

## ENERGY BALANCE OF TWO SYNCHRONIZED SELF-EXCITED PENDULUMS WITH DIFFERENT MASSES

TOMASZ KAPITANIAK, KRZYSZTOF CZOLCZYŃSKI, PRZEMYSŁAW PERLIKOWSKI,  
ANDRZEJ STEFAŃSKI

*Technical University of Lodz, Division of Dynamics, Łódź, Poland*

*e-mail: tomasz.kapitaniak@p.lodz.pl; krzysztof.czolczyński@p.lodz.pl; przemyslaw.perlikowski@p.lodz.pl;*

*andrzej.stefanski@p.lodz.pl*

We consider the synchronization of two self-excited pendulums with different masses. We show that such pendulums hanging on the same beam can show almost-complete (in-phase) and almost-antiphase synchronizations in which the difference of the pendulums displacements is small. Our approximate analytical analysis allows one to derive the synchronization conditions and explains the observed types of synchronizations as well as gives an approximate formula for amplitudes of both the pendulums and the phase shift between them. We consider the energy balance in the system and show how the energy is transferred between the pendulums via the oscillating beam allowing synchronization of the pendulums.

*Key words:* coupled oscillators, pendulum, synchronization

### 1. Introduction

Currently, we observe growing interest in the studies of coupled oscillatory systems which are stimulated by various applications in physics, engineering, biology, medicine, *etc.* (Andronov *et al.*, 1966; Blekhman, 1988; Pikovsky *et al.*, 2001). Synchronization is commonly observed to occur between oscillators. It is a process where two or more systems interact with each other and come to oscillate together. Groups of oscillators are observed to synchronize in a diverse variety of systems, despite inevitable differences between the oscillators. The phenomenon of synchronization of clocks hanging on a common movable beam (Kapitaniak *et al.*, 2019) has been recently the subject of research by a number of authors (Bennet *et al.*, 2002; Czolczynski *et al.*, 2009a,b, 2011; Dilao, 2009; Fradkov and Andrievsky, 2007; Huygens, 1893; Kanunnikov *et al.*, 2003; Kumon *et al.*, 2002; Pantaleone, 2002; Perlikowski *et al.*, 2012; Senator, 2006; Ulrichs *et al.*, 2009). These studies give the definite answer to the question; what Huygens was able to observe, e.g., Bennet *et al.* (2002) state that to repeat Huygens' results, high precision (the precision that Huygens certainly could not achieve) is necessary, and Kanunnikov *et al.* (2003) show that the precise antiphase motion of different pendulums noted by Huygens cannot occur. Our studies (Czolczynski *et al.*, 2009a,b, 2011; Dilao, 2009) prove that in the case of nonidentical clocks, only almost-antiphase synchronization can be observed.

In this paper, we consider the synchronization of two self-excited pendulums which have the the same length but different masses. Oscillations of each pendulum are self-excited by van der Pol's type of damping. We show that two such pendulums hanging on the same beam, besides the complete (in-phase) and antiphase synchronizations already demonstrated for the case of pendulums with the same masses in Blekhman (1988), Czolczynski *et al.* (2009b), Fradkov and Andrievsky (2007), Perlikowski *et al.* (2012), Ulrichs *et al.* (2009), perform almost-complete and almost-antiphase synchronization in which the phase differences of oscillations are respectively close (but not equal) to 0 or  $\pi$ . We perform an approximate analytical analysis which allows one

to derive the synchronization conditions and explains the observed types of synchronizations. The energy balance in the system allows one to show how the energy is transferred between the pendulums via the oscillating beam.

This paper is organized as follows. Section 2 describes the considered model of the coupled pendulums. In Section 3 we derive the energy balance of the synchronized pendulums. Section 4 presents the results of numerical simulations and describes the observed synchronization states. Finally, we summarize our results in Section 5.

## 2. Model

The analyzed system is shown in Fig. 1. It consists of a rigid beam and two pendulums suspended on it. The beam of mass  $M$  can move in the horizontal direction, its movement is described by the coordinate  $x$ . The mass of the beam is connected to the refuge of a linear spring and linear damper  $k_x$  and  $c_x$ . Each pendulum consists of a light beam of length  $l$  and a mass mounted at its end. We consider the pendulums with the same length  $l$  but different masses  $m_1$  and  $m_2$ . The motion of the pendulums is described by angles  $\varphi_1$  and  $\varphi_2$  and is self-excited by van der Pol's type of damping (not shown in Fig. 1) given by momentum (torgue)  $c_{\varphi vdp}\dot{\varphi}l(1 - \zeta_l^2)$ , where  $c_{\varphi vdp}$  and  $\zeta$  are constant.

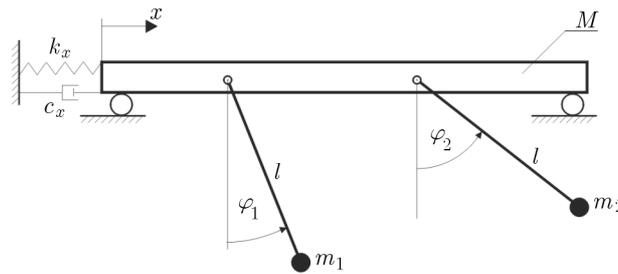


Fig. 1. The model of the system – two self-excited pendulums mounted to the beam which can move horizontally

The object of studies, whose results are presented in this paper differ from the earlier ones (Czolczynski *et al.*, 2009a,b, 2011; Kapitaniak *et al.*, 2012; Perlikowski *et al.*, 2012) – as instead of the clocks with pendulums driven by a discontinuous escapement mechanism, we consider two self-excited pendulums with van der Pol's type of damping. The mathematical description of these pendulums contains the self-excited component  $c_{\varphi vdp}\dot{\varphi}$  and energy-dissipating component  $-c_{\varphi vdp}\zeta\dot{\varphi}\varphi^2$ . The balance of these components results in creation of a stable limit cycle (Andronov *et al.*, 1966).

The equations of motion of the considered system are as follows

$$\begin{aligned} m_1 l^2 \ddot{\varphi}_1 + m_1 \ddot{x} l \cos \varphi_1 + c_{\varphi vdp} \dot{\varphi}_1 (1 - \zeta \varphi_1^2) + m_1 g l \sin \varphi_1 &= 0 \\ m_2 l^2 \ddot{\varphi}_2 + m_2 \ddot{x} l \cos \varphi_2 + c_{\varphi vdp} \dot{\varphi}_2 (1 - \zeta \varphi_2^2) + m_2 g l \sin \varphi_2 &= 0 \end{aligned} \quad (2.1)$$

and

$$\left( M + \sum_{i=1}^2 m_i \right) \ddot{x} + c_x \dot{x} + k_x x + \sum_{i=1}^2 m_i l (\ddot{\varphi}_i \cos \varphi_i - \dot{\varphi}_i^2 \sin \varphi_i) = 0 \quad (2.2)$$

Equations (2.1) and (2.2), contrary to the equations considered in Czolczynski *et al.* (2009a,b, 2011), Kapitaniak *et al.* (2012), Perlikowski *et al.* (2012), are continuous.

