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GAUSSIAN LIGHT BEAM PROPAGATION IN INHOMOGENEOUS MEDIUM

In this work the results of numerical calculations describing propagation of Gaussian light beam in inhomogeneous media is presented. Numerical calculation was based on scalar wave approximation. Additionally the split-step beam propagation method was exploited.

Keywords: Gaussian light beam, propagation, inhomogeneous media.

1. INTRODUCTION

In many cases processes occurring in medium affect its properties, in particular its refractive index. For instance, diffusion or compression of air. As a consequence of the refractive index variations, parameters of the light beam are changed.

It is possible to obtain these parameters using either wave optics or geometrical optics. It would seem that geometrical optics is simpler and proper method for such aims. But actually the ray equation [1,2] that describes the ray trajectory is neither analytically solvable nor in all cases true. Geometrical optics due to diffraction can yield significant errors when light beam is spatially limited. Therefore in this paper the wave optics was applied.

Practically one of the most exploited light beam is a laser beam. The laser can be set to work in TEM₀₀, that is, generates Gaussian light beam [3]. For this reason the trajectory and intensity distribution of such Gaussian light beam was considered below.

In particular, the behavior of Gaussian light beam emitted by He-Ne laser (632.8nm) propagating in inhomogeneous media was examined. Numerical calculation was based on scalar wave approximation, additionally exploiting the split-step beam propagation method [4,5].

2. THEORETICAL FOUNDATIONS OF NUMERICAL CALCULATIONS

The best description of light propagation in classical electromagnetism is given by the Maxwell's equations

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j} \quad (1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (2)$$

$$\nabla \cdot \vec{D} = \rho \quad (3)$$

$$\nabla \cdot \vec{B} = 0 \quad (4)$$

Because, in this paper, only linear and non-magnetic ($\mu_r = 1$) media without electric currents ($\vec{j} = 0$) and free charges ($\rho = 0$) are taken into account the electric flux density \vec{D} and magnetic flux density \vec{B} can be expressed in simple form by the electric field \vec{E} and magnetic field \vec{H}

$$\vec{D} = \epsilon_o \epsilon_r \vec{E} \quad (5)$$

$$\vec{B} = \mu_o \vec{H} \quad (6)$$

Where:

ϵ_o – vacuum permittivity or the electric constant

ϵ_r – relative permittivity

μ_o – permeability or magnetic constant

Let us assume ϵ_r to be time-independent. Since additionally ϵ_o and μ_o are universal constants then the Maxwell's equations take form:

$$\nabla \times \vec{H} = \epsilon_o \epsilon_r \frac{\partial \vec{E}}{\partial t} \quad (7)$$

$$\nabla \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \quad (8)$$

$$\epsilon_o \nabla \cdot \epsilon_r \vec{E} = 0 \quad (9)$$

$$\nabla \cdot \vec{H} = 0 \quad (10)$$

By taking the curl of both sides Eq.(8) and using the vector identity Eq.(11) (\vec{X} is any vector) Eq. (12) can be derived where Laplacian is denoted by mark Δ .

$$\nabla \times \nabla \times \vec{X} = \nabla(\nabla \cdot \vec{X}) - \Delta \vec{X} \quad (11)$$

$$\Delta \vec{E} - \mu_o \cdot \epsilon_o \cdot \epsilon_r(x, y, z) \cdot \frac{\partial^2 \vec{E}}{\partial t^2} = -\nabla \left(\vec{E} \cdot \frac{\nabla \epsilon_r(x, y, z)}{\epsilon_r(x, y, z)} \right) \quad (12)$$

The right hand side of the above equation in general (in inhomogeneous media) does not equal zero. For instance, during diffusion process in materials or very frequently in many kinds of fluid flows. Fortunately, when the variation of the relative permittivity ϵ_r is small over the distance of one wavelength of electromagnetic wave described by the above equation, the following approximation can be proved [6].

$$\nabla \left(\frac{\nabla \epsilon_r(x, y, z)}{\epsilon_r(x, y, z)} \right) \cong 0 \quad (13)$$

Since refractive index of medium n and its variation δn satisfy following formula

$$n = \sqrt{\epsilon_r(x, y, z)} \Rightarrow \delta n = -\frac{\delta \epsilon_r(x, y, z)}{2n} \Rightarrow \delta n = -\frac{\delta \epsilon_r(x, y, z)}{2n} \quad (14)$$

