{i=1}^{3}\frac{a_{i4}}{\lambda_i^3}\frac{\partial^2\varphi_i}{\partial z^2} \\ \sigma_{\theta\theta} &= -\sum_{i=1}^{3}\frac{a_{i4}}{\lambda_i^2}\frac{\partial^2\varphi_i}{\partial z^2} - (c_{11}-c_{12})\frac{\partial u_r}{\partial r} \qquad\qquad \sigma_{zr} = \sum_{i=1}^{3}\frac{a_{i4}}{\lambda_i}\frac{\partial^2\varphi_i}{\partial r\partial z} \end{aligned} \tag{2.26}$$

where

$$a_{i4} = \frac{e_{31}c_{44}\lambda_i^2 + e_{15}c_{11}}{e_{31} + e_{15}} a_{i1} + \frac{c_{44}e_{31} - c_{13}e_{15}}{e_{31} + e_{15}} \tag{2.27}$$

The components of the electric field vector E_r and E_z are obtained from relations

$$E_r = -\frac{\partial \phi}{\partial r} = \sum_{i=1}^{3} \frac{a_{i3}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial r \partial z} \qquad E_z = -\frac{\partial \phi}{\partial z} = \sum_{i=1}^{3} \frac{a_{i3}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial z^2} \tag{2.28}$$

The electric displacements are defined by equations

$$\begin{aligned} D_r &= e_{15}\Big(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\Big) + \varepsilon_{11} E_r \\ D_z &= e_{31}\Big(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\Big) + e_{33}\frac{\partial u_z}{\partial z} + \varepsilon_{33} E_z \end{aligned} \tag{2.29}$$

and presented by potentials as follows

$$D_r = \sum_{i=1}^{3} a_{i5}\lambda_i \frac{\partial^2 \varphi_i}{\partial r \partial z} \qquad D_z = \sum_{i=1}^{3} \frac{a_{i5}}{\lambda_i} \frac{\partial^2 \varphi_i}{\partial z^2} \tag{2.30}$$

where

$$a_{i5} = \frac{e_{33}\varepsilon_{11} - e_{15}\varepsilon_{33}}{\varepsilon_{11} - \varepsilon_{33}\lambda_i^2} - \frac{e_{31}\varepsilon_{11} - e_{15}\varepsilon_{33}\lambda_i^2}{\varepsilon_{11} - \varepsilon_{33}\lambda_i^2} a_{i1} \tag{2.31}$$

The form of the solution is very simple. It can be used to solve various kinds of mixed boundary - value problems of electroelasticity of a piezoelectric material, such as crack and punch problems.

It can be easily verified that:
Gauss'law (Parton and Kudryatvsev, 1988)

$$\frac{\partial D_r}{\partial r} + \frac{D_r}{r} + \frac{\partial D_z}{\partial z} = 0 \tag{2.32}$$

and equilibrium equations for stresses (Nowacki, 1973)

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0 \\ \frac{\partial \sigma_{zr}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} &= 0 \end{aligned} \tag{2.33}$$

are satisfied.

In the vacuum, constitutive equations (2.29) and governing equations (2.32) become

$$
\begin{gathered}
D_r = \varepsilon_0 E_r \qquad\qquad D_z = \varepsilon_0 E_z \\
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0
\end{gathered}
\tag{2.34}
$$

where ε_0 is the electric permittivity of the vacuum.

For axially symmetric problems, the very useful is Hankel transform method.

Assume the solutions to differential equations (2.23) in the form of Hankel's integrals as follows

$$
\varphi_i(r,z) = \int\limits_0^\infty A_i(\xi)\exp(-\lambda_i \xi z) J_0(r\xi)\, d\xi \tag{2.35}
$$

where $A_i(\xi)$ $(i = 1,2,3)$ are arbitrary functions of the transform parameter ξ, which is to be determined from the boundary conditions and λ_i are the roots of equations (2.8), which have positive real parts to ensure the regularity conditions at $z \to \infty$. Then we can easily obtain mechanical and electrical quantities (2.21), (2.26), (2.28) and (2.30).