### 3. Energy balance of the system

Multiplying both sides of Eq. (2.1) by the angular velocity  $\dot{\varphi}_i$ , one gets

$$m_i l^2 \ddot{\varphi}_i \dot{\varphi}_i + m_i g l \dot{\varphi}_i \sin \varphi_i = -c_{\varphi vdp} \dot{\varphi}_i^2 + c_{\varphi vdp} \zeta \dot{\varphi}_i^2 \varphi_i^2 - m_i \ddot{x} l \cos \varphi_i \dot{\varphi}_i \quad i = 1, 2 \quad (3.1)$$

In the case of periodic oscillations with period  $T$ , integration of Eq. (2.2) gives the following energy balance

$$\begin{aligned} \int_0^T m_i l^2 \ddot{\varphi}_i \dot{\varphi}_i dt + \int_0^T m_i g l \dot{\varphi}_i \sin \varphi_i dt &= - \int_0^T c_{\varphi vdp} \dot{\varphi}_i^2 dt + \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_i^2 \varphi_i^2 dt \\ &- \int_0^T m_i \ddot{x} l \cos \varphi_i \dot{\varphi}_i dt \quad i = 1, 2 \end{aligned} \quad (3.2)$$

The left hand side of Eq. (3.2) represents the increase of the total energy of  $i$ -th pendulum, which in the case of periodic oscillations is equal to zero

$$\int_0^T m_i l^2 \ddot{\varphi}_i \dot{\varphi}_i dt + \int_0^T m_i g l \dot{\varphi}_i \sin \varphi_i dt = 0 \quad i = 1, 2 \quad (3.3)$$

The energy supplied to the system by van der Pol's damper in one period of oscillations is given by

$$W_i^{SELF} = - \int_0^T c_{\varphi vdp} \dot{\varphi}_i^2 dt \quad i = 1, 2 \quad (3.4)$$

The next component on the right hand side of Eq. (3.2) represents the energy dissipated by the van der Pol damper

$$W_i^{VDP} = - \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_i^2 \varphi_i^2 dt \quad i = 1, 2 \quad (3.5)$$

The last component of Eq. (3.2) represents the energy transfer from the pendulum to the beam or to the second pendulum (via the beam)

$$W_i^{SYN} = \int_0^T m_i \ddot{x} l \cos \varphi_i \dot{\varphi}_i dt \quad i = 1, 2 \quad (3.6)$$

Substituting Eqs. (3.3)-(3.6) into Eq. (3.2), one gets energy balances of the pendulums in the form

$$\begin{aligned} W_1^{SELF} - W_1^{VDP} - W_1^{SYN} &= 0 \\ W_2^{SELF} - W_2^{VDP} - W_2^{SYN} &= 0 \end{aligned} \quad (3.7)$$

Multiplying equation of motion (2.2) by the beam velocity  $\dot{x}$ , one gets

$$\left( M + \sum_{i=1}^2 m_i \right) \ddot{x} \dot{x} + c_x \dot{x}^2 + k_x x \dot{x} + \left( \sum_{i=1}^2 m_i l (\ddot{\varphi}_i \cos \varphi_i - \dot{\varphi}_i^2 \sin \varphi_i) \right) \dot{x} = 0 \quad (3.8)$$

Integrating Eq. (3.8) over the period of oscillations, we obtain the following energy balance

$$\int_0^T \left( M + \sum_{i=1}^2 m_i \right) \ddot{x} \dot{x} dt + \int_0^T k_x x \dot{x} dt = - \int_0^T \left( \sum_{i=1}^2 m_i l (\ddot{\varphi}_i \cos \varphi_i - \dot{\varphi}_i^2 \sin \varphi_i) \right) \dot{x} dt - \int_0^T c_x \dot{x}^2 dt \quad (3.9)$$

The left hand side of Eq. (3.9) represents the increase of the total energy of the beam, which for the periodic oscillations is equal to zero

$$\int_0^T \left( M + \sum_{i=1}^2 m_i \right) \ddot{x} \dot{x} dt + \int_0^T k_x x \dot{x} dt = 0 \quad (3.10)$$

The first component on the right-hand side of Eq.(3.9) represents the work performed by the horizontal component of the force with which the pendulums act on the beam causing its motion

$$W_{beam}^{DRIVE} = - \int_0^T \left( \sum_{i=1}^2 m_i l (\ddot{\varphi}_i \cos \varphi_i - \dot{\varphi}_i^2 \sin \varphi_i) \right) \dot{x} dt \quad (3.11)$$

The second component on the right hand side of Eq.(3.9) represents the energy dissipated by the damper  $c_x$

$$W_{beam}^{DAMP} = \int_0^T c_x \dot{x}^2 dt \quad (3.12)$$

Substituting Eqs. (3.10)-(3.12) into Eq. (3.9), one gets the energy balance in the following form

$$W_{beam}^{DRIVE} - W_{beam}^{DAMP} = 0 \quad (3.13)$$

In the case of periodic oscillations, it is possible to prove that

$$W_1^{SYN} + W_2^{SYN} = W_{beam}^{DRIVE} = W_{beam}^{DAMP} \quad (3.14)$$

so adding Eqs. (3.7) and (3.13) and considering Eq.(3.14) one obtains

$$W_1^{DRIVE} + W_2^{DRIVE} - W_1^{DAMP} - W_2^{DAMP} - W_{beam}^{DAMP} = 0 \quad (3.15)$$

Equation (3.15) represents the energy balance of the whole system (1,2).

#### 4. Numerical results

We perform a series of numerical simulations in which Eqs. (2.1) and (2.2) have been integrated using the Runge-Kutta method. The primary objective of these simulations is to investigate the influence of nonidentity of the pendulums on the observed types of synchronization.

