JOURNAL OF APPLIED
COMPUTER SCIENCE
Vol. 19 No. 2 (2011), pp. 125-139

On Properties of a Lattice Structure for a Wavelet Filter Bank Implementation: Part II

Jan Stolarek

Institute of Information Technology
Technical University of Lodz
Wólczańska 215, 90-924 Łódź, Poland
Jan.Stolarek@p.lodz.pl

Abstract. *This paper continues discussion of a lattice structure for parametrization and implementation of a Discrete Wavelet Transform. Based on an algorithm for converting the lattice structure to a wavelet filter bank coefficients, developed in the first part of this paper, second part of the proof demonstrating that filters implemented by the lattice structure fulfil conditions imposed on an orthogonal wavelet filter bank is carried out.*
Keywords: *wavelet transform, filter parametrization, lattice structure.*

1. Introduction

In the first part of this paper [1] a lattice structure for parametrization and implementation of a Discrete Wavelet Transform was defined. It was based on two-point base operations:

$$D_l = \begin{bmatrix} w_{11}^{(l)} & w_{12}^{(l)} \\ w_{21}^{(l)} & w_{22}^{(l)} \end{bmatrix} , \tag{1}$$

where l is the index of a layer in the lattice structure, while $w_{11}^{(l)}$, $w_{12}^{(l)}$, $w_{21}^{(l)}$ and $w_{22}^{(l)}$ represent the parameters. Based on this general form two specific types of base operations have been defined. Symmetric forward base operation was defined as:

$$P_l = \begin{bmatrix} w_{11}^{(l)} & w_{12}^{(l)} \\ w_{12}^{(l)} & -w_{11}^{(l)} \end{bmatrix} \quad , \tag{2}$$

while asymmetric forward base operation was given by:

$$O_l = \begin{bmatrix} w_{11}^{(l)} & w_{12}^{(l)} \\ -w_{12}^{(l)} & w_{11}^{(l)} \end{bmatrix} \quad . \tag{3}$$

To produce orthogonal base operations the following substitution was used:

$$\begin{aligned} w_{11}^{(l)} &= cos(\alpha_l) \quad , \\ w_{12}^{(l)} &= sin(\alpha_l) \quad , \end{aligned} \tag{4}$$

giving symmetric orthogonal forward/inverse base operation:

$$S_l = S_l^{-1} = S_l^T = \begin{bmatrix} cos(\alpha_l) & sin(\alpha_l) \\ sin(\alpha_l) & -cos(\alpha_l) \end{bmatrix} \quad , \tag{5}$$

and asymmetric orthogonal forward and inverse base operations:

$$F_l = \begin{bmatrix} cos(\alpha_l) & sin(\alpha_l) \\ -sin(\alpha_l) & cos(\alpha_l) \end{bmatrix} \quad , \tag{6}$$

$$F_l^{-1} = F_l^T = \begin{bmatrix} cos(\alpha_l) & -sin(\alpha_l) \\ sin(\alpha_l) & cos(\alpha_l) \end{bmatrix} \quad . \tag{7}$$

In Section 4 of [1] the following algorithm for converting a lattice structure parameters to wavelet filter coefficients was constructed:

$$\left\{ \begin{array}{ll} H_0^{(2)} = [w_{11}^{(1)}, w_{12}^{(1)}] & \\ H_1^{(2)} = [w_{21}^{(1)}, w_{22}^{(1)}] & \text{for layer } l = 1 \\ & \\ H_0^{(2l)} = [H_1^{(2l-2)} w_{11}^{(l)}, 0, 0] + [0, 0, H_0^{(2l-2)} w_{12}^{(l)}] & \\ H_1^{(2l)} = [H_1^{(2l-2)} w_{21}^{(l)}, 0, 0] + [0, 0, H_0^{(2l-2)} w_{22}^{(l)}] & \text{for layer } l = 2, ..., \frac{L}{2} \end{array} \right. \quad , \tag{8}$$

where $H_0^{(L)}$ and $H_1^{(L)}$ are low-pass and high-pass filters of length L respectively. Based on this algorithm a proof was carried out, demonstrating that filters implemented by the lattice structure based on symmetric orthogonal base operations fulfil conditions for an orthogonal wavelet filter bank.