We have:

$$
\begin{aligned}
& u_r(r,z) = -\sum_{i=1}^{3} a_{i1}\lambda_i I_{i1}(r,z) && u_z(r,z) = -\sum_{i=1}^{3} I_{i0}(r,z) \\
& \phi(r,z) = \sum_{i=1}^{3} a_{i3} I_{i0}(r,z) \\
& \sigma_{rr}(r,z) = -\sum_{i=1}^{3} a_{i4}\lambda_i J_{i0}(r,z) - (c_{11}-c_{12})\frac{u_r}{r} \\
& \sigma_{\theta\theta}(r,z) = -\sum_{i=1}^{3} a_{i4}\lambda_i J_{i0}(r,z) - (c_{11}-c_{12})\frac{\partial u_r}{\partial r} \\
& \sigma_{zz}(r,z) = \sum_{i=1}^{3} \frac{a_{i4}}{\lambda_i} J_{i0}(r,z) && \sigma_{zr}(r,z) = \sum_{i=1}^{3} a_{i4} J_{i1}(r,z) \\
& E_r(r,z) = \sum_{i=1}^{3} a_{i3} J_{i1}(r,z) && E_z(r,z) = \sum_{i=1}^{3} a_{i3}\lambda_i J_{i0}(r,z) \\
& D_r(r,z) = \sum_{i=1}^{3} a_{i5}\lambda_i^2 J_{i1}(r,z) && D_z(r,z) = \sum_{i=1}^{3} a_{i5}\lambda_i J_{i0}(r,z)
\end{aligned}
\tag{2.36}
$$

where

$$I_{i\nu} = \int\limits_0^\infty \xi A_i(\xi) \exp(-\lambda_i \xi z) J_\nu(r\xi)\, d\xi$$
$$J_{i\nu} = \int\limits_0^\infty \xi^2 A_i(\xi) \exp(-\lambda_i \xi z) J_\nu(r\xi)\, d\xi \qquad \nu = 0,1 \tag{2.37}$$

As an application of the obtained fundamental solution, the punch problem will be considered in the next Section.

3. Punch problem

We assume that the circular punch is flat ended, maintained at a constants electric potential and loaded centrally by a concentrated force. On these assumptions, it is known that both the electric potential ϕ and the punch penetration δ are constants inside the contact region (Fig. 1).

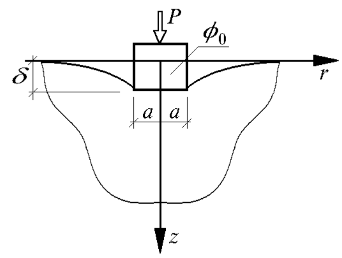

Fig. 1. Circular punch problem

In this case, we have the boundary conditions

$$\begin{aligned}
&(a) \qquad u_z(r,0) = \delta \qquad && 0 \leqslant r \leqslant a\\
&(b) \qquad \phi(r,0) = \phi_0 \qquad && 0 \leqslant r \leqslant a\\
&(c) \qquad \sigma_{rz}(r,0) = 0 \qquad && r \geqslant 0\\
&(d) \qquad \sigma_{zz}(r,0) = 0 \qquad && r > a
\end{aligned} \tag{3.1}$$

As usual (Fan *et al.*, 1996), the displacement and electric potential are prescribed in the contact region as δ and φ_0, respectively. For the sake of practical convenience, the punch can be grounded and the electric potential will be zero. Introducing two new unknown functions $D_1(\xi)$ and $D_2(\xi)$ for simplicity of the formulae for u_z and ϕ and using boundary condition (3.1c), we may obtain the following system of equations

$$\begin{aligned}
&A_1(\xi) + A_2(\xi) + A_3(\xi) = D_1(\xi) \\
&a_{13}A_1(\xi) + a_{23}A_2(\xi) + a_{33}A_3(\xi) = D_2(\xi) \\
&a_{14}A_1(\xi) + a_{24}A_2(\xi) + a_{34}A_3(\xi) = 0
\end{aligned} \tag{3.2}$$