In our numerical studies, we consider the following parameters: mass of pendulum 1  $m_1 = 1.0$  kg; pendulums length  $l = g/4\pi^2 = 0.2485$  m ( $g = 9.81$  m/s<sup>2</sup>) (chosen so that their period of free oscillations in the case of unmovable beam is  $T = 1.0$  s and the frequency of free oscillations  $\alpha = 2\pi$  s<sup>-1</sup>), negative damping coefficient causing self-excited oscillations  $c_{\varphi vdp} = -0.01$  Nms; van der Pol coefficient  $\zeta = 60.0$ ; beam mass  $M = 10.0$  kg, beam damping coefficient  $c_x = 1.53$  Ns/m, beam stiffness coefficient  $k_x = 4.0$  N/m. We assume the mass of the second pendulum  $m_2$  as a control parameter.

Note that because the coefficients of self-oscillations  $c_{\varphi vdp}$  and damping  $\zeta$  of the two pendulums are the same, in the case of an unmovable beam both pendulums have the same amplitude  $\Phi = 0.26$  ( $\approx 15^\circ$ ), regardless of their masses. The motion of the beam may change both the period and amplitude of oscillations.

#### 4.1. From complete to almost-antiphase synchronization

The evolution of system (1,2) behavior starting from the complete synchronization of identical pendulums ( $m_1 = m_2 = 1.0$  kg) and increasing the value of control parameter  $m_2$  is illustrated in Figs. 2a-2f. Figure 2a presents the bifurcation diagram for the increasing values of  $m_2$  ( $m_2 \in [1.0, 6.0]$ ). On the vertical axis, we show the maximum displacement  $\varphi_1$  of pendulum 1, and the displacements of pendulum 2 –  $\varphi_2$  as well as of the beam  $x$  recorded at moments when  $\varphi_1$  is maximum. Creating this diagram, we start with the state of complete synchronization of the pendulums with masses  $m_1 = m_2 = 1.0$  kg, during which they are moving in the same way ( $\varphi_1 = \varphi_2$ ) in antiphase to the movement of the beam.

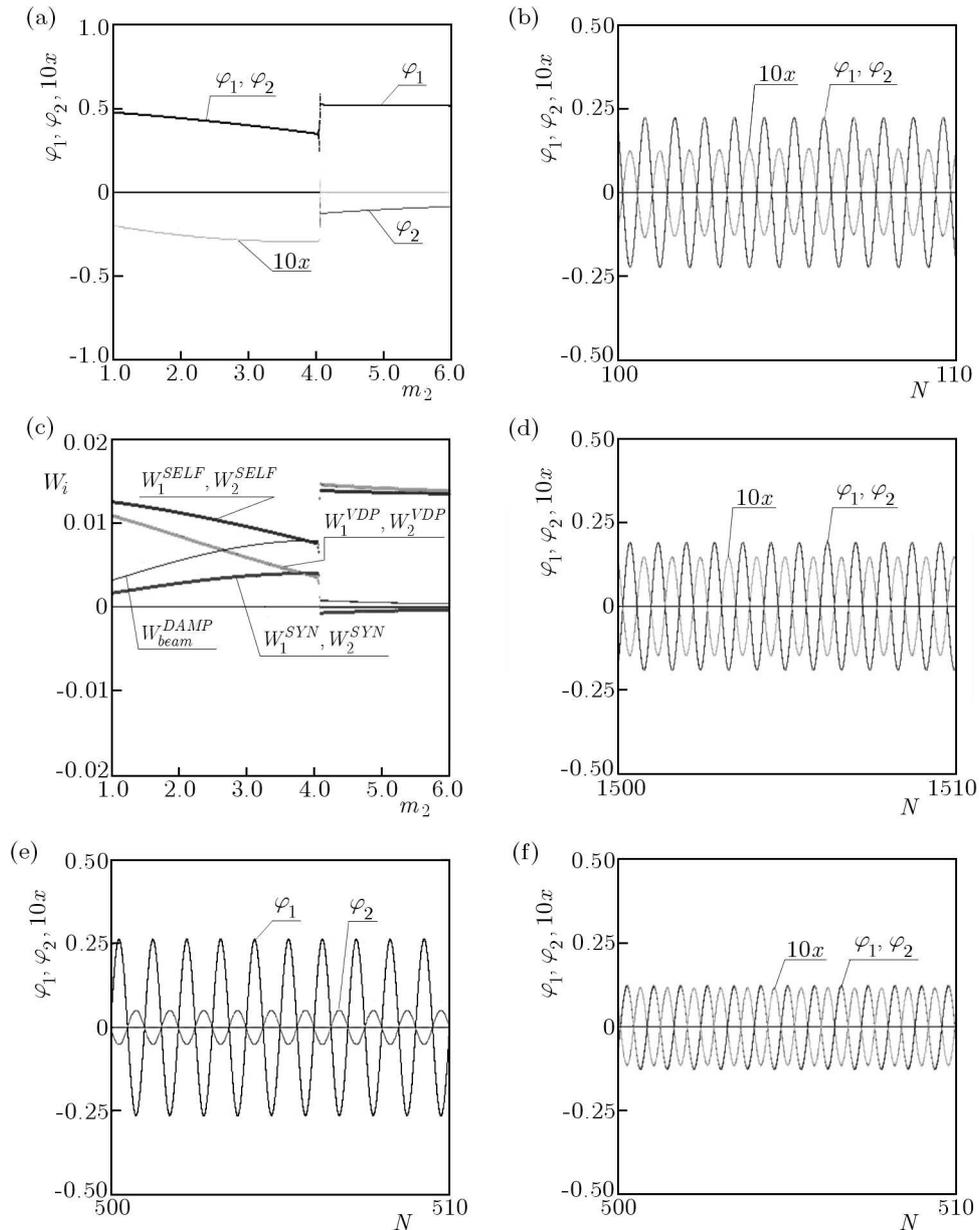


Fig. 2. Evolution from the complete to almost-antiphase synchronization; (a) bifurcation diagram for increasing values of  $m_2$ , (b) time histories of almost-complete synchronization  $m_1 = 1.0$  kg and  $m_2 = 2.0$  kg; (c) plots of system energy; (d) time histories of almost-complete synchronization for  $m_1 = 1.0$  kg and  $m_2 = 3.5$  kg; (e) time histories of almost-antiphase synchronization:  $m_1 = 1.0$  kg,  $m_2 = 5.0$  kg; (f) nonstationary complete synchronization:  $m_1 = m_2 = 3.0$  kg

Increasing the value of  $m_2$ , we observe that initially both pendulums are in the state of almost-complete synchronization. Figure 2b found for  $m_1 = 1.0$  kg,  $m_2 = 2.0$  kg shows the displacements  $\varphi_1 \approx \varphi_2$  and the displacement of the beam  $x$  (for better visibility enlarged 10 times) as a function of time (on the horizontal axis, the time is expressed as the number of periods of free oscillations of pendulums suspended on an unmovable beam –  $N$ ). Notice that the differences  $\varphi_1 - \varphi_2$  are hardly visible.