It is possible to state that when the variation of the refractive index is small over the distance of one wavelength of optical wave Eq.(15) can be written.

$$\Delta \vec{E} - \mu_0 \cdot \epsilon_0 \cdot \epsilon_r(x, y, z) \cdot \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad (15)$$

Due to the facts that the analogous equation as above can be derived for magnetic field and the scalar wave approximation will exploit in this paper, Eq.(15) can be rewritten in the form.

$$\Delta U - \mu_0 \cdot \epsilon_0 \cdot \epsilon_r(x, y, z) \cdot \frac{\partial^2 U}{\partial t^2} = 0 \quad (16)$$

For the reason, that for monochromatic light beams generated by lasers, the harmonic time dependence of electric and magnetic fields can be assumed and such beams propagate predominantly in one direction (for instance along z-axis), the scalar function U can be factorized in following form.

$$U = u(x, y, z) \cdot \exp [i(\omega_0 t - k_0 z)] \quad (17)$$

$$k_0 = \frac{n \cdot \omega_0}{c} = \mu_0 \cdot \epsilon_0 \cdot \omega_0^2 - \text{propagation constant.}$$

ω – angular frequency,

c – speed of light in vacuum.

Substitution Eq.(17) to Eq.(16) yields

$$\Delta_t u + \frac{\partial^2 u}{\partial z^2} - 2 \cdot i \cdot k_0 \frac{\partial u}{\partial z} - (k_0^2 - \mu_0 \cdot \epsilon_0 \cdot \epsilon_r(x, y, z) \cdot \omega_0^2) u = 0 \quad (18)$$

Where $\Delta_t = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ denotes so called “transverse Laplacian”. In many cases it is reasonable to assume that function u very slowly varies along z direction. That means

$$\left| \frac{\partial u}{\partial z} \right| \ll k \cdot u \quad (19)$$

Since the right hand side of Eq.(18) can be rewritten in different shape

$$k_0^2 - \mu_0 \cdot \varepsilon_0 \cdot \varepsilon_r(x, y, z) \cdot \omega_0^2 = -\mu_0 \cdot \varepsilon_0 \cdot \omega_0^2 \cdot (\varepsilon_r(x, y, z) - 1) = -k_0^2 \cdot \delta\varepsilon_r \quad (20)$$

The substitution of Eq.(20) to Eq.(18) leads to Eq.(21) on condition expressed by Eq.(19).

$$\frac{\partial u}{\partial z} = \frac{\Delta_z u}{2 \cdot i \cdot k_0} + \frac{k_0}{2 \cdot i} \cdot \delta\varepsilon_r \cdot u \quad (21)$$

For the reason, that in optical applications refractive index is more convenient in use than relative permittivity, the relationship between variation of refractive index δn and variation of relative permittivity $\delta\varepsilon_r$ has to be found.

$$\begin{aligned} n &= n_0 (1 + \delta n) = \sqrt{\varepsilon_r (1 + \delta\varepsilon_r)} \cong \sqrt{\varepsilon_r} + 0.5 \cdot \delta\varepsilon_r \\ \delta\varepsilon_r &\cong \frac{2n_0}{\sqrt{\varepsilon_r}} \cdot \delta n = 2 \cdot \delta n \end{aligned} \quad (22)$$

Exploiting this outcome Eq.(21) can be rewritten as follows

$$\frac{\partial u}{\partial z} = \frac{\Delta_z u}{2 \cdot i \cdot k_0} - i \cdot k_0 \cdot \delta n \cdot u \quad (23)$$

The split-step beam propagation method will be used to solve above equation. First what is necessary to use this method is to rewrite Eq.(23) in the operator form.

$$\frac{\partial u}{\partial z} = (\hat{D} + \hat{S})u \quad (24)$$

Where operator $\hat{D} = \frac{1}{2 \cdot i \cdot k_0} \Delta_z$ and operator $\hat{S} = -i \cdot k_0 \cdot \delta n$.

The solution of Eq.(24) is described by

$$u(x, y, z + \delta z) = u(x, y, z) \exp[(\hat{D} + \hat{S}) \cdot \delta z] \quad (25)$$

From above equation it is seen that if the field $u(x, y, 0)$ is known in the plane XY at $z = 0$ it is possible to find the field $u(x, y, \delta z)$ in the plane XY at $z = \delta z$ then knowing the field $u(x, y, \delta z)$ the field $u(x, y, 2 \cdot \delta z)$ at $z = 2 \cdot \delta z$ can be found and so on. This recursive formula enables to determine step by step field in long distance from the source plane of light.

