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THE TRIPLE PENDULUM WITH BARRIERS AND THE PISTON – CONNECTING ROD – CRANCKSHAFT MODEL

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A model of a flat triple pendulum with obstacles imposed on its position is used for the modelling of a piston-connecting rod-crankshaft system of a mono-cylinder four-stroke combustion engine. The introduced self-excited system can be only treated as the first step in more advanced modelling of real processes, but some examples of numerical simulations exhibit well known six stages of the piston sliding along the cylinder surface per cycle.

Key words: triple pendulum, multibody system, unilateral constraints, impact, piston dynamics

1. Introduction

Single degree-of-freedom models are only the first step to understand real behaviour of either natural or engineering systems, usually modelled by a few degrees of freedom. On the other hand, it is well known that impact and friction accompanies almost all real behaviour, leading to non-smooth dynamics. The non-smooth dynamical systems are analysed in both pure (Kunze, 2000) and applied sciences (Brogliato, 1999). The non-classical bifurcations are analysed in systems with dry friction (Leine *et al.*, 2000) and in systems with impacts (Ivanov, 1996; Lenci and Rega, 2000).

The scope of the paper contains the modelling of a flat triple physical pendulum with arbitrary situated barriers imposed on the position of the system (including modelling of the impact and sliding motion), numerical schemes for system simulation, methods for stability investigation of the orbit analysis in the case of the non-smooth system (Müller, 1995) and its application in the investigated system in order to study non-smooth dynamics as well as classical and non-classical bifurcations (Awrejcewicz *et al.*, 2002, 2004; Kudra, 2002).

This report is devoted to another goal of the research focused on applications of the investigated system and an example of a piston-connecting rod-crankshaft system of a mono-cylinder combustion engine modelled as an inverted triple pendulum with impacts (Kudra, 2002). The mathematical model is described and some numerical simulations are presented and discussed.

2. Mathematical model

The special case of a triple pendulum with barriers: the piston – connecting rod – crankshaft system of a mono–cylinder combustion engine is presented in Fig. 1. The first link of the pendulum represents crankshaft (1), the second one is connecting rod (2) and the third one is piston (3). The links (with masses m_i and with moments of inertia J_{zi} with respect to the principal central axes perpendicular to the plane of motion including points O_i) are connected by rotational joints with viscous damping (with real coefficients $\overline{c}_i$). The cylinder barrel imposes restrictions on the position of the piston, which moves in the cylinder with backlash. It is assumed that in the contact zone between surfaces of the piston and the cylinder, a tangent force does not appear.

It is assumed that the gas pressure force $\overline{F}(\varphi_1)$ is a function of the angular position φ_1 of the crankshaft and can be reduced to a force acting along the line parallel to the axis of the system and containing the piston pin axis O_3. Moreover, the crankshaft is externally driven by the moment $\overline{M}_0$ originating from an external power receiver (brake) and acting contrary to the positive sense of the angle φ_1. We also assume that the rotational speed of the crankshaft is positive. In that way we obtaine a self-excited system.

The non-dimensional governing equations (if none of the obstacles is active) are as follows

$$\mathbf{M}(\boldsymbol{\phi})\ddot{\boldsymbol{\phi}} + \mathbf{N}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}^2 + \mathbf{C}\boldsymbol{\phi} + \boldsymbol{p}(\boldsymbol{\phi}) = \boldsymbol{f}_e(\boldsymbol{\phi}) \tag{2.1}$$

where

$$\boldsymbol{\phi} = \begin{Bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{Bmatrix} \qquad \ddot{\boldsymbol{\phi}} = \begin{Bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \\ \ddot{\varphi}_3 \end{Bmatrix} \qquad \dot{\boldsymbol{\phi}}^2 = \begin{Bmatrix} \dot{\varphi}_1^2 \\ \dot{\varphi}_2^2 \\ \dot{\varphi}_3^2 \end{Bmatrix} \qquad \dot{\boldsymbol{\phi}} = \begin{Bmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{\varphi}_3 \end{Bmatrix}$$

$$\mathbf{M}(\boldsymbol{\phi}) = \begin{bmatrix} 1 & \nu_{12}\cos(\varphi_1-\varphi_2) & \nu_{13}\cos(\varphi_1-\varphi_3) \\ \nu_{12}\cos(\varphi_1-\varphi_2) & \beta_2 & \nu_{23}\cos(\varphi_2-\varphi_3) \\ \nu_{13}\cos(\varphi_1-\varphi_3) & \nu_{23}\cos(\varphi_2-\varphi_3) & \beta_3 \end{bmatrix}$$