The solution to this system of algebraic equations is

$$m_2 A_i(\xi) = d_i D_1(\xi) + l_i D_2(\xi) \tag{3.3}$$

where

$$\begin{array}{lll}
l_1 = -a_{24} + a_{34} & l_2 = -a_{34} + a_{14} & l_3 = -a_{14} + a_{24} \\
d_1 = a_{24}a_{33} - a_{34}a_{23} & d_2 = a_{13}a_{34} - a_{14}a_{33} & \\
d_3 = a_{14}a_{23} - a_{13}a_{24} & m_2 = \displaystyle\sum_{i=1}^{3} d_i &
\end{array} \tag{3.4}$$

Boundary conditions (3.1a), (3.1b) and (3.1d) yield

$$\begin{aligned}
&-\int\limits_0^\infty \xi D_1(\xi) J_0(r\xi)\, d\xi = \delta && 0 \leqslant r \leqslant a \\
&\int\limits_0^\infty \xi D_2(\xi) J_0(r\xi)\, d\xi = \phi_0 && 0 \leqslant r \leqslant a \\
&\frac{m}{m_2}\int\limits_0^\infty \xi^2 D_1(\xi) J_0(r\xi)\, d\xi + \frac{m_6}{m_2}\int\limits_0^\infty \xi^2 D_2(\xi) J_0(r\xi)\, d\xi = 0 && r > a
\end{aligned} \tag{3.5}$$

where m and m_6 are defined by equations (3.14) and m_2 by equations (3.4).

Dual integral equations (3.5) are converted to the Abel integral equation by means of the following integral representation for $\xi D_i(\xi)$ (Sneddon, 1972)

$$\xi D_i(\xi) = \sqrt{\frac{2}{\pi}} \int\limits_0^a \varPhi_i(x) \cos(\xi x)\, dx \qquad i = 1, 2 \tag{3.6}$$

Using the integrals involving Bessel and trigonometric functions, we may verify that equation $(3.5)_3$ is satisfied identically, while equations $(3.5)_{1,2}$ give

$$-\sqrt{\frac{2}{\pi}} \int\limits_0^r \frac{\varPhi_1(x)}{\sqrt{r^2 - x^2}}\, dx = \delta \qquad \sqrt{\frac{2}{\pi}} \int\limits_0^r \frac{\varPhi_2(x)}{\sqrt{r^2 - x^2}}\, dx = \phi_0 \tag{3.7}$$

These equations are of Abel's type and have the following solutions

$$\varPhi_1(x) = -\sqrt{\frac{2}{\pi}}\delta \qquad\qquad \varPhi_2(x) = \sqrt{\frac{2}{\pi}}\phi_0 \tag{3.8}$$

Substituting (3.8) into equation (3.6) and integrating, we obtain

$$\xi D_1(\xi) = -\frac{2}{\pi}\delta\frac{\sin\xi a}{\xi} \qquad\qquad \xi D_2(\xi) = \frac{2}{\pi}\phi_0\frac{\sin\xi a}{\xi} \tag{3.9}$$

The stress σ_{zz} and electric displacement D_z on the crack plane $z = 0$ are obtained as $(r < a)$

$$\begin{aligned}
\sigma_{zz}(r,0) &= \frac{2}{\pi}\Big(-\frac{m}{m_2}\delta + \frac{m_6}{m_2}\phi_0\Big)\frac{1}{\sqrt{a^2-r^2}}\\
D_z(r,0) &= \frac{2}{\pi}\Big(-\frac{m_5}{m_2}\delta + \frac{m_7}{m_2}\phi_0\Big)\frac{1}{\sqrt{a^2-r^2}}
\end{aligned} \tag{3.10}$$

where m_5 and m_7 are defined by equations (3.14).