The simplest manner of proceeding that enable us to figure out the right hand side of Eq.(25) is to use successively Fourier transform and inverse Fourier transform. But for this purpose the below relationship must be satisfied.

$$\exp[(\hat{D} + \hat{S}) \cdot \delta z] = \exp[\hat{D} \cdot \delta z] \cdot \exp[\hat{S} \cdot \delta z] \quad (26)$$

The relationship (26) is satisfied when operators \hat{D} and \hat{S} commute. In this case it is not perfectly true but when refractive index varies very small in all direction the Eq.(26) is approximately fulfilled.

3. RESULTS OF NUMERICAL CALCULATIONS

Using the theory briefly presented above, the intensity distribution of the light on subsequent XY planes along Z axis with increment $\delta z = 0,5\text{m}$ was calculated. For clarity of plots only some of them are shown. In all cases the light beam at the origin, that is, at the input plane at $z = 0$ had the constant phase and Gaussian distribution of its amplitude. That means that input plane coincide with the beam waist w_0 . The profile of the amplitude at the origin was taken in the form

$$u(x, y, 0) = \frac{1}{w_0 \cdot \sqrt{\pi}} \exp \left[-\frac{(x - 0,005\text{m})^2 + y^2}{w_0^2} \right] \quad (27)$$

All calculations was made on assumption that light propagates in the air and its refractive index n_0 is roughly equal to 1.0003.

All figures below present how optical inhomogeneity of the medium affects Gaussian light beam propagation. In calculations that enabled drawing Figs. 1-3 the following assumptions were made: the refractive index of the medium is a function of x , that is, $n = n_0(1 - 5 \cdot 10^{-5} \cdot x)$ and the waist of beam $w_0 = 0,001\text{m}$. One can notice that the light beam, in Figs. 1-3, deflects and diverges but the characteristic Gaussian profile does not change.

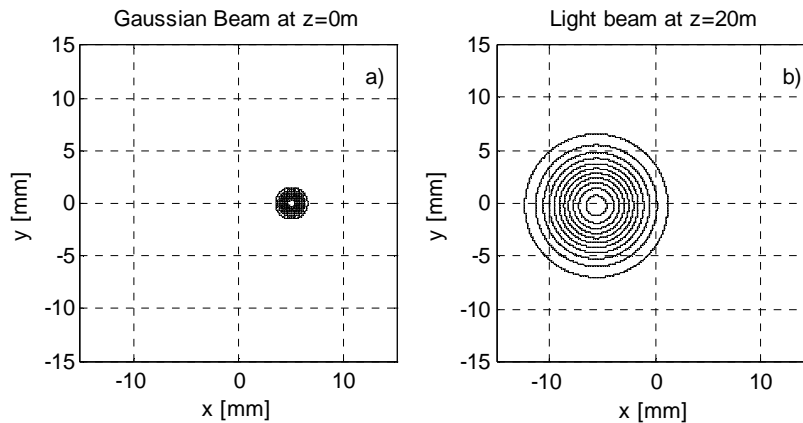


Fig. 1. Contour plots show the intensity distribution of the light across the laser beam on input plane at $z = 0$ and on the output plane at $z = 20$ m. On the total path of the light beam the refractive index varies linearly in x direction.

The set of Figs. 4-6 similarly to the set of Figs. 1-3 shows the laser beam in inhomogeneous medium. But now the waist of beam at the input plane $w_0 = 0,004\text{m}$ and the refractive index of the medium is described by the following function $n = n_0(1 - 5 \cdot 10^{-3} \cdot (x^2 + y^2))$. Analysis of Figs. 4-6 evidently demonstrate that in contrast to the former case, the light beam does not tend to constantly deflect from

previous direction of propagation. Instead of constant deflection the beam undergoes oscillations. One can perceive, in Figs. 4-6, two types of oscillations: longitudinal and transverse.

