

## NON-STATIONARY HEAT TRANSFER IN A HOLLOW CYLINDER WITH FUNCTIONALLY GRADED MATERIAL PROPERTIES

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The unidirectional non-stationary heat conduction in a two-phase hollow cylinder is considered. The conductor is made of two-phase stratified composites and has smooth gradation of effective properties in the radial direction. Therefore, we deal here with a special case of functionally graded materials, FGM. The formulation of mathematical model of the conductor is based on a tolerance averaging approach (TAA). Application to the non-stationary heat conduction and a comparison of the tolerance model with the asymptotic one is shown. The effect of geometry and material properties of the conductor on the temperature field is examined.

*Key words:* heat transfer, functionally graded material, tolerance averaging approach

### 1. Introduction

The main aim of this paper is to consider an effect of geometry and material properties on the temperature field in a two-phase hollow cylinder. This consideration deals with a non-stationary heat transfer problem in a composite conductor with a deterministic microstructure which is periodic along the angular axis and has smooth and slow gradation of effective properties in the radial direction (Fig. 1). Therefore, we deal here with a special case of functionally graded materials, FGM (Suresh and Mortensen, 1998).

Functionally graded materials are a new class of composite materials where composition of constituents generates continuous and smooth gradation of apparent properties of the composite. The analysis of the heat transfer in a

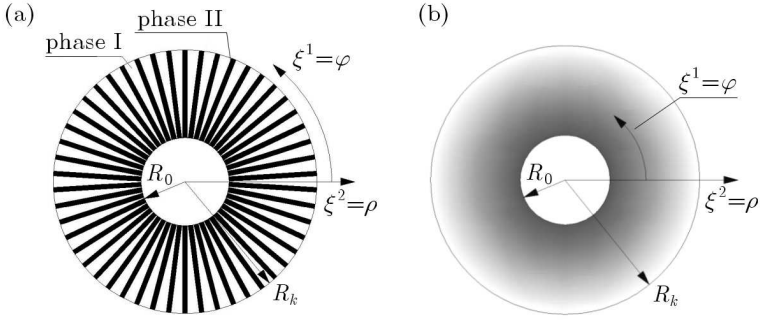


Fig. 1. Structure of the two-phase functionally graded composite in (a) micro- and (b) macro-scale

hollow cylinder made of functionally graded materials can be found in Hosseini *et al.* (2008), Ootao and Tanigawa (2006), Sladek *et al.* (2003), Wang and Mai (2005), where material properties are expressed as power or exponential functions of the radial coordinate. The hollow cylinder presented in Hosseini *et al.* (2008) has a heterogeneous microstructure and it is divided into many subcylinders (layers) across the thickness. In the paper by Aboudi *et al.* (1999), one can find applications of higher-order theory for thermal analysis in functionally graded materials.

The physical phenomenon of the heat transfer is described by the well known Fourier equation

$$c\dot{\Theta} - \nabla \cdot (\mathbf{K} \cdot \nabla \Theta) = 0 \quad (1.1)$$

which contains (in this case) highly oscillating and discontinuous coefficients;  $\mathbf{K}$  – heat conduction tensor, and  $c$  – specific heat. Therefore, different averaged models have been proposed. The modelling problem is how to describe a microheterogeneous conductor by certain averaged equations. The solution to the above problem for periodic structures based on homogenization technique for differential equations with highly oscillating coefficients has an extensive list. Here we can mention monographs by Jikov *et al.* (1994) and the paper by Lewinski and Kucharski (1992). Homogenization can be also realised using a concept of micro-local parameters, c.f. Matysiak (1991). However, because the formulation of averaged models by using the asymptotic homogenization is rather complicated from the computational point of view, these asymptotic methods are restricted to the first approximation. Hence, the averaged model obtained by using this method neglects the effect of the microstructure size on the heat transfer in a FGM-conductor. The formulation of the macroscopic mathematical model for the analysis of heat transfer in the conductor under

consideration will be based on the tolerance averaging technique, c.f. Woźniak *et al.* (2008), Woźniak and Wierzbicki (2000). The general description of this technique and application to analysis of longitudinally graded stratified media can be found in Michalak *et al.* (2007), Woźniak *et al.* (2008).