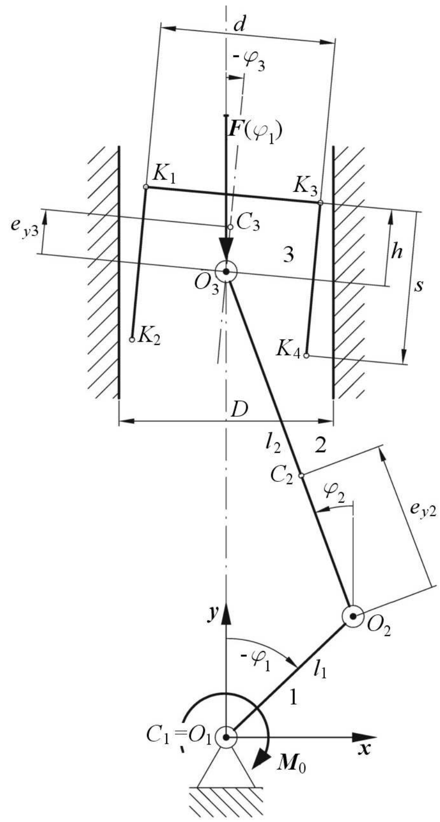

Fig. 1. Piston – connecting rod – crankshaft system

$$\mathbf{N}(\boldsymbol{\phi}) = \begin{bmatrix} 0 & \nu_{12}\sin(\varphi_1-\varphi_2) & \nu_{13}\sin(\varphi_1-\varphi_3) \\ -\nu_{12}\sin(\varphi_1-\varphi_2) & 0 & \nu_{23}\sin(\varphi_2-\varphi_3) \\ -\nu_{13}\sin(\varphi_1-\varphi_3) & -\nu_{23}\sin(\varphi_2-\varphi_3) & 0 \end{bmatrix} \qquad (2.2)$$

$$\mathbf{C} = \overline{\alpha}_1^{-1} c_1 \begin{bmatrix} 1+c_{21} & -c_{21} & 0 \\ -c_{21} & c_{21}+c_{31} & -c_{31} \\ 0 & -c_{31} & c_{31} \end{bmatrix}$$

$$\boldsymbol{f}_e(\boldsymbol{\phi}) = \overline{\alpha}_1^{-2} \begin{bmatrix} F_0 p(\varphi_1)\sin\varphi_1 - M_0 \\ \lambda_2 F_0 p(\varphi_1)\sin\varphi_2 \\ 0 \end{bmatrix}$$

In the above the symbols $(\dot{\cdot})$ and $(\ddot{\cdot})$ denote respectively the first and second derivatives with respect to the non-dimensional time $\overline{t}$ (such that the angular velocity φ_1 in the post-transient motion is approximately equal to 1) and $p(\varphi_1)$ is the non-dimensional function of gas pressure distribution.

The restrictions on the piston position imposed by the cylinder barrel can be described using the following set of non-dimensional inequalities

$$\begin{aligned} h_1(\boldsymbol{\phi}) &= \frac{\Delta}{2} - \sin\varphi_1 - \lambda_2 \sin\varphi_2 - \eta\sin\varphi_3 - \frac{\delta}{2}\cos\varphi_3 \geqslant 0 \\ h_2(\boldsymbol{\phi}) &= \frac{\Delta}{2} - \sin\varphi_1 - \lambda_2 \sin\varphi_2 + (\sigma-\eta)\sin\varphi_3 - \frac{\delta}{2}\cos\varphi_3 \geqslant 0 \\ h_3(\boldsymbol{\phi}) &= \frac{\Delta}{2} + \sin\varphi_1 + \lambda_2 \sin\varphi_2 + \eta\sin\varphi_3 - \frac{\delta}{2}\cos\varphi_3 \geqslant 0 \\ h_4(\boldsymbol{\phi}) &= \frac{\Delta}{2} + \sin\varphi_1 + \lambda_2 \sin\varphi_2 - (\sigma-\eta)\sin\varphi_3 - \frac{\delta}{2}\cos\varphi_3 \geqslant 0 \end{aligned} \tag{2.3}$$

where

$$\lambda_2 = \frac{l_2}{l_1} \qquad \eta = \frac{h}{l_1} \qquad \sigma = \frac{s}{l_1}$$
$$\delta = \frac{d}{l_1} \qquad \Delta = \frac{D}{l_1}$$