Further increase of the mass  $m_2$  causes an increase of the amplitude of pendulums oscillations and an increase of the amplitude of beam oscillations as can be seen in Fig. 2d ( $m_2 = 3.5$  kg). One also observes an increase of the period of pendulum oscillations (Fig. 2b presents 11.25 periods of oscillations while Fig. 2d – 12 periods in the same time). This is due to the fact that with the increasing mass of pendulum 2, the center of mass moves towards the ends of the pendulums, i.e., towards the material points with masses  $m_1$  and  $m_2$ , and moves away from the beam with the constant mass.

Noteworthy is the fact that in the state of complete synchronization, when the displacements of both pendulums fulfill the relation  $\varphi_1(t) = \varphi_2(t)$ , the energy transmitted to the beam by each pendulum is proportional to its mass. Therefore, these energies satisfy the following equations

$$\begin{aligned} W_1^{SELF} &= \int_0^T c_{\varphi vdp} \dot{\varphi}_1^2 dt = \int_0^T c_{\varphi vdp} \dot{\varphi}_2^2 dt = W_2^{SELF} \\ W_1^{VDP} &= \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_1^2 \varphi_1^2 dt = \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_2^2 \varphi_2^2 dt = W_2^{VDP} \\ W_1^{SYN} &= \int_0^T m_1 \ddot{x} l \cos \varphi_1 \dot{\varphi}_1 dt = \frac{m_1}{m_2} \int_0^T m_2 \ddot{x} l \cos \varphi_2 \dot{\varphi}_2 dt = \frac{m_1}{m_2} W_2^{SYN} \end{aligned} \quad (4.1)$$

After substituting Eqs. (4.1) into Eqs. (3.7), Eqs. (3.7) become contradictory (except for special non-robust case of two identical pendulums when  $m_1 = m_2$ ). In the general case when  $m_1 \neq m_2$ , instead of the complete synchronization, an almost-complete synchronization occurs during which the displacements and velocities of the pendulums are almost-equal, and appropriate energies satisfy the following equations

$$\begin{aligned} W_1^{DAMP} &= \int_0^T c_{\varphi} \dot{\varphi}_1^2 dt \approx \int_0^T c_{\varphi} \dot{\varphi}_2^2 dt = W_2^{DAMP} \\ W_1^{SYN} &= \int_0^T m_1 \ddot{x} l \cos \varphi_1 \dot{\varphi}_1 dt \approx \int_0^T m_2 \ddot{x} l \cos \varphi_2 \dot{\varphi}_2 dt = W_2^{SYN} \\ W_1^{SELF} &= \int_0^T c_{\varphi vdp} \dot{\varphi}_1^2 dt \approx \int_0^T c_{\varphi vdp} \dot{\varphi}_2^2 dt = W_2^{SELF} \\ W_1^{VDP} &= \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_1^2 \varphi_1^2 dt \approx \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_2^2 \varphi_2^2 dt = W_2^{VDP} \\ W_1^{SYN} &= \int_0^T m_1 \ddot{x} l \cos \varphi_1 \dot{\varphi}_1 dt \approx \int_0^T m_2 \ddot{x} l \cos \varphi_2 \dot{\varphi}_2 dt = W_2^{SYN} \end{aligned} \quad (4.2)$$

After substitution of Eqs. (4.2), the energy equations (3.7) are satisfied for pendulums of different masses. Figure 2c shows the values of all energies as a function of the mass  $m_2$ . As one can see,

for  $m_2 < 4.0$  kg all energies are positive. This means that both pendulums transfer a part of their energy to the beam, causing its motion (see Eq. (3.14)).

For  $m_2 = 4.0$  kg, the system undergoes bifurcation, an attractor of an almost-complete synchronized state loses its stability and we observe the jump to the co-existing attractor of almost-antiphase synchronization as shown in Fig. 2e ( $m_2 = 5.0$  kg). The amplitudes of oscillations are different but the phase shift between the pendulums is close to  $\pi$ . The oscillations of the beam are so small that they are not visible in the scale of Fig. 2e.

One can show that when one changes the mass of pendulum 1 to  $m_1 = 2.0$  kg,  $m_1 = 3.0$  kg,  $m_1 = 4.0$  kg, the bifurcation from almost-complete to almost-antiphase synchronization occurs respectively for  $m_2 = 3.0$  kg,  $m_2 = 2.0$  kg and  $m_2 = 1.0$  kg. This bifurcation occurs when the total mass of both pendulums reaches the critical value  $m_{cr} = 5.0$  kg, which depends on the system parameters, particularly on the beam ones  $M$ ,  $c_x$  and  $k_x$ .

Figure 2f shows the time histories of beam vibrations and oscillations of two pendulums in the case of identical masses  $m_1 = m_2 = 3.0$  kg in the state of complete synchronization. These results have been obtained for identical initial conditions, so that they constitute *de facto* the pendulum of mass  $m = 6.0 > m_{cr}$ . It is easy to see that this synchronized state is unstable: small disturbances lead to a stable coexisting attractor of antiphase synchronization. Notice that for the pendulums with slightly different masses (e.g.,  $m_1 = 2.99$  kg and  $m_2 = 3.01$  kg) it is impossible to obtain a result similar to that shown in Fig. 2f, even for the identical initial conditions. Different pendulum masses cause that initially an almost-complete synchronization is observed, but small differences in  $\varphi_1$  and  $\varphi_2$  lead to the stable almost-antiphase synchronization.

To summarize the bifurcation diagram in Fig. 2a, the existence of three different types of synchronization can be distinguished; (i) complete for  $m_1 = m_2 = 1.0$  kg, (ii) almost-complete for  $1.0 \text{ kg} < m_2 < 4.0 \text{ kg}$ , (iii) almost-antiphase for  $m_2 > 4.0$  kg.

#### 4.2. From complete synchronization to quasiperiodic oscillations

Evolution of the behavior of system (1,2), starting from the complete synchronization of identical pendulums ( $m_1 = m_2 = 1.0$  kg) and decreasing the values of the control parameter  $m_2$ , is illustrated in Figs. 3a-3d. Figure 3a shows the bifurcation diagram for decreasing values of mass  $m_2$  ( $m_2 \in [0.01, 1.00]$ ). In the interval  $1.0 \text{ kg} > m_2 > 0.0975$  kg, both pendulums are in the state of almost-complete synchronization. Their oscillations are “almost-identical” as can be seen in Fig. 3b for  $m_1 = 1.0$  kg and  $m_2 = 0.01$  kg – the differences between the amplitudes and phases of  $\varphi_1$  and  $\varphi_2$  are close to zero, both pendulums remain in (almost) antiphase to the oscillations of the beam.