This paper continues the theoretical analysis of a lattice structure. In Section 2 we carry out a proof demonstrating that lattice structure based on asymmetric base operations also fulfils conditions imposed on an orthogonal wavelet filter bank, although the required sum of angles is different than in case of symmetric base operations. Section 3 discusses representation of a lattice structure in a form that ensures fulfilment of conditions imposed on the sum of angles.

2. Implementation of the orthogonal filter bank using the lattice structure - asymmetric base operations

Let us consider orthogonal lattice structure based on asymmetric orthogonal base operations given by Equation (6). Let H_0 and H_1 be a pair of filters implemented by such structure and created according to Equation (8). For such filters it will be proved that the following conditions imposed on an orthogonal wavelet filter bank are fulfilled:

1. Equation

$$\sum_{n=0}^{L-1} h_0(n) = \sqrt{2} \quad , \tag{9}$$

   holds true if the following condition is fulfilled:

$$\sum_{l=1}^{L/2} (-1)^{l+1} \alpha_l = (-1)^{(L/2+1)} \cdot \frac{\pi}{4} \quad . \tag{10}$$

2. Equation

$$\sum_{n=0}^{L-1} h_1(n) = 0 \quad , \tag{11}$$

   holds true, if Equation (10) holds true.

3. Coefficients of H_1 satisfy

$$h_1(n) = -(-1)^n h_0(L-n-1) = (-1)^{n+1} h_0(L-n-1) \quad , \tag{12}$$

for a lattice structure with odd number of layers. In case of a lattice structure with even number of layers, the coefficients of H_1 satisfy:

$$h_1(n) = (-1)^n h_0(L-n-1) \quad . \tag{13}$$

4. H_0 filter is orthogonal to its shifts by 2:

$$\sum_{n=0}^{L-1} h_0(n)h_0(n+2m) = 0 \text{ for } m \in \mathbb{Z} \setminus \{0\} \quad . \tag{14}$$

5. H_1 filter is orthogonal to its shifts by 2:

$$\sum_{n=0}^{L-1} h_1(n)h_1(n+2m) = 0 \text{ for } m \in \mathbb{Z} \setminus \{0\} \quad . \tag{15}$$

6. H_0 and H_1 filters are orthogonal to each other:

$$\sum_{n=0}^{L-1} h_0(n)h_1(n+2m) = 0 \text{ for } m \in \mathbb{Z} \quad . \tag{16}$$

7. H_0 and H_1 filters have unit norm:

$$\sum_{n=0}^{L-1} h_0(n)^2 = 1 \quad , \tag{17}$$

$$\sum_{n}^{L-1} h_1(n)^2 = 1 \quad . \tag{18}$$

The following dependencies between the trigonometric functions will be used in the proof:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad , \tag{19}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad . \tag{20}$$

For the asymmetric base operation, the following substitutions into (1) are used:

$$\begin{array}{llll} w_{11}^{(l)} = & \cos(\alpha_l) \quad , & w_{12}^{(l)} = \sin(\alpha_l) \quad , \\ w_{21}^{(l)} = & -\sin(\alpha_l) \quad , & w_{22}^{(l)} = \cos(\alpha_l) \quad . \end{array} \tag{21}$$

The following notation will be used:

$$\gamma_j = \sum_{i=1}^{j} (-1)^{i+1} \alpha_i \quad . \tag{22}$$

where α_l are the angles of the lattice structure, as defined in Equation (21).

Proof. Let us begin by proving Equations (9) and (11). We will analyze how does the sum of angles in the lattice structure depend on the number of layers, assuming that Equation (9) is to be fulfilled. According to Equation (8), for a one-layer lattice structure we have:

$$\begin{aligned} H_0^{(2)} &= [\ \ \cos(\alpha_1), \sin(\alpha_1)] \quad , \\ H_1^{(2)} &= [-\sin(\alpha_1), \cos(\alpha_1)] \quad . \end{aligned} \tag{23}$$

As was already demonstrated in the first part of this paper, for a filter $H_0^{(2)}$ given by the above equation, condition (9) is fulfilled for $\alpha_1 = \frac{\pi}{4}$. Let us substitute that angle into Equation (11) for a filter $H_1^{(2)}$ given by Equation (23):

$$-\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0 \quad . \tag{24}$$

Therefore, in order for Equations (9) and (11) to hold true for a one-layer lattice structure, angle α_1 in the matrix of base operation F_1 must have a value of $\frac{\pi}{4}$, which is in accordance with Equation (10).