## 2. Model equations

The object of our considerations is a hollow conductor with microstructure given in Fig. 2. Let us introduce the orthogonal curvilinear coordinate system  $O\rho\varphi z$  in the physical space  $\Omega$  occupied by a conductor under consideration. The region  $\Omega$  occupied by the conductor is given by  $\Omega = \Pi \times I$ , where  $\Pi$  is a region in the  $O\rho\varphi$  plane. The time coordinate will be denoted by  $t$ . The microstructure is determined by the unit cell  $\Delta$  with the diameter of  $\lambda = 2\pi/N$ , where  $N$  is a number of cells in the considered composite. What is most important, the walls width  $g$  is constant along the radial axis, which implies smooth variation of macroscopic material properties in this direction. Volume fractions of homogeneous layers are denoted by  $\nu'(\rho) = \delta(\rho)/\lambda$  and  $\nu''(\rho) = g/\lambda\rho$ . Dimensionless function  $\nu = \sqrt{\nu'\nu''} \in [0, 0.5]$  is referred to as the distribution of heterogeneity.

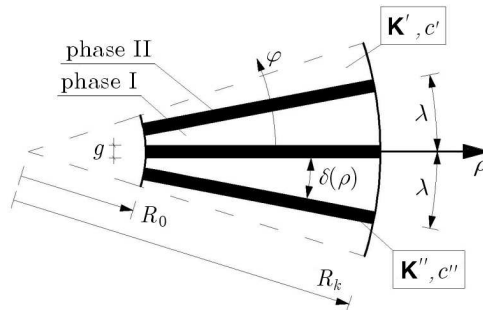


Fig. 2. Fragment of a cross-section of the hollow conductor

One of the fundamental assumptions in the tolerance averaging approach concerns the temperature field decomposition

$$\Theta(\rho, \varphi, t) = \theta(\rho, \varphi, t) + h(\rho, \varphi)\psi(\rho, \varphi, t) \quad (2.1)$$

where  $\rho \in [R_0, R_k]$ ,  $\varphi \in [0, 2\pi)$  and  $t \geq 0$  s. Functions of averaged temperature  $\theta$  and temperature fluctuation amplitude  $\psi$  are assumed to be slowly varying, i.e.  $\theta(\rho, \cdot, t), \psi(\rho, \cdot, t) \in SV_d^1(\Omega, \Delta)$ . The exact definition of the slowly

varying and tolerance periodic function can be found in Woźniak *et al.* (2008). The expected form of the temperature oscillations, caused by discontinuity of the coefficients in (2.1), is assured by the "saw-type" *locally periodic function*, which would be called the fluctuation shape function  $h$ .

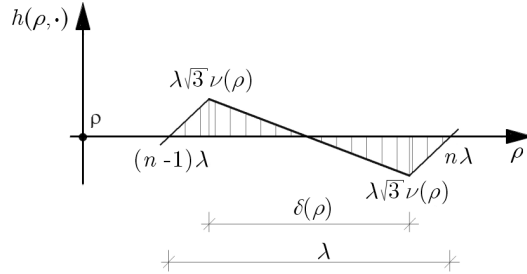


Fig. 3. Fluctuation shape function

The second concept of the modelling technique is the averaging operation

$$\langle f \rangle(\rho, \varphi) = \frac{1}{|\Delta|} \int_{\varphi-\lambda/2}^{\varphi+\lambda/2} f(\rho, z) dz \tag{2.2}$$

where  $|\Delta| = \lambda$ . On the grounds of this definition, we can formulate the second modelling assumption, the tolerance averaging approximation. In the course of modelling it is assumed that terms  $O(d)$  are negligibly small, where  $d$  is a certain tolerance parameter, c.f. Woźniak *et al.* (2008). For an arbitrary tolerance periodic function  $f \in TP_d^1(\Omega, \Delta)$ , slowly varying function  $F \in SV_d^1(\Omega, \Delta)$  and fluctuation shape function  $h \in FS_d^1(\Omega, \Delta)$ , we have

$$\begin{aligned} \langle fF \rangle &= \langle f \rangle F + O(d) \\ \langle f \nabla(hF) \rangle &= \langle f \partial h \rangle F + \langle fh \rangle \bar{\nabla} F + O(d) \end{aligned} \tag{2.3}$$