Figure 3c shows values of different energies. Like in the interval  $1.0 \text{ kg} < m_2 < 4.0$  kg of Fig. 2c, all energies are positive and both pendulums drive the beam. Further reduction of the mass  $m_2$  leads to the loss of synchronization, and motion of the system becomes quasiperiodic. Figure 3d presents the Poincaré map (the displacements and velocities of the pendulums have been taken at the moments of greatest positive displacement of the first pendulum) for  $m_2 = 0.07$  kg. The mechanism of the loss of stability is explained in Fig. 3c. In the interval  $0.35 \text{ kg} > m_2 > 0.07$  kg, the energy dissipated by the first pendulum  $W_1^{VDP}$  approaches the level of the energy supplied by the self-excited component of this pendulum  $W_1^{SELF}$ . Consequently, the energy supplied by the first pendulum to the beam  $W_1^{SYN}$  decreases. The energy supplied to the system by the second pendulum also decreases  $W_2^{SELF}$ , which drives the pendulum from the beam. For  $m_2 < 0.07$  kg, the energy balance is disrupted: pendulum 2 has not enough energy to cause its oscillations, the oscillations of the beam additionally support the oscillations of pendulum 1. In this case, the almost-antiphase synchronization is not possible (see Section 3.4), and system (1,2) exhibits quasiperiodic oscillations.

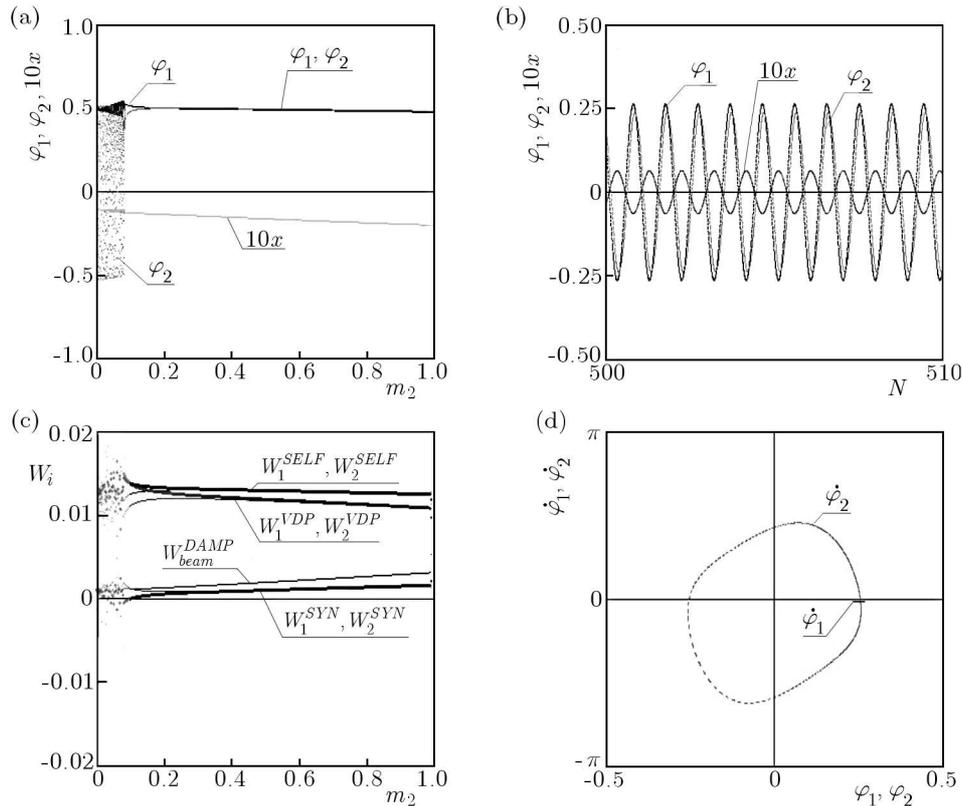


Fig. 3. Evolution from the complete synchronization to quasiperiodic oscillations; (a) bifurcation diagram for increasing values of  $m_2$ , (b) time histories of almost-complete synchronization for  $m_1 = 1.0$  kg and  $m_2 = 0.1$  kg, (c) energy plots, (d) Poincaré map showing quasiperiodic oscillations for  $m_1 = 1.0$  kg and  $m_2 = 0.07$  kg

In summary, the bifurcation diagram in Fig. 3a shows the existence of: (i) complete synchronization for  $m_1 = m_2 = 1.0$  kg, (ii) almost-complete synchronization for  $1.0$  kg  $> m_2 > 0.0975$  kg, (iii) the lack of synchronization and quasi-periodic oscillations for  $m_2 < 0.0975$  kg.

#### 4.3. From antiphase to almost-antiphase synchronization

The evolution of the system (1,2) behavior starting from antiphase synchronization of identical pendulums ( $m_1 = m_2 = 1.0$  kg) and the increase of the values of the control parameter  $m_2$  are illustrated in Figs. 4a-4d. Figure 4a presents another bifurcation diagram for the increasing values of  $m_2$  ( $m_2 \in [1.0, 6.0]$ ). This time we start with a state of antiphase synchronization of the pendulums with masses  $m_1 = m_2 = 1.0$  kg, during which two pendulums are moving in the same way ( $\varphi_1 = -\varphi_2$ ) and the beam is at rest.

The increase of the control parameter  $m_2$  leads to the reduction of pendulum 2 amplitude of oscillations but the amplitude of oscillations of pendulum 1 remains nearly constant. The pendulums remain in a state of almost-phase synchronization: the phase shift between the displacements is close to  $\pi$ , as shown in Fig. 4b ( $m_1 = 1.0$  kg,  $m_2 = 1.5$  kg). The displacement of the beam is practically equal to zero.