Let us calculate the sum of angles in a two-layer lattice structure required to fulfil Equation (10). Based on Equation (8), we can write:

$$\begin{aligned} \sum_{n=0}^{3} h_0(n) &= [\cos(\alpha_1) - \sin(\alpha_1)]\cos(\alpha_2) + [\cos(\alpha_1) + \sin(\alpha_1)]\sin(\alpha_2) = \\ &= \cos(\alpha_1)\cos(\alpha_2) - \sin(\alpha_1)\cos(\alpha_2) + \cos(\alpha_1)\sin(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2) = \\ &= -\sin(\alpha_1 - \alpha_2) + \cos(\alpha_1 - \alpha_2) = \\ &= -\sin(\gamma_2) + \cos(\gamma_2) \quad . \end{aligned} \tag{25}$$

For a two-layer lattice structure Equation (9) can be therefore given as

$$-\sin(\gamma_2) + \cos(\gamma_2) = \sqrt{2} \;,$$

and is fulfilled for $\gamma_2 = -\frac{\pi}{4}$.[1] The sum of coefficients of a 4-tap high-pass filter $H_1^{(4)}$ can be written as:

$$\begin{aligned}\sum_{n=0}^{3} h_0(n) &= -\left[\cos(\alpha_1) - \sin(\alpha_1)\right]\sin(\alpha_2) + \left[\cos(\alpha_1) + \sin(\alpha_1)\right]\cos(\alpha_2) = \\ &= -\cos(\alpha_1)\sin(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2) + \cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\cos(\alpha_2) = \\ &= \sin(\alpha_1 - \alpha_2) + \cos(\alpha_1 - \alpha_2) = \\ &= \sin(\gamma_2) + \cos(\gamma_2) \;.\end{aligned} \tag{26}$$

By substituting $\gamma_2 = -\frac{\pi}{4}$ into Equation (26) we get:

$$\sin\left(-\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0 \;. \tag{27}$$

Therefore, for a two-layer lattice structure, Equations (9) and (11) hold true if and only if the sum $\alpha_1 - \alpha_2$ equals to $-\frac{\pi}{4}$, which stays in accordance with Equation (10).

Let us analyze a three-layer lattice structure. Equations (25) and (26) show, that sum of $H_0^{(4)}$ and $H_1^{(4)}$ coefficients can be shortly written as:

$$\cos(\gamma_2) \mp \sin(\gamma_2) \;. \tag{28}$$

Since the multiplication distributes over addition, the above results can be used to calculate the sum of $H_0^{(6)}$ and $H_1^{(6)}$ coefficients:

$$\begin{aligned}\sum_{n=0}^{5} h_0(n) &= \left[\cos(\gamma_2) + \sin(\gamma_2)\right]\cos(\alpha_3) + \left[-\sin(\gamma_2) + \cos(\gamma_2)\right]\sin(\alpha_3) = \\ &= \cos(\gamma_2)\cos(\alpha_3) + \sin(\gamma_2)\cos(\alpha_3) - \sin(\gamma_2)\sin(\alpha_3) + \cos(\gamma_2)\sin(\alpha_3) = \\ &= \cos(\gamma_2 + \alpha_3) + \sin(\gamma_2 + \alpha_3) = \\ &= \cos(\gamma_3) + \sin(\gamma_3) \;.\end{aligned} \tag{29}$$

[1]Step by step solutions to this and similar equations are given in the first part of this paper.

We notice, that the sum of coefficients of a $H_0^{(6)}$ low-pass filter is given by the same formula as for a one-layer lattice structure. This implies, that for a three-layer lattice structure Equation (9) is fulfilled for angle $\gamma_3 = \gamma_1 = \frac{\pi}{4}$. Let us now calculate the sum of coefficients of a high-pass filter $H_1^{(6)}$:

$$\begin{aligned}\sum_{n=0}^{5} h_0(n) = -\left[\cos(\gamma_2) + \sin(\gamma_2)\right]\sin(\alpha_3) + \left[-\sin(\gamma_2) + \cos(\gamma_2)\right]\cos(\alpha_3) = \\ = -\cos(\gamma_2)\sin(\alpha_3) - \sin(\gamma_2)\sin(\alpha_3) - \sin(\gamma_2)\cos(\alpha_3) + \cos(\gamma_2)\cos(\alpha_3) = \\ = \cos(\gamma_2 + \alpha_3) - \sin(\gamma_2 + \alpha_3) = \\ = \cos(\gamma_3) - \sin(\gamma_3) \quad .\end{aligned} \tag{30}$$