The condition of complete contact

$$\sigma_{zz}(r,0) \leqslant 0 \qquad\qquad 0 \leqslant r < a$$

requires

$$\frac{m}{m_2}\delta \geqslant \frac{m_6}{m_2}\phi_0$$

The total force P and the concentrated electric charge Q are obtained by integrating equations (3.10) over a circle. We obtain

$$P = 4\Big(\frac{m}{m_2}\delta - \frac{m_6}{m_2}\phi_0)a \qquad\qquad Q = 4\Big(\frac{m_5}{m_2}\delta - \frac{m_7}{m_2}\phi_0)a \tag{3.11}$$

Using the above equations in (3.10), we obtain $(r < a)$

$$\sigma_{zz}(r,0) = -\frac{P}{2\pi a}\frac{1}{\sqrt{a^2-r^2}} \qquad\qquad D_z(r,0) = -\frac{Q}{2\pi a}\frac{1}{\sqrt{a^2-r^2}} \tag{3.12}$$

On the other hand, solving (3.11) with respect to δ and ϕ_0, we obtain

$$\delta = \frac{m_2}{4\widetilde{m}a}(Pm_7 - Qm_6) \qquad\qquad \phi_0 = \frac{m_2}{4\widetilde{m}a}(Pm_5 - Qm) \tag{3.13}$$

where $\widetilde{m}$, m_2, m_6, m_7, m_5 are defined as follows

$$
\begin{aligned}
m &= \sum_{i=1}^{3} \frac{a_{i4}d_i}{\lambda_i} & \qquad m_2 &= \sum_{i=1}^{3} d_i \\
m_5 &= \sum_{i=1}^{3} a_{i5}\lambda_i d_i & \qquad m_6 &= \sum_{i=1}^{3} \frac{a_{i4}l_i}{\lambda_i} \\
m_7 &= \sum_{i=1}^{3} a_{i5}\lambda_i l_i & \qquad \widetilde{m} &= mm_7 - m_5m_6
\end{aligned}
\tag{3.14}
$$

It is seen that the stress singularity has identically the same form as that for pure elasticity. In other words, the coupling effect of piezoelectric nature has no effect on the contact stress. As regards to the penetration depth δ and electric potential ϕ_0, these quantities depend on the elastic, piezoelectric and dielectric constants.

The displacement components u_r and u_z, electric potential ϕ, stress σ_{zr}, $\sigma_{\theta\theta}$, σ_{rr} and singular stress σ_{zz}, electric field intensities E_r and E_z and electric displacements D_r and D_z in the piezoelectric half space are obtained as

$$
\begin{aligned}
u_r(r,z) &= \frac{2a}{\pi r m_2} \sum_{i=1}^{3} a_{i1}\lambda_i(\delta d_i - \phi_0 l_i)(1-\eta_i) \\
u_z(r,z) &= \frac{1}{m_2} \sum_{i=1}^{3} (\delta d_i - \phi_0 l_i)\Big(1 - \frac{2}{\pi}\tan^{-1}\xi_i\Big) \\
\phi(r,z) &= -\frac{1}{m_2} \sum_{i=1}^{3} a_{i3}(\delta d_i - \phi_0 l_i)\Big(1 - \frac{2}{\pi}\tan^{-1}\xi_i\Big) \\
\sigma_{zr}(r,z) &= -\frac{2}{\pi r m_2} \sum_{i=1}^{3} a_{i4}(\delta d_i - \phi_0 l_i)\frac{\xi_i(1-\eta_i^2)}{\xi_i^2+\eta_i^2} \\
\sigma_{zz}(r,z) &= -\frac{2}{\pi a m_2} \sum_{i=1}^{3} \frac{a_{i4}}{\lambda_i}(\delta d_i - \phi_0 l_i)\frac{\eta_i}{\xi_i^2+\eta_i^2} \\
\sigma_{rr}(r,z) &= \frac{2}{\pi a m_2} \sum_{i=1}^{3} a_{i4}\lambda_i(\delta d_i - \phi_0 l_i)\frac{\eta_i}{\xi_i^2+\eta_i^2} - (c_{11}-c_{12})\frac{u_r}{r} \\
\sigma_{\theta\theta}(r,z) &= \frac{2}{\pi a m_2} \sum_{i=1}^{3} a_{i4}\lambda_i(\delta d_i - \phi_0 l_i)\frac{\eta_i}{\xi_i^2+\eta_i^2} - (c_{11}-c_{12})\frac{\partial u_r}{\partial r} \\
E_r(r,z) &= -\frac{2}{\pi r m_2} \sum_{i=1}^{3} a_{i3}(\delta d_i - \phi_0 l_i)\frac{\xi_i(1-\eta_i^2)}{\xi_i^2+\eta_i^2}
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
E_z(r,z) &= -\frac{2}{\pi a m_2}\sum_{i=1}^{3} a_{i3}\lambda_i(\delta d_i - \phi_0 l_i)\frac{\eta_i}{\xi_i^2+\eta_i^2}\\
D_r(r,z) &= -\frac{2}{\pi r m_2}\sum_{i=1}^{3} a_{i5}\lambda_i^2(\delta d_i - \phi_0 l_i)\frac{\xi_i(1-\eta_i^2)}{\xi_i^2+\eta_i^2}\\
D_z(r,z) &= -\frac{2}{\pi a m_2}\sum_{i=1}^{3} a_{i5}\lambda_i(\delta d_i - \phi_0 l_i)\frac{\eta_i}{\xi_i^2+\eta_i^2}
\end{aligned}
$$