In the state of antiphase synchronization when the pendulums' oscillations satisfy the condition  $\varphi_1(t) = -\varphi_2(t)$ , two van der Pol's dampers dissipate the same amount of energy. The energies transmitted by both pendulums to the beam have absolute values proportional to pendulums masses and opposite signs

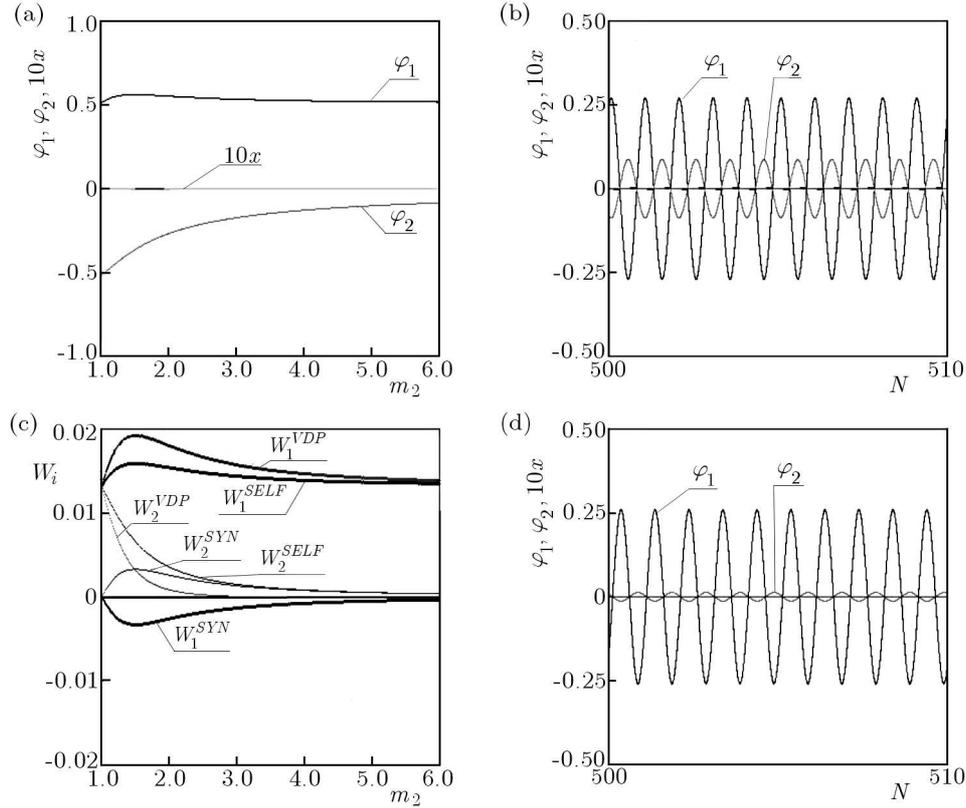


Fig. 4. Evolution from antiphase to almost-antiphase synchronization; (a) bifurcation diagram for increasing values of  $m_2$ , (b) time series of almost-antiphase synchronization for  $m_1 = 1.0$  kg and  $m_2 = 1.5$  kg, (c) energy plots, (d) time series of almost-antiphase synchronization for  $m_1 = 1.0$  kg and  $m_2 = 20.0$  kg

$$\begin{aligned}
 W_1^{SELF} &= \int_0^T c_{\varphi vdp} \dot{\varphi}_1^2 dt = \int_0^T c_{\varphi vdp} \dot{\varphi}_2^2 dt = W_2^{SELF} \\
 W_1^{VDP} &= \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_1^2 \varphi_1^2 dt = \int_0^T c_{\varphi vdp} \zeta \dot{\varphi}_2^2 \varphi_2^2 dt = W_2^{VDP} \\
 W_1^{SYN} &= \int_0^T m_1 \ddot{x} \cos \varphi_1 \dot{\varphi}_1 dt = -\frac{m_1}{m_2} \int_0^T m_2 \ddot{x} \cos \varphi_2 \dot{\varphi}_2 dt = -\frac{m_1}{m_2} W_2^{SYN}
 \end{aligned} \tag{4.3}$$

After substituting the energy values satisfying Eqs. (4.3) into Eqs. (3.7), Eqs.(3.7) are not contradictory equations only when the beam acceleration is zero, which implies the zero value of its velocity and acceleration (in the synchronization state of the behavior of the system is periodic). This condition requires the balancing of the forces which act on the pendulum beam, and this in turn requires that the pendulums have the same mass. If the pendulums' masses are different, instead of antiphase synchronization we observe an almost-antiphase synchronization, during which the pendulums' displacements have different amplitudes and phase shift between these displacements is close, but not equal to  $\pi$ . Hence

$$W_1^{SELF} \neq W_2^{SELF} \quad W_1^{VDP} \neq W_2^{VDP} \quad W_1^{SYN} \neq W_2^{SYN} \tag{4.4}$$

The values of each considered energy is shown in Fig. 4c. In a state of almost-antiphase synchronization we have  $W_1^{SELF} < W_1^{VDP}$  and  $W_2^{SELF} > W_2^{VDP}$ .  $W_2^{SELF}$  part of the energy  $W_2^{SELF}$

supplied by van der Pol's damper of pendulum 2 (with a greater mass) is transferred via the beam as  $W_2^{SYN}$  to the pendulum 1 (for this pendulum it is negative energy denoted by  $W_1^{SYN}$ ) and together with the energy  $W_1^{SELF}$  dissipated as  $W_1^{VDP}$  by van der Pol damper. Van der Pol's component of pendulum 2 dissipates the rest of the energy  $W_2^{SELF}$ , as  $W_2^{VDP}$ . The energy dissipated by the beam damper is negligibly small, because the beam virtually does not move.

Figure 4d shows the time series of the system oscillations for the  $m_2 = 20.0$  kg. We observe that further increase  $m_2$  causes the reduction of the amplitude of pendulum 2 oscillations, the amplitude of oscillations of pendulum 1 remains unchanged. It can be observed that when the mass  $m_2$  increases, the equality of forces, with which the pendulums act on the beam occurs at decreasing amplitude of oscillations of pendulum 2. Pendulum 1 (with a smaller mass) has a virtually constant amplitude of oscillations and works here as a classical dynamical damper.

The comparison of Fig. 2a and Fig. 4a indicates that in the interval  $1.0 \text{ kg} < m_2 < 4.0 \text{ kg}$  almost-complete and almost-antiphase synchronization coexist (which of them takes place the initial conditions decide).

In summary, the diagram shown in Fig. 4a shows the existence of: (i) antiphase synchronization for  $m_1 = m_2 = 1.0$  kg, (ii) almost-antiphase synchronization for  $1.0 \text{ kg} < m_2 < 6.0$  kg (our research shows that this state is preserved for larger values  $m_2$ ).

#### 4.4. From antiphase synchronization to quasiperiodic oscillations

The evolution of system (1,2) behavior starting from antiphase synchronization of identical pendulums ( $m_1 = m_2 = 1.0$  kg) and the decrease of the values of the control parameter  $m_2$  are illustrated in Figs. 5a-5d. Figure 5a shows the bifurcation diagram of the system (1,2) for decreasing values of  $m_2$  ( $m_2$  decreases from an initial value 1.0 up to 0.01). We start from the state of antiphase synchronization observed for  $m_1 = m_2 = 1.0$  kg. In the interval  $1.0 \text{ kg} > m_2 > 0.45$  kg, both pendulums are in the state of almost-antiphase synchronization, as shown in Fig. 5b for  $m_2 = 0.5$  kg. We observe a phenomenon similar to that of Fig. 4a, i.e., when decreasing mass  $m_2$ , the amplitude of oscillations of pendulum 1 decreases (in this case pendulum 1 has a larger mass), the amplitude of pendulum 2 oscillations is practically constant and pendulum 2 acts as a dynamical damper. In Fig. 5c one can see the negative energy  $W_2^{SYN}$  – there is a transfer of energy from pendulum 1 to pendulum 2.