*Averaging description*

Bearing in mind the model assumptions, we derive from equation (1.1) the following system of averaged equations for the unknowns  $\theta(\rho, \varphi, t)$  and  $\psi(\rho, \varphi, t)$ , which can be found in Woźniak *et al.* (2008)

$$\begin{aligned} \nabla \cdot (\langle \mathbf{K} \rangle \nabla \theta + \langle \mathbf{K} \partial h \rangle \psi) - \langle c \rangle \dot{\theta} &= 0 \\ \bar{\nabla} \cdot (\langle \mathbf{K} h h \rangle \bar{\nabla} \psi) - \langle \mathbf{K} \partial h \rangle \nabla \theta - \langle \mathbf{K} \partial h \partial h \rangle \psi - \langle ch h \rangle \dot{\psi} &= 0 \end{aligned} \tag{2.4}$$

The above equations describe two-dimensional heat conduction in the two-phase hollow cylinder. The coefficients

$$\begin{aligned} \langle \mathbf{K} \rangle &= k' \nu' + k'' \nu'' & \langle c \rangle &= c' \nu' + c'' \nu'' \\ \langle \mathbf{K}hh \rangle &= \lambda^2 \nu^2 \langle \mathbf{K} \rangle & \langle chh \rangle &= \lambda^2 \nu^2 \langle c \rangle \\ \langle \mathbf{K}\partial h \rangle &= 2\sqrt{3}\nu(k' - k'') & \langle \mathbf{K}\partial h\partial h \rangle &= 12(k' \nu'' + k'' \nu') \end{aligned} \quad (2.5)$$

are continuous and functional. The gradient operators in the above equations have the form

$$\nabla = (\partial_1, \partial_2) \quad \bar{\nabla} = (\partial_1, 0) \quad \partial = (0, \partial_2) \quad (2.6)$$

where  $\partial_\alpha = \partial/\partial\xi_\alpha$  for  $\alpha = 1, 2$ .

The obtained averaged differential equations, (2.4), have smooth functional coefficients in contrast to coefficients in equation (1.1), hence in some special cases (stationary unidirectional conduction) analytical solution can be obtained. In other cases, numerical methods have to be used. Here we shall use the finite difference method (Cranck-Nicholson method for time integration) to derive solutions to boundary/initial value problems formulated in the framework of the proposed tolerance model. This model takes into account the effect of microstructure size on the overall heat transfer behaviour.

### 3. Examples of application

The main aim of this section is to present the effect of some parameters on the temperature field and relative velocity of achieving the steady state problem – denoted in figures with the subscript *st*. Hence, we consider in all three following examples the ratio of the temperature value in selected time  $t$  to the temperature value for a steady state problem. We restrict the analysis to the unidirectional heat transfer for a conductor with deterministic microstructure shown in Fig. 1. In general, we write the full anisotropic tensor of conductivity for each component

$$\mathbf{K} = k \begin{bmatrix} 1 & b \\ b & a \end{bmatrix} \quad (3.1)$$

where  $a \in (0, 1]$ ,  $b \in [0, \sqrt{a}]$ . Fixed values of conductivity and specific heat for both components are listed in Table 1.

Initial-boundary conditions would be given *a priori*. For the temperature field given by equation (2.1), two unknown functions  $\theta$  and  $\psi$  must be defined

**Table 1.** Material properties

	Conductivity $k$ [ $\text{Wm}^{-1}\text{K}^{-1}$ ]	Specific heat $c$ [ $\text{Jm}^{-3}\text{K}^{-1}$ ]
phase I	58	3432000
phase II	0.045	14600

on the boundary. Let the initial conditions for  $\theta$  and  $\psi$  be assumed in the form

$$\theta(\rho, 0) = \psi(\rho, 0) = 0^\circ\text{C} \quad (3.2)$$

and the boundary conditions for every time  $t \geq 0$  s

$$\begin{aligned} \theta(R_0, t) &= 100^\circ\text{C} \\ \theta(R_k, t) &= \psi(R_0, t) = \psi(R_k, t) = 0^\circ\text{C} \end{aligned} \quad (3.3)$$

All the above conditions and formulations will be used in the subsequent part of this paper.