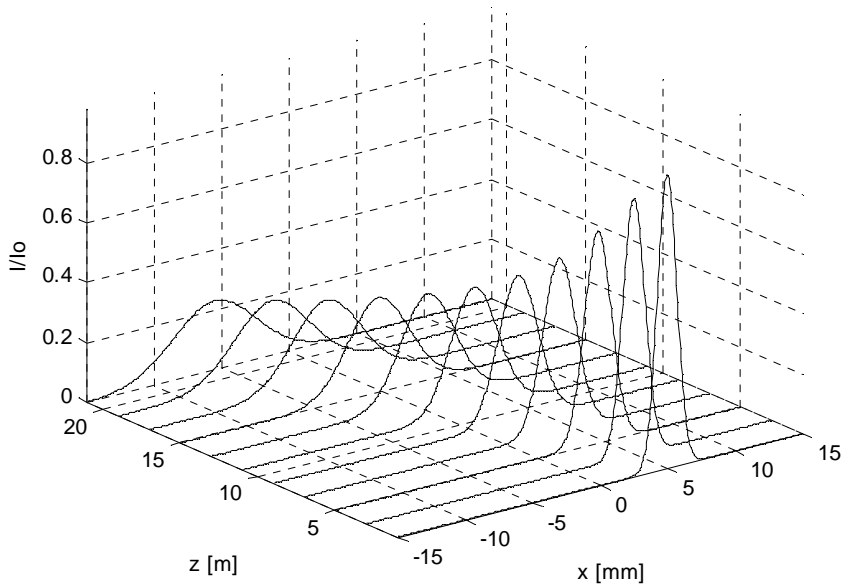


Fig. 2. Normalized intensity profile of light beam along Z axis. The intensity was normalized with respect to maximal intensity I_0 on the input plane at $z = 0$

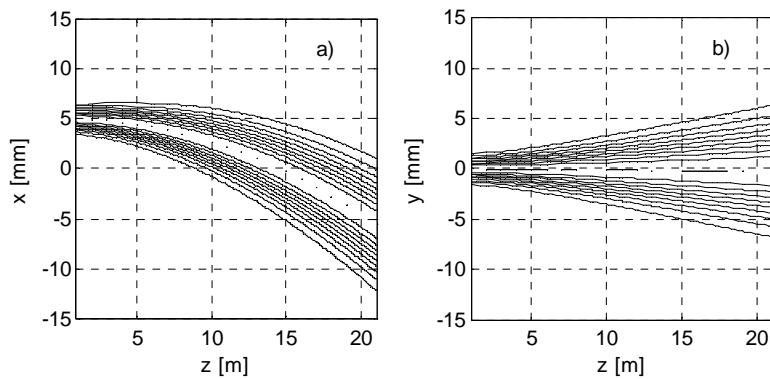


Fig. 3. The plots show contours of the laser beam along Z axis: **a)** projection on the plane XZ. **b)** projection on the plane YZ

Longitudinal oscillations cause periodically focusing of the light. Transverse oscillations caused by the fact that the symmetry axis of light beam was shifted

with respect to the symmetry axis (Z axis) of medium refractive index distribution stimulate periodically moving away and towards Z axis.

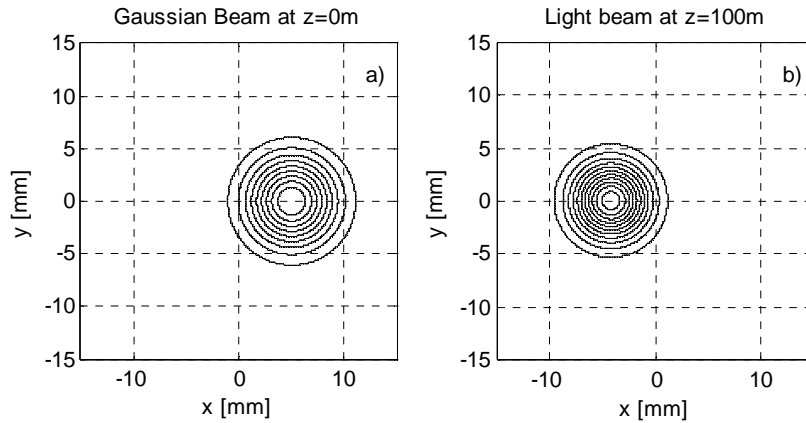


Fig. 4. Contour plots show the intensity distribution of the light across the laser beam on input plane at $z = 0$ and on the output plane at $z = 100$ m. On the total path of the light beam the refractive index varies linearly in x direction