For  $m_2 = 0.45$  kg we observe the loss of synchronization due to the fact that energy  $W_1^{SELF}$  becomes equal to energy  $W_1^{SYN}$  which means that all the energy supplied to pendulum 1 by van der Pol's damper is transmitted to pendulum 2. For smaller values of  $m_2$ , pendulum 2 is not able to supply the energy needed to maintain a state of almost-antiphase synchronization and the system first obtains the state of almost-complete synchronization, and next when  $m_2 < 0.095$  kg exhibits unsynchronized quasi-periodic oscillations. The behavior of the system for  $m_2 < 0.415$  kg has been described in Section 2.2.

In the narrow interval between the state of almost-antiphase and the state of almost-complete synchronization, i.e., for  $0.45 \text{ kg} > m_2 > 0.415$  kg we observe quasiperiodic oscillations of the system, as shown on the Poincaré map of Fig. 5d ( $m_2 = 0.44$  kg).

The bifurcation diagram of Fig. 5a shows the existence of: (i) antiphase synchronization for  $m_1 = m_2 = 1.0$  kg, (ii) almost-antiphase synchronization for  $1.0 \text{ kg} > m_2 > 0.45$  kg, (iii) the lack of synchronization and quasi-periodic oscillations for  $0.45 \text{ kg} > m_2 > 0.415$  kg, (iv) almost-complete synchronization for  $0.415 \text{ kg} > m_2 > 0.095$  kg, (v) the lack of synchronization and quasi-periodic oscillations for  $m_2 < 0.095$  kg.

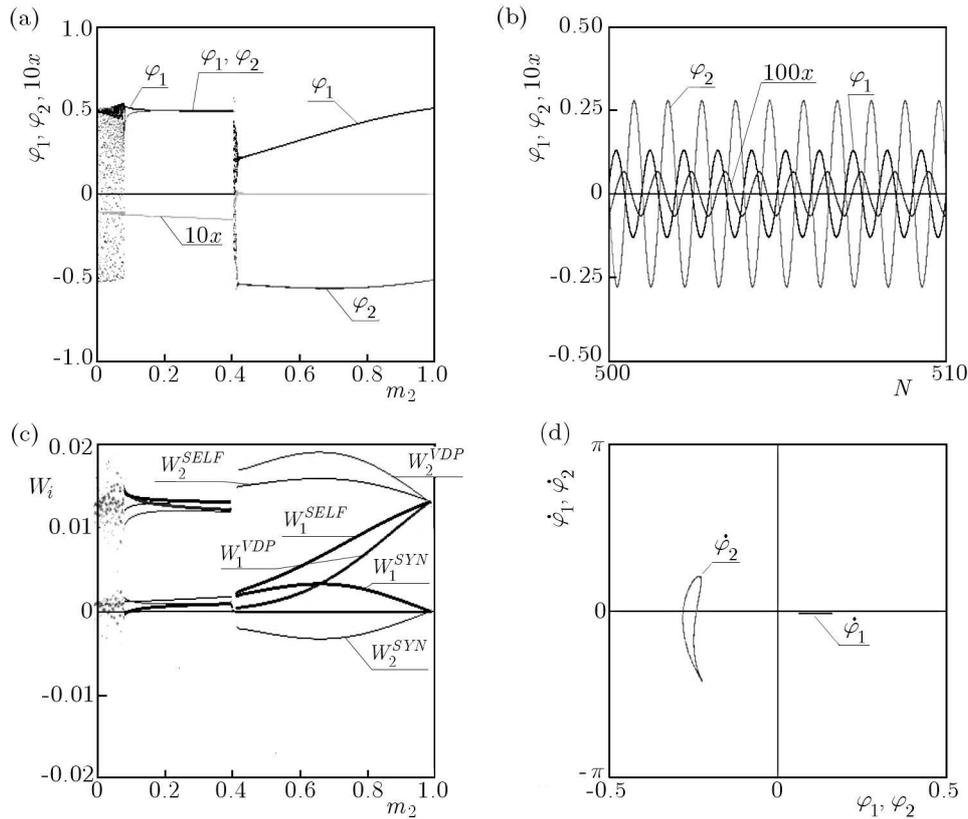


Fig. 5. Evolution from antiphase synchronization to quasiperiodic oscillations; (a) bifurcation diagram of system (1,2) for decreasing  $m_2$ , (b) time series of almost-antiphase synchronization for  $m_1 = 1.0$  kg,  $m_2 = 0.5$  kg, (c) energy plots, (d) Poincaré maps showing quasiperiodic oscillations for  $m_1 = 1.0$  kg and  $m_2 = 0.44$  kg

## 5. Conclusions

Our studies show that the system consisting of a beam and two self-excited pendulums with van der Pol's type of damping can perform four types of synchronization: (i) complete synchronization (possible only for nonrobust case of identical masses of both pendulums), i.e., the periodic motion of the system during which the displacements of both pendulums are identical ( $\varphi_1(t) = \varphi_2(t)$ ), (ii) almost-complete synchronization of the pendulums with different masses, in which phase difference between the displacements  $\varphi_1(t)$  and  $\varphi_2(t)$  is small (not larger than a few degrees), (iii) antiphase synchronization (possible only for nonrobust case of identical masses of both pendulums), i.e., the periodic motion of the system, during which the phase difference between the displacements  $\varphi_1(t)$  and  $\varphi_2(t)$  is equal to  $180^\circ$ , (iv) almost-antiphase synchronization, during which the phase difference between the displacements  $\varphi_1(t)$  and  $\varphi_2(t)$  is close to  $180^\circ$  and the amplitude of oscillations of both pendulums are different.

The observed behavior of the system (1,2) can be explained by the energy expressions derived in Section 3. The examples of the energy flow diagrams are shown in Figs. 6a,b. In the state (ii) both pendulums drive the beam (transferring to it part of the energy obtained from van der Pol's dampers) as seen in Fig. 6a. In the case (iv) the pendulum with larger mass and smaller amplitude of oscillations transmits part of its energy to the pendulum lower mass. The beam motion is negligibly small and the pendulum with lower mass reduces the amplitude of vibration of the pendulum with larger mass, acting on the classical model of the dynamic damper.