### 3.1. Benchmark solutions

**Case 1.** For verification of the postulated value of the step time parameter in the Cranck-Nicholson method for time integration, we compare three independent methods, i.e. finite difference method (FDM) for the tolerance model, finite element method (FEM) for the microheterogeneous conductor and analytical solution (AS) of the tolerance model equations. Let us consider a uniform hollow cylinder with conductivity  $K = 58 \text{ Wm}^{-1}\text{K}^{-1}$  and specific heat  $c = 3432000 \text{ Jm}^{-3}\text{K}^{-1}$ . Geometry as shown in Fig. 1 for  $R_0 = 1$  m,  $R_k = 3$  m. Analytical solution is expressed by

$$\begin{aligned} \theta(\rho, t) &= \theta_0 + (\theta_k - \theta_0) \frac{\ln \rho - \ln R_0}{\ln R_k - \ln R_0} + \\ &+ \pi \sum_{n=1}^{\infty} \frac{C_0(\rho, \alpha_n)}{F(\alpha_n)} J_0(R_k \alpha_n) [\theta_k J_0(R_0 \alpha_n) - \theta_0 J_0(R_k \alpha_n)] e^{-\kappa \alpha_n^2 t} \end{aligned} \quad (3.4)$$

where  $\theta_0 = \theta(R_0, t)$ ,  $\theta_k = \theta(R_k, t)$  and

$$\begin{aligned} C_0(\rho, \alpha_n) &= J_0(R_0 \alpha_n) Y_0(\rho \alpha_n) - J_0(\rho \alpha_n) Y_0(R_0 \alpha_n) \\ F(\alpha_n) &= J_0^2(R_0 \alpha_n) - J_0^2(R_k \alpha_n) \end{aligned} \quad (3.5)$$

for  $\alpha_n, n = 1, 2, \dots$  as roots of the equation

$$J_0(R_0\alpha)Y_0(R_k\alpha) - J_0(R_k\alpha)Y_0(R_0\alpha) = 0 \tag{3.6}$$

where functions  $J_0, Y_0$  are well known Bessel functions. Comparison of obtained results is made for  $t = 3600$  s and  $t = 7200$  s. The amplitude of temperature fluctuation in this case equals zero. All diagrams for every method and at every time  $t$  are overlapped.

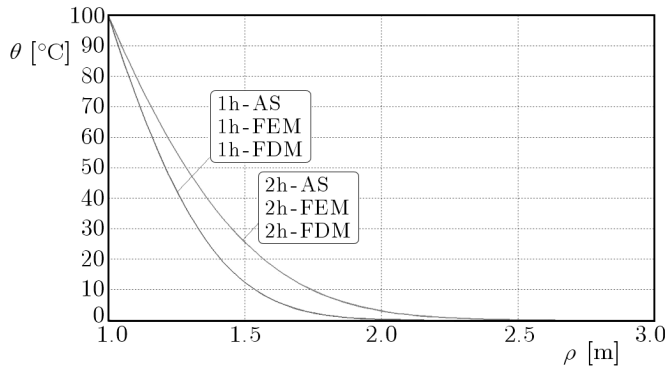


Fig. 4. Comparison of averaged temperature for AS – analytical solution, FEM – finite element method and FDM – finite difference method

**Case 2.** Additionally, a comparison of the tolerance model with the asymptotic one, which does not include the effect of microstructure size will be shown. The governing equations of the asymptotic model are expressed by

$$\nabla \cdot \left[ \left( \langle \mathbf{K} \rangle - \frac{\langle \mathbf{K} \partial h \rangle^2}{\langle \mathbf{K} \partial h \partial h \rangle} \right) \nabla \theta \right] - \langle c \rangle \dot{\theta} = 0 \tag{3.7}$$

and the temperature fluctuation amplitude is given by the equation

$$\psi = - \frac{\langle \mathbf{K} \partial h \rangle}{\langle \mathbf{K} \partial h \partial h \rangle} \nabla \theta \tag{3.8}$$