The closed form solutions for elastic and electric fields (3.15) are obtained accordingly to the improper integrals presented analytically in Appendix. In the above equations, three sets of oblate spherodial coordinates ξ_i, η_i are defined by equations

$$
r^2 = a^2(1+\xi_i^2)(1-\eta_i^2) \qquad \lambda_i z = a\xi_i\eta_i \qquad i=1,2,3 \tag{3.16}
$$

and are related to r, $\lambda_i z$ by equations

$$
\begin{aligned}
\xi_i(r,z,a,\lambda_i) &= \frac{1}{\sqrt{2}a}\sqrt{\sqrt{(r^2+\lambda_i^2z^2-a^2)^2+4\lambda_i^2z^2a^2}+(r^2+\lambda_i^2z^2-a^2)}\\
\eta_i(r,z,a,\lambda_i) &= \frac{1}{\sqrt{2}a}\sqrt{\sqrt{(r^2+\lambda_i^2z^2-a^2)^2+4\lambda_i^2z^2a^2}-(r^2+\lambda_i^2z^2-a^2)}
\end{aligned}
\tag{3.17}
$$

On the plane $z=0$ we have

$$
\begin{aligned}
u_r(r,0) &= \frac{2a}{\pi r m_2}(m_1\delta - m_1^*\phi_0)\Big[1-\sqrt{1-\frac{r^2}{a^2}}H(a-r)\Big]\\
u_z(r,0) &= \delta\Big[1-\Big(1-\frac{2}{\pi}\sin^{-1}\frac{a}{r}\Big)H(r-a)\Big]\\
\phi(r,0) &= \phi_0\Big[1-\Big(1-\frac{2}{\pi}\sin^{-1}\frac{a}{r}\Big)H(r-a)\Big]\\
\sigma_{zr}(r,0) &\equiv 0\\
\sigma_{zz}(r,0) &= -\frac{P}{2\pi a}\frac{H(a-r)}{\sqrt{a^2-r^2}}\\
\sigma_{rr}(r,0) &= \frac{2}{\pi m_2}(\delta m_4^* - \phi_0 m_4^{**})\frac{H(a-r)}{\sqrt{a^2-r^2}} - (c_{11}-c_{12})\frac{u_r}{r}\\
\sigma_{\theta\theta}(r,0) &= \frac{2}{\pi m_2}(\delta m_4^* - \phi_0 m_4^{**})\frac{H(a-r)}{\sqrt{a^2-r^2}} - (c_{11}-c_{12})\frac{\partial u_r}{\partial r}
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
E_r(r,0) &= -\frac{2a}{\pi r m_2}(\delta m_3 - \phi_0 m_2)\frac{H(r-a)}{\sqrt{r^2-a^2}} = \frac{2a}{\pi r}\phi_0\frac{H(r-a)}{\sqrt{r^2-a^2}} \\
E_z(r,0) &= -\frac{2}{\pi m_2}(\delta m_3^* - \phi_0 m_3^{**})\frac{H(a-r)}{\sqrt{a^2-r^2}} \\
D_r(r,0) &= -\frac{2a}{\pi r m_2}(\delta m_5^* - \phi_0 m_7^*)\frac{H(r-a)}{\sqrt{r^2-a^2}} \\
D_z(r,0) &= -\frac{Q}{2\pi a}\frac{H(a-r)}{\sqrt{a^2-r^2}}
\end{aligned}
$$