We identified two reasons for the sudden change of the attractor in system (1,2); (i) loss of stability of one type of synchronization after which the system trajectory jumps to the coexisting

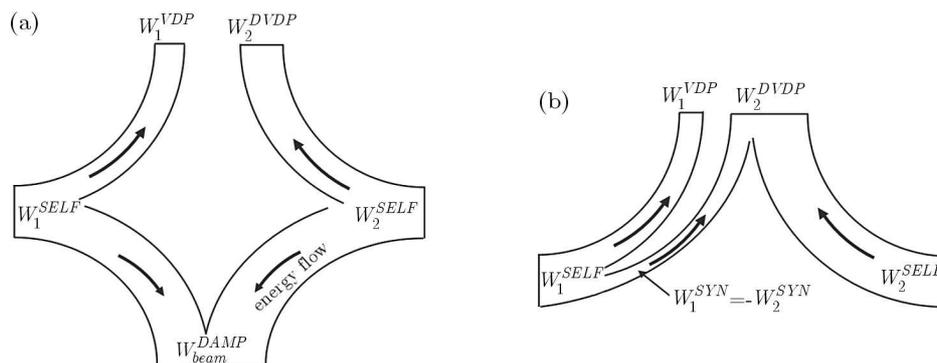


Fig. 6. Energy balances of the system (1,2); (a) almost-complete synchronization – both pendulums driver the beam, (b) almost-antiphase synchronization – pendulum 1 drives pendulum 2 via the beam

synchronization state, (ii) inability of van der Pol's damper of one of the pendulums energy necessary to drive the second pendulum.

#### Acknowledgement

This work has been supported by the Foundation for Polish Science, TEAM Programme – Project No. TEAM/2010/5/5.

#### References

1. ANDRONOV A., WITT A., KHAIKIN S., 1966, *Theory of Oscillations*, Pergamon, Oxford
2. BENNET M., SCHATZ M.F., ROCKWOOD H., WIESENFELD K., 2002, Huygens's clocks, *Proc. Roy. Soc. London*, **A 458**, 563-579
3. BLEKHMEN I.I., 1988, *Synchronization in Science and Technology*, ASME, New York
4. CZOLCZYNSKI K., PERLIKOWSKI P., STEFANSKI A., KAPITANIAK T., 2009a, Clustering and Synchronization of Huygens' Clocks, *Physica A*, **388**, 5013-5023
5. CZOLCZYNSKI K., PERLIKOWSKI P., STEFANSKI A., KAPITANIAK T., 2009b, Clustering of Huygens' Clocks, *Prog. Theor. Phys.*, **122**, 1027-1033
6. CZOLCZYNSKI K., PERLIKOWSKI P., STEFANSKI A., KAPITANIAK T., 2011, Why two clocks synchronize: Energy balance of the synchronized clocks, *Chaos*, **21**, 023129
7. DILAO R., 2009, Antiphase and in-phase synchronization of nonlinear oscillators: The Huygens's clocks system, *Chaos*, **19**, 023118
8. FRADKOV A.L., ANDRIEVSKY B., 2007, Synchronization and phase relations in the motion of twopendulum system, *Int. J. Non-linear Mech.*, **42**, 895
9. HUYGENS C., 1893, Letter to de Sluse, [In:] *Oeuvres Completes de Christian Huygens*, (letters; no. 1333 of 24 February 1665, no. 1335 of 26 February 1665, no. 1345 of 6 March 1665), Societe Hollandaise Des Sciences, Martinus Nijhoff, La Haye
10. KAPITANIAK M., CZOLCZYNSKI K., PERLIKOWSKI P., STEFANSKI A., KAPITANIAK T., 2012, Synchronization of clocks, *Physics Report*, published on-line, <http://dx.doi.org/10.1016/j.physrep.2012.03.002>
11. KANUNNIKOV A.YU., LAMPER R.E., 2003, Synchronization of pendulum clocks suspended on an elastic beam, *J. Appl. Mech. and Theor. Phys.*, **44**, 748-752
12. KUMON M., WASHIZAKI R., SATO J., MIZUMOTO R.K.I., IWAI Z., 2002, Controlled synchronization of two 1-DOF coupled oscillators, *Proceedings of the 15th IFAC World Congress*, Barcelona
13. PANTALEONE J., 2002, Synchronization of metronomes, *Am. J. Phys.*, **70**, 992

14. PERLIKOWSKI P., KAPITANIAK M., CZOLCZYNSKI K., STEFANSKI A., KAPITANIAK T., 2012, Chaos on coupled clocks, *Int. J. Bif. Chaos*, in press
15. PIKOVSKY A., ROESENBLUM M., KURTHS J., 2001, *Synchronization: An Universal Concept in Nonlinear Sciences*, Cambridge University Press, Cambridge
16. POGROMSKY A.YU., BELYKH V.N., NIJMEIJER H., 2003, Controlled synchronization of pendula, *Proceedings of the 42nd IEEE Conference on Design and Control*, Maui, Hawaii, 4381-4385
17. SENATOR M., 2006, Synchronization of two coupled escapement-driven pendulum clocks, *Journal Sound and Vibration*, **291**, 566-603
18. STROGATZ S.H., 2004, *Sync: The Emerging Science of Spontaneous Order*, Penguin Science, London
19. ULRICHS H., MANN A., PARLITZ U., 2009, Synchronization and chaotic dynamics of coupled mechanical metronomes, *Chaos*, **19**, 043120

### Bilans energii dwóch zsynchronizowanych wahadeł samowzbudnych o różnych masach

#### Streszczenie

Artykuł prezentuje analizę zjawiska synchronizacji dwóch wahadeł samowzbudnych o różnych masach. Pokazano, że jeśli takie wahadła zostaną zawieszona na wspólnej, ruchomej podstawie, zachodzi zjawisko ich (prawie) zupełnej lub (prawie) antyfazowej synchronizacji. Analiza bilansu energetycznego układu pozwala na określenie parametrów układu w stanie synchronizacji (amplitudy drgań i przesunięcia fazowe). Analiza bilansu energetycznego wyjaśnia także mechanizm synchronizowania się ruchu wahadeł: stały przepływ strumienia energii od jednego wahadła, *via* wspólna ruchoma podstawa, do drugiego wahadła powoduje, że ruch układu jest okresowy, a przesunięcia fazowe pomiędzy wahadłami przyjmują stałe, charakterystyczne wartości.

*Manuscript received February 15, 2012; accepted for print March 9, 2012*