The above formulas can be found in Woźniak *et al.* (2008). Let us consider the two-phase hollow cylinder (Fig. 1) for  $R_0 = 1$  m,  $R_k = 3$  m and material properties as in Table 1. The number of cells is fixed at  $N = 60$  and the width of the walls  $g = 0.5\lambda R_0$ . Calculations were made for  $a = 1$  and  $b = 0.25$  in (3.1). Initial-boundary conditions are given by (3.2) and (3.3). Let us notice that for the asymptotic model there is no need to impose conditions on the temperature fluctuation amplitude  $\psi$ . The obtained results for both models are covered. However, since for the asymptotic model function of  $\psi$  is expressed by (3.8), the differences between two models occur but only nearby inner boundary.

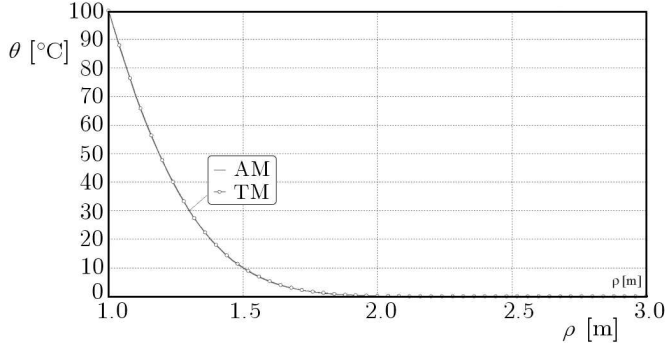


Fig. 5. Diagram of averaged temperature after  $t = 1$  h for tolerance (TM) and asymptotic (AM) model

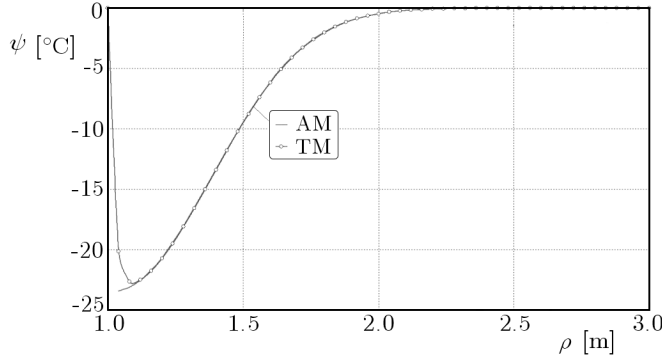


Fig. 6. Diagram of on temperature fluctuation after  $t = 1$  h for tolerance (TM) and asymptotic (AM) model

**3.2. Effect of the walls width on the temperature field**

Let us consider a composite with geometry as in Fig.1 with  $R_0 = 1$  m,  $R_k = 3$  m. Initial-boundary conditions as in (3.2) and (3.3), and material properties are as in Table 1 for  $a = 1$  and  $b = 0$  in (3.1). We denote the width of the wall by

$$g(\eta) = \frac{2\pi R_0}{N} \eta \tag{3.9}$$

where  $N$  stands for the number of cells. In this case  $N = 60$ . Diagrams of the ratio of the averaged temperature value in a selected time  $t$  to the averaged temperature value for a steady state problem are shown in Fig.7. Similar diagrams for the temperature fluctuation amplitude are shown in Fig.8. The walls width ratio  $\eta$  is taken as a parameter. We consider only two values of the parameter  $\eta$ , i.e.  $\eta = 0.25$  and  $\eta = 0.75$ .



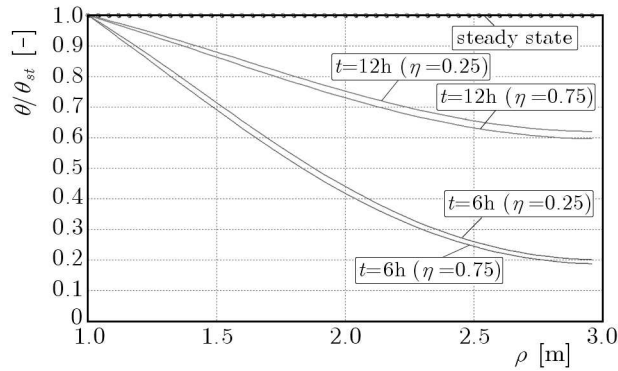


Fig. 7. Diagram of change in time of the averaged temperature;  $\eta$  is taken as a parameter