where $H(r-a)$ is the Heaviside function and

$$
\begin{array}{lll}
m_1 = \sum\limits_{i=1}^{3} a_{i1}\lambda_i d_i & m_1^* = \sum\limits_{i=1}^{3} a_{i1}\lambda_i l_i & m_3^* = \sum\limits_{i=1}^{3} a_{i3}\lambda_i d_i \\
m_3^{**} = \sum\limits_{i=1}^{3} a_{i3}\lambda_i l_i & m_4^* = \sum\limits_{i=1}^{3} a_{i4}\lambda_i d_i & m_4^{**} = \sum\limits_{i=1}^{3} a_{i4}\lambda_i l_i \\
m_5^* = \sum\limits_{i=1}^{3} a_{i5}\lambda_i^2 d_i & m_7^* = \sum\limits_{i=1}^{3} a_{i5}\lambda_i^2 l_i &
\end{array}
\tag{3.19}
$$

Note that

$$
\begin{array}{lll}
\sum\limits_{i=1}^{3} a_{i4} d_i = 0 & \sum\limits_{i=1}^{3} a_{i4} l_i = 0 & \sum\limits_{i=1}^{3} a_{i3} l_i = m_2 \\
m_3 = \sum\limits_{i=1}^{3} a_{i3} d_i = 0 & \sum\limits_{i=1}^{3} l_i = 0 &
\end{array}
\tag{3.20}
$$

4. Numerical results

The piezoelectric material being considered is PZT-4 due to its popularity. The non-zero constitutive coefficients of PZT-4 are (Park and Sun, 1995)

$$
\left.\begin{array}{lll}
c_{11} = 13.90 & c_{33} = 11.30 & c_{44} = 2.56 \\
c_{12} = 7.78 & c_{13} = 7.43 &
\end{array}\right\} \quad (\times 10^{10},\ \text{in}\ \ \mathrm{N/m^2})
$$

$$
\begin{array}{llll}
e_{15} = 13.44 & e_{31} = -6.98 & e_{33} = 13.84 & (\text{in}\ \ \mathrm{C/m^2}) \\
\varepsilon_{11} = 60.00 & \varepsilon_{33} = 54.70 & & (\times 10^{-10},\ \text{in}\ \ \mathrm{C/Vm})
\end{array}
$$

Also two other piezoelectric ceramics PZT-5H and P-7 as the comparative model materials for our numerical calculation are used. The properties of PZT-5H are

$$\left.\begin{array}{lll} c_{11} = 12.60 & c_{33} = 11.70 & c_{44} = 3.53 \\ c_{12} = 5.50 & c_{13} = 5.30 & \end{array}\right\} \quad (\times 10^{10}, \text{ in } \mathrm{N/m^2})$$

$$e_{15} = 17.00 \qquad e_{31} = -6.50 \qquad e_{33} = 23.30 \qquad (\text{in } \mathrm{C/m^2})$$

$$\varepsilon_{11} = 151.00 \qquad \varepsilon_{33} = 130.00 \qquad (\times 10^{-10}, \text{ in } \mathrm{C/Vm})$$

and the properties of P-7 are

$$\left.\begin{array}{lll} c_{11} = 13.00 & c_{33} = 11.90 & c_{44} = 2.50 \\ c_{12} = 8.30 & c_{13} = 8.30 & \end{array}\right\} \quad (\times 10^{10}, \text{ in } \mathrm{N/m^2})$$

$$e_{15} = 13.50 \qquad e_{31} = -10.30 \qquad e_{33} = 14.70 \qquad (\text{in } \mathrm{C/m^2})$$

$$\varepsilon_{11} = 171.00 \qquad \varepsilon_{33} = 186.00 \qquad (\times 10^{-10}, \text{ in } \mathrm{C/Vm})$$