More details concerning described the model of the piston – connecting rod – crankshaft system can be found in work by Kudra (2002).

Observe that the proposed dynamical model of the piston – connecting rod – crankshaft system can be treated as a simplified model, since some very important technological details are neglected. The most important simplifications are:

- tangent forces of interaction between the piston and cylinder surfaces are neglected;
- interaction of the piston-cylinder introduced by the piston rings (by means of friction forces in the ring grooves in the direction perpendicular to the cylinder surface) is neglected;
- simplified friction model in every joint of the system (i.e. linear damping) is assumed.

In addition, the modelling of impact between the piston and the cylinder, where an oil layer exists, requires an approach different from the generalized restitution coefficient rule.

3. Numerical examples

In Figures 2-4, exemplary solutions of the piston – connecting rod – crankshaft system described in the previous section are presented. The gas pressure function $p(\varphi_1)$ is developed into the Fourier series with $K = 25$ terms using

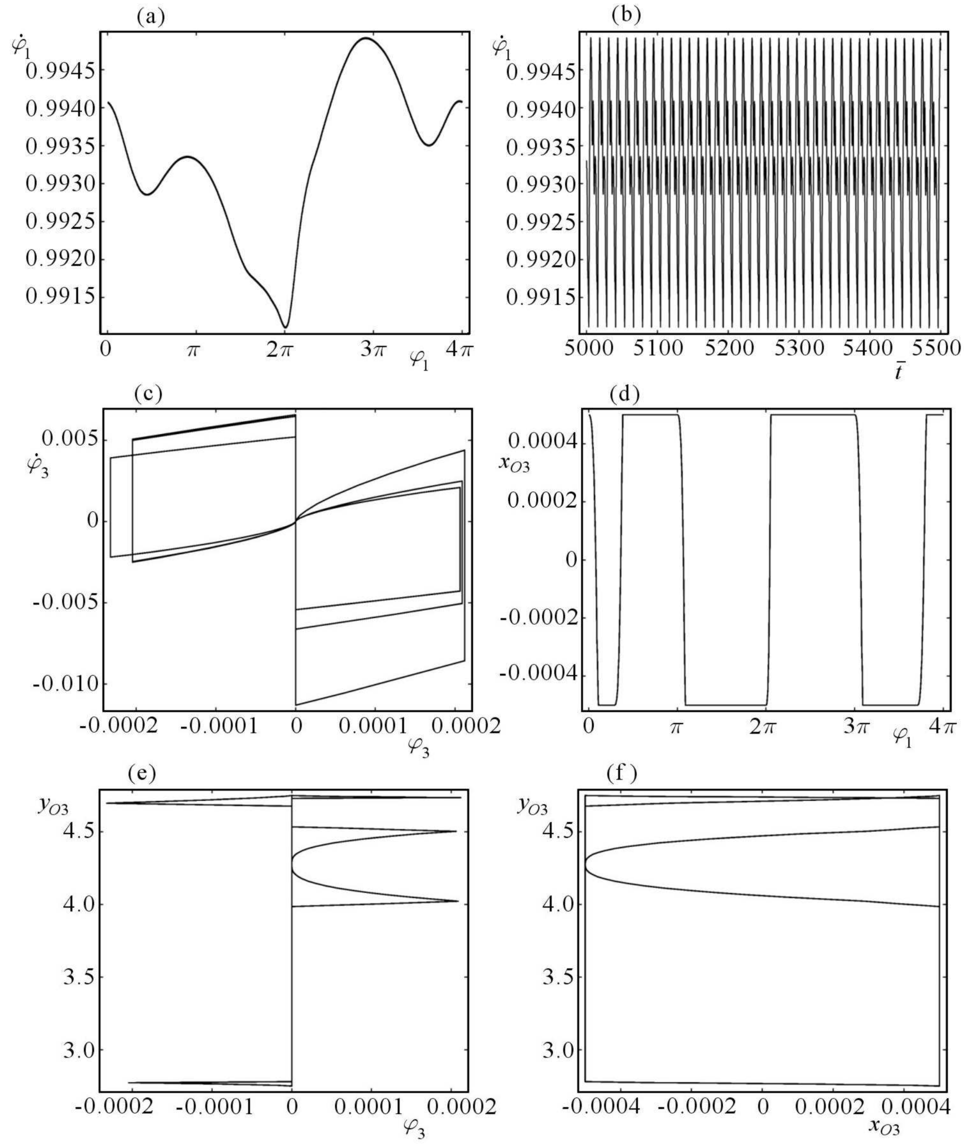

Fig. 2. System response for $e = 0$ and $D = 0.08004\,\mathrm{m}$

real data presented in work by Sygniewicz (1999) for a four stroke engine. The maximal gas pressure is choosen as $p_{max} = 3\,\mathrm{MPa}$ for the rotational crankshaft speed $n = 2000\,\mathrm{rot/min}$. The rest of real parameters are as follows: $m_1 = 10\,\mathrm{kg}$, $m_2 = 1\,\mathrm{kg}$, $m_3 = 0.4\,\mathrm{kg}$, $J_{z1} = 1\,\mathrm{kg\,m^2}$, $J_{z2} = 0.0075\,\mathrm{kg\,m^2}$, $J_{z3} = 0.001\,\mathrm{kg\,m^2}$, $l_1 = 0.04\,\mathrm{m}$, $l_2 = 0.15\,\mathrm{m}$, $e_{y1} = 0\,\mathrm{m}$, $e_{y2} = 0.12\,\mathrm{m}$, $e_{y3} = 0.01\,\mathrm{m}$, $d = 0.08\,\mathrm{m}$, $s = 0.08\,\mathrm{m}$, $h = 0.04\,\mathrm{m}$, $\overline{c}_1 = 0.00432\,\mathrm{Nm^{-1}\,s}$, $c_{21} = \overline{c}_2/\overline{c}_1 = 0.2$, $c_{31} = \overline{c}_3/\overline{c}_1 = 0.2$, $\overline{M}_0 = 9.807\,\mathrm{Nm}$.