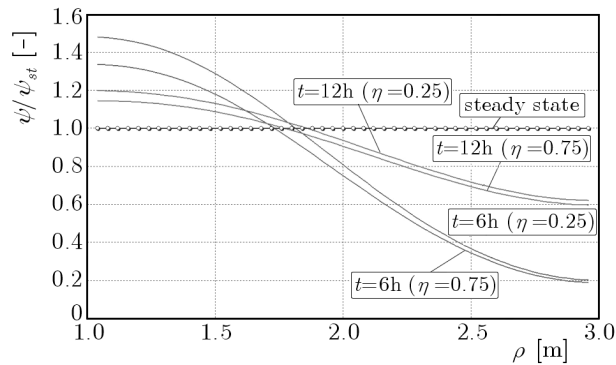


Fig. 8. Diagram of change in time of the temperature fluctuation amplitude;  $\eta$  is taken as a parameter

### 3.3. Effect of material properties on the temperature field

Let us consider a composite with geometry as in Fig. 1 with  $R_0 = 1$  m,  $R_k = 3$  m. Initial-boundary conditions as in (3.2) and (3.3). The number of cells  $N = 60$ . In this example, we consider two different values of the parameter  $a$  in (3.1), i.e.  $a = 0.75$  and  $a = 1$ , by a fixed value of  $b = 0.5$ . Diagrams of the ratio of the averaged temperature value in a selected time  $t$  to the averaged temperature value for a steady state problem are shown in Fig. 9. Similar diagrams for the temperature fluctuation amplitude are shown in Fig. 10. It can be observed in Figs. 9 and 10 that for materials with stronger anisotropic conductivity the temperature fields achieve the steady state slower.

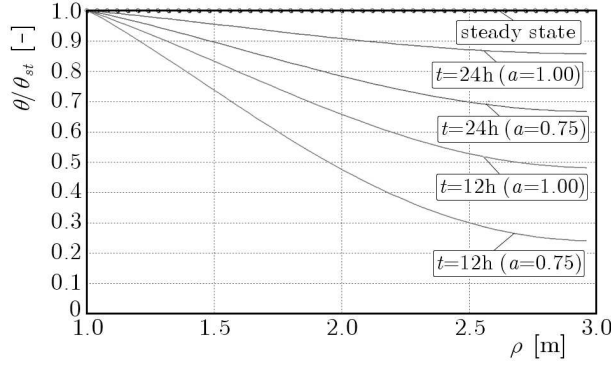


Fig. 9. Diagram of change in time of the averaged temperature;  $a$  is taken as a conductivity parameter

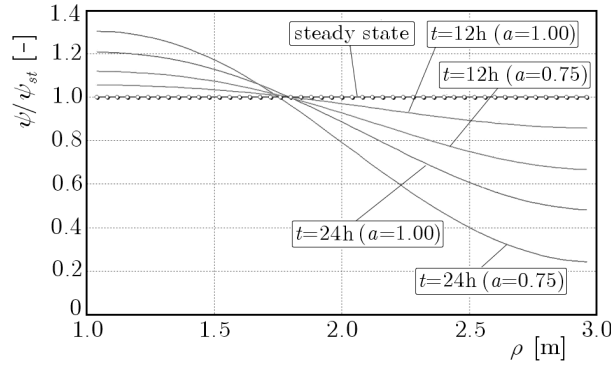


Fig. 10. Diagram of change in time of the temperature fluctuation amplitude;  $a$  is taken as a conductivity parameter

### 3.4. Effect of inner radius size on the temperature field

Let us consider a composite with geometry as in Fig.1 with a constant width of the hollow cylinder  $R_k - R_0 = 1$  m. Initial-boundary conditions as in (3.2) and (3.3), and material properties are as in Table 1 for  $a = 1$  and  $b = 0$  in (3.1). We demand also, by various radius  $R_0$ , constant effective material properties on the inner boundary. That is why the number of cells  $N$  must be a function of  $R_0$

$$N(R_0) = \frac{\pi}{g} R_0 \tag{3.10}$$

where we assumed  $g = \pi/20$  m. We consider both cases where  $R_0 = 5$  m and  $R_0 = 10$  m. From the above figures it can be observed that the inner

radius does not influence the rate of achieving the steady state for averaged temperature.