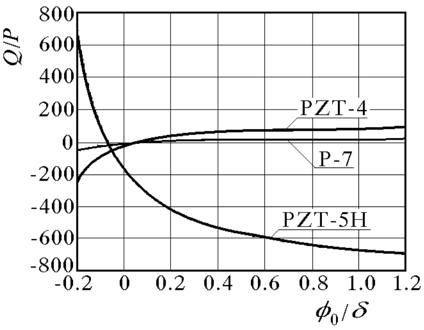

Fig. 2. Variation of the ratio of concentrated electric charge Q to total force P with the ratio of boundary electric potential ϕ_0 to depth of indentation of the punch δ

Figure 2 shows the dependence of Q/P [C/N] versus φ_0/δ [V/m]. From the condition of complete contact and equation $(3.10)_1$, we conclude that $\phi_0/\delta \geqslant -0.3938$; -0.3583; -0.3240 for PZT-4, PZT-5H and P-7 piezoelectric materials, respectively. In this figure we can notice that, firstly, the material dissimilarity is more visible, secondly, these curves either increase for one material or decreases for another with the increasing ratio ϕ_0/δ. These curves tend to some asymptotic values, which are positive or negative depending on physical properties of the material. Note that the significant role in the problem under consideration plays the piezoelectric constant e_{33}. This constant for PZT-5H piezoelectric material is significant larger from the ones for other materials. Therefore, the behaviour of this material is opposite to the other ones.

A. Appendix

The following integrals are used

$$\int_0^\infty \frac{d}{d\xi}\Big(\frac{\sin\xi a}{\xi}\Big)\mathrm{e}^{-\xi\lambda_i z} J_0(r\xi)\,d\xi = a\eta_i\Big[1-\xi_i\Big(\frac{\pi}{2}-\tan^{-1}\xi_i\Big)\Big] \tag{A.1}$$

$$\int_0^\infty \frac{d}{d\xi}\Big(\frac{\sin\xi a}{\xi}\Big)\mathrm{e}^{-\xi\lambda_i z} J_1(r\xi)\,d\xi = -\frac{r}{2}\Big(\frac{\pi}{2}-\tan^{-1}\xi_i-\frac{\xi_i}{1+\xi_i^2}\Big) \tag{A.2}$$

$$\int_0^\infty \xi\frac{d}{d\xi}\Big(\frac{\sin\xi a}{\xi}\Big)\mathrm{e}^{-\xi\lambda_i z} J_0(r\xi)\,d\xi = -\frac{\pi}{2}+\tan^{-1}\xi_i+\frac{\xi_i}{\xi_i^2+\eta_i^2} \tag{A.3}$$

$$\int_0^\infty \xi\frac{d}{d\xi}\Big(\frac{\sin\xi a}{\xi}\Big)\mathrm{e}^{-\xi\lambda_i z} J_1(r\xi)\,d\xi = \frac{r}{a}\frac{\eta_i}{(1+\xi_i^2)(\xi_i^2+\eta_i^2)} \tag{A.4}$$

where ξ_i and η_i are defined by equations (3.17) and λ_i are the roots of equation (2.8), which have positive real parts.

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O zagadnieniu kontaktowym gładkiego stempla w piezoelektrosprężystości

Streszczenie

Rozpatrzono osiowo symetryczne zagadnienie elektrosprężystości dla materiałów piezoelektrycznych. Wprowadzono trzy potencjały opisujące przemieszczenia, naprężenia, elektryczny potencjał, składowe wektora pola elektrycznego i elektrycznych przemieszczeń. Znalezione fundamentalne rozwiązania wykorzystano do analizy zagadnienia kontaktowego gładkiego stempla. Znaleziono ścisłe rozwiązania opisujące sprężyste i elektryczne pola w rozpatrywanym zagadnieniu kontaktowym. Wyniki obliczeń przedstawiono na wykresie w celu pokazania wpływu mechanicznych i elektrycznych obciążeń na analizowane wielkości. Efekt anizotropii materiałów piezoelektrycznych w omawianym zagadnieniu jest znaczący.

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