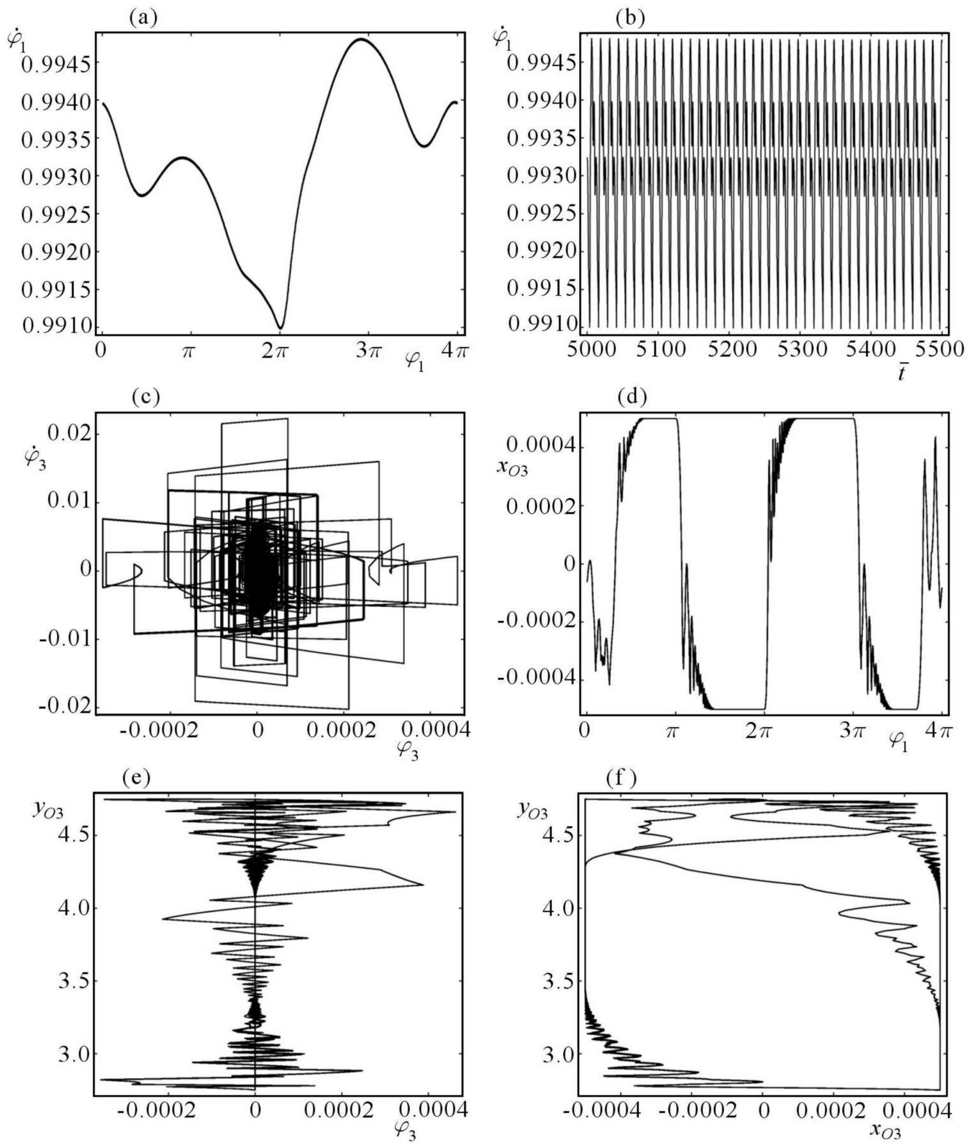

Fig. 3. System response for $e = 0.9$ and $D = 0.08004\,\mathrm{m}$

The calculations are performed for different values of the restitution coefficient e and the external diameter D. The quantities x_{03} and y_{03} describe the non-dimensional position of the piston pin axis O_3. It is seen in Fig. 2 that the piston moves six times from one side of the cylinder to the second side during one cycle of the engine work, and most of the time the piston adjoins either to one or second side of the cylinder surface. This result confirms the investigations presented by Sygniewicz (1999). However, the piston loses its contact