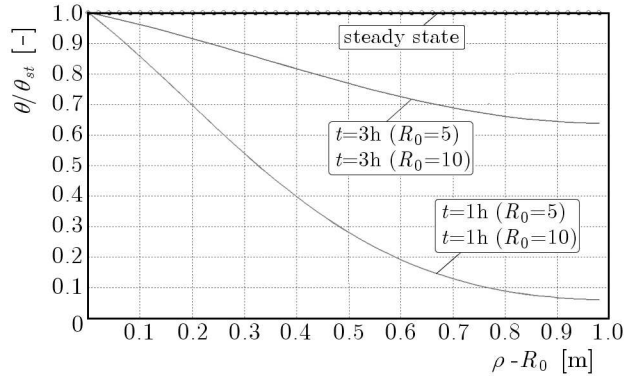


Fig. 11. Diagram of change in time of the averaged temperature; inner radius is taken as a parameter

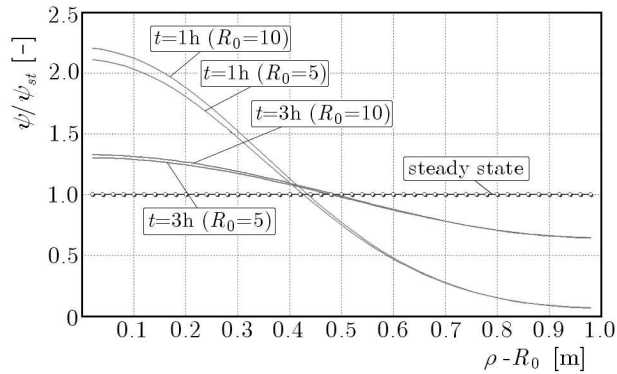


Fig. 12. Diagram of change in time of the temperature fluctuation amplitude; inner radius is taken as a parameter

#### 4. Conclusions

The tolerance averaging approximation leads to the mathematical model of composite conductors with functionally graded material properties. The obtained model equations have continuous coefficients in opposition to discrete models, where they are strongly oscillating. Since the proposed model equations have smooth functional coefficients then, in most cases, solutions to the

specific problem, for the heat conductor under consideration, have to be obtained using well known numerical methods. The tolerance model takes into account the effect of microstructure size on the temperature field, particularly on the temperature oscillation amplitude. Moreover, by changing volume fractions or material properties of every component, we can obtain the desirable temperature field inside composite. For different geometry and material properties, the temperature fields for the conductor under consideration have a slow relative velocity of achieving the steady state:

- For materials with anisotropic conductivity, the temperature fields achieve steady state slower than for materials with isotropic conductivity
- The inner radius does not influence the rate of achieving the steady state for averaged temperature but has a low influence on the rate for temperature fluctuation amplitude.

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### Niestacjonarny przepływ ciepła w wydrążonym cylindrze wykonanym z materiału o funkcyjnej gradacji własności

#### Streszczenie

W pracy rozważano niestacjonarne przewodzenia ciepła w dwu-składnikowym wydrążonym cylindrze. Przewodnik jest wykonany z dwuskładnikowego warstwowego kompozytu mającego łagodną zmienność efektywnych własności w kierunku promieniowym. Stąd mamy tutaj do czynienia ze specjalnym przypadkiem materiału o funkcyjnej gradacji własności (ang. *functionally graded materiale*, FGM). Zbudowanie uśrednionego modelu matematycznego rozpatrywanego przewodnika jest oparte na technice tolerancyjnej aproksymacji. W pracy pokazano zastosowanie otrzymanego modelu tolerancyjnego i porównanie wyników z wynikami dla modelu asymptotycznego w przypadku niestacjonarnego przewodzenia ciepła. Zbadano wpływ zmienności geometrii i własności materiałowych przewodnika na pole tempertatury.

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