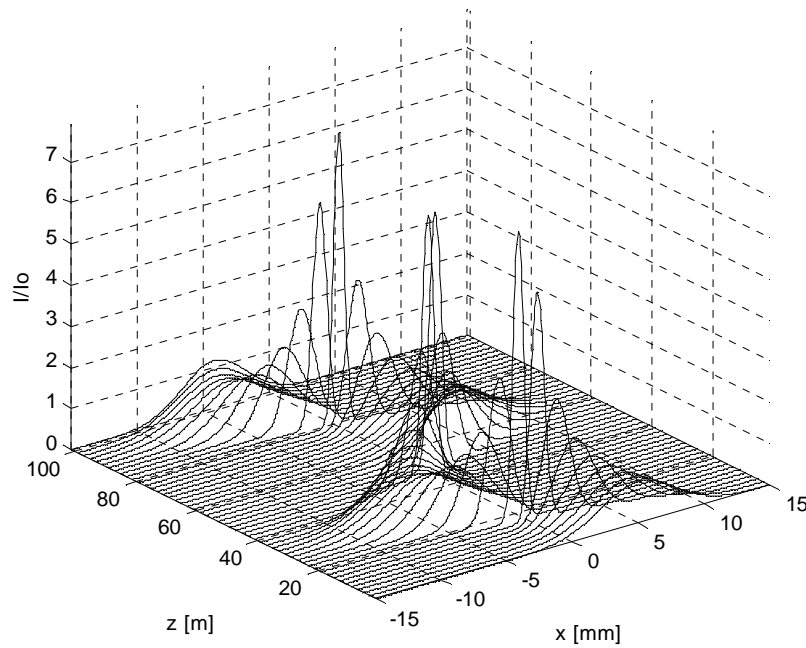


Fig. 5. Normalized intensity profile of light beam along Z axis. The intensity was normalized with respect to maximal intensity I_0 on the input plane at $z = 0$

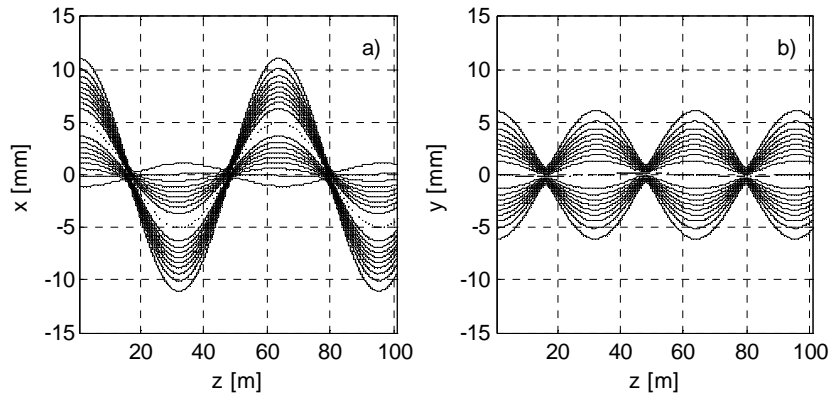


Fig. 6. The plots show contours of the laser beam along Z axis: **a)** projection on the plane XZ. **b)** projection on the plane YZ

The set of Figs. 7-9. shows also the laser beam in inhomogeneous medium and the the waist of beam at input plane is the same as previous ($w_0 = 0,004\text{m}$) but the refractive index of the medium is described by function $n = n_0(1 - 5 \cdot 10^{-3} \cdot x^3)$.

Due to the asymmetry of the function describing refractive index of the medium the light beam is constantly deflected from its previous direction. Additionally the light beam loses its characteristic Gaussian profile.

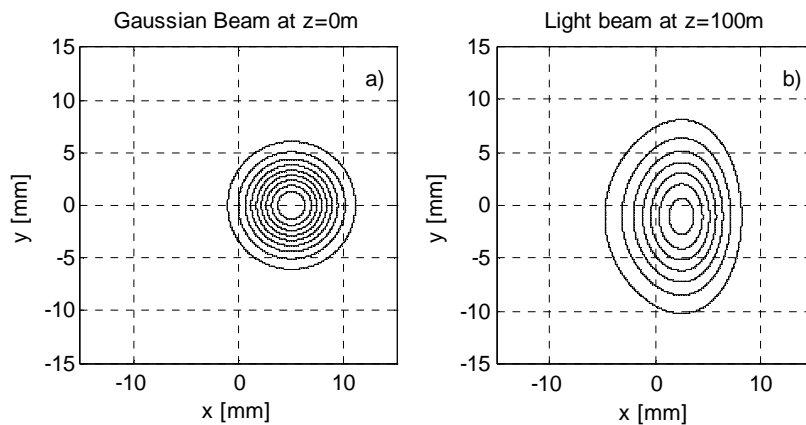


Fig. 7. Contour plots show the intensity distribution of the light across the laser beam on input plane at $z = 0$ and on the output plane at $z = 100\text{m}$. On the total path of the light beam the refractive index varies linearly in x direction