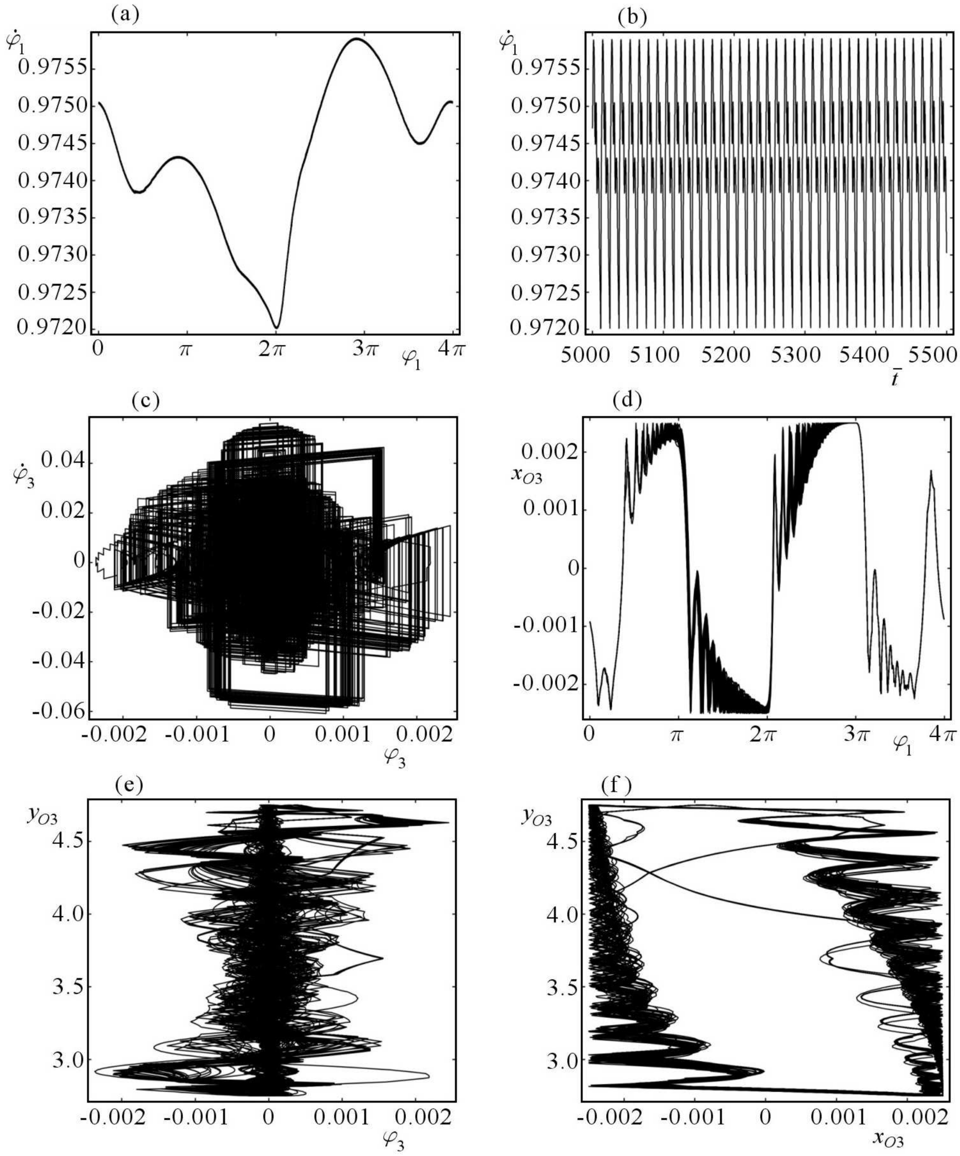

Fig. 4. System response for $e = 0.9$ and $D = 0.0802\,\mathrm{m}$

with the cylinder moving from one side to the second side of the cylinder with a small rotation angle. This phenomenon differs from the results presented by Sygniewicz (1999), where it was assumed that the piston does not loose the contact with the cylinder. The crankshaft angular positions at the beginning and end of the phases of the piston adjoining and sliding along the cylinder also differ from the results presented by Sygniewicz (1999) up to $35°$. In the latter case, the exhibited differences follow from negligence of some essential technological details mentioned in the previous section. For a larger restitution coefficient (Fig. 3) and larger backlash (Fig. 4), we also observe tendency of the piston to slide six times along the cylinder per one cycle of the engine work, but since multiple impacts between the piston and the cylinder occur before each of the sliding state, it happens that before the piston stablizes its motion at one cylinder side, it rapidly leaves the contact and transits into the other side of the cylinder.

4. Concluding remarks

The developed general model of the triple physical pendulum with barriers can be useful in the modelling of many real processes in nature and engineering. The presented model of the piston – connecting rod – crankshaft system modelled as a triple physical pendulum with impacts (in spite of some differences) behaves in a way similar to that described and illustrated in the monograph by Sygniewicz (1999). In particular, six piston movements from one side of the cylinder to its opposite side (during one cycle of the engine work) have been detected. The presented model can be treated as the first step to describe the real piston-connecting rod-crankshaft system, and after taking account of some technological details, a better convergence with the real system can be expected. Moreover, the proposed model describes full dynamics of piston motion in the cylinder, and thus it can be very useful for analysis of noise generated by impacts between the piston and cylinder barrel.

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Potrójne wahadło z przeszkodami i model mechanizmu korbowego

Streszczenie

Model płaskiego potrójnego wahadła fizycznego z przeszkodami ograniczającymi jego ruch został użyty do zamodelowania mechanizmu korbowego silnika spalinowego. Wprowadzony samowzbudny model może być traktowany jedynie jako bardzo zgrabne przybliżenie rzeczywistych zjawisk zachodzących w cylindrze silnika spalinowego. Pomimo tego, kilka zaprezentowanych przykładów symulacji numerycznych wykazuje bardzo dobrą zgodność z danymi doświadczalnymi prezentowanymi w literaturze.

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