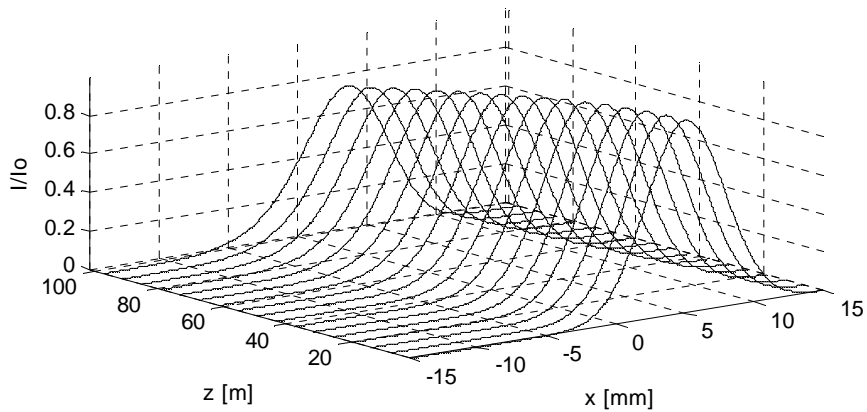


Fig. 8. Normalized intensity profile of light beam along Z axis. The intensity was normalized with respect to maximal intensity I_0 on the input plane at $z = 0$

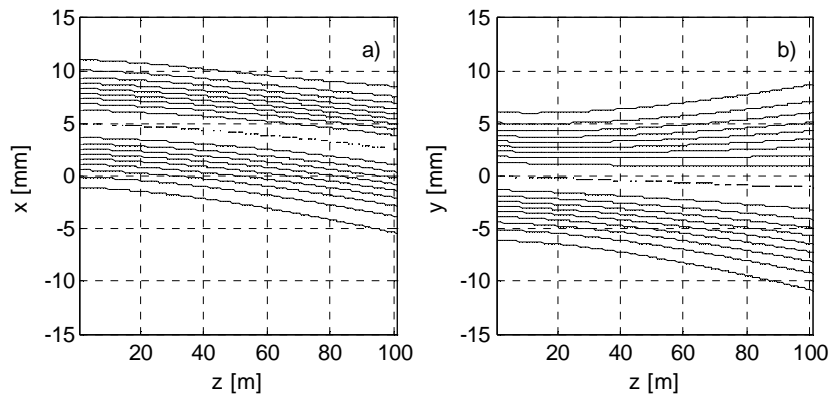


Fig. 9. The plots show contours of the laser beam along Z axis: **a)** projection on the plane XZ. **b)** projection on the plane YZ

4. CONCLUSIONS

Above propagation of Gaussian light beam in inhomogeneous media was numerically examined. The numerical calculations were performed for three cases. Each case differs from others refractive index distribution in the medium.

The analysis of the numerical calculations presented in Figs.1-9 shows that when the function describing refractive index of the medium is odd function the

light beam undergoes deflection from previous direction of propagation, but only when mentioned earlier function is linear, the light beam keeps its Gaussian characteristic shape. But when the function describing the refractive index of the medium is quadratic-type and has maximum at axis along which the light propagates, both the path of the light beam and its intensity experiences oscillations. What is more surprising in this case, in spite of the fact, that the function describing the refractive index of the medium is nonlinear the beam keeps its Gaussian shape.

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PROPAGACJA WIĄZKI GAUSOWSKIEJ W OŚRODKU NIEJEDNORODNYM

Streszczenie

Praca przedstawia wyniki obliczeń numerycznych rozkładu natężenia wiązki lasera He-Ne w trakcie propagacji przez trzy niejednorodne ośrodki różniące się charakterem tejże niejednorodności. Do obliczeń wykorzystano przybliżenie pola skalarnego i rekurencyjnie liczone kolejne profile intensywności wiązki światła.