We notice, that the sum of coefficients of a $H_1^{(6)}$ high-pass filter is given by the same formula as for a one-layer lattice structure. This implies, that for a three-layer lattice structure Equation (11) is fulfilled for angle $\gamma_3 = \gamma_1 = \frac{\pi}{4}$.

For a three-layer lattice structure, the sum of filter coefficients is given by the same formula as for a one-layer lattice structure. Therefore adding more layers to the structure will produce the same results as the ones obtained above for a one and two-layer structure. Hence it has been proved, that the low-pass and high-pass filters implemented by the orthogonal lattice structure based on the asymmetric base operations fulfil Equation (9) and (11), assuming that condition (10) holds true.

The Equation (12) (in case of odd number of layers) and Equation (13) (in case of even number of layers) will be proved using the mathematical induction method. This requires proving three claims:

1. filters $H_0^{(2)}$ and $H_1^{(2)}$ fulfil Equation (12),

2. the fact that filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil Equation (12) leads to conclusion that filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ fulfil Equation (13),

3. the fact that filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil Equation (13) leads to conclusion that filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ fulfil Equation (12).

Let us begin by proving first claim. According to Equations (8) and (21), filters $H_0^{(2)}$ and $H_1^{(2)}$ take form:

$$H_0^{(2)} = [\quad \cos(\alpha_1), \sin(\alpha_1)] \quad , \tag{31}$$
$$H_1^{(2)} = [-\sin(\alpha_1), \cos(\alpha_1)] \quad . \tag{32}$$

It can be easily noticed, that coefficients of $H_1^{(2)}$ filter are created by reversing the order of $H_0^{(2)}$ filter coefficients and changing the sign of every odd coefficient:

$$\begin{aligned} h_1(0) &= (-1)^1 h_0(2-0-1) = -h_0(1) = -\sin(\alpha_1) \quad , \\ h_1(1) &= (-1)^2 h_0(2-1-1) = \quad h_0(0) = \quad \cos(\alpha_1) \quad , \end{aligned} \tag{33}$$

which proves the first claim. Let us now prove the second claim. We assume that Equation (12) holds true for filters of length L. This means, that there's a following dependency between the $h_0(n)$ and $h_1(n)$ coefficients:

$$\begin{aligned} h_1(0) &= -h_0(L-1) \quad , \\ h_1(1) &= \quad h_0(L-2) \quad , \\ h_1(2) &= -h_0(L-3) \quad , \\ &\vdots \\ h_1(L-3) &= \quad h_0(2) \quad , \\ h_1(L-2) &= -h_0(1) \quad , \\ h_1(L-1) &= \quad h_0(0) \quad . \end{aligned} \tag{34}$$

It will be demonstrated, that assuming the inductive hypothesis (34) and using the Equation (8) leads to obtaining filter coefficients that fulfil Equation (13). The coefficients of $H_1^{(L+2)}$ take form[2]:

$$\begin{aligned} H_1^{(L+2)} &= [-h_1(0)\sin(\alpha),\ -h_1(1)\sin(\alpha),\ -h_1(2)\sin(\alpha) + h_0(0)\cos(\alpha), \\ &-h_1(3)\sin(\alpha) + h_0(1)\cos(\alpha),\ \ldots,\ -h_1(L-2)\sin(\alpha) + h_0(L-4)\cos(\alpha), \\ -h_1(L-1)&\sin(\alpha) + h_0(L-3)\cos(\alpha),\ h_0(L-2)\cos(\alpha),\ h_0(L-1)\cos(\alpha)] \overset{((34))}{=} \\ &= [h_0(L-1)\sin(\alpha),\ -h_0(L-2)\sin(\alpha),\ h_0(L-3)\sin(\alpha) + h_0(0)\cos(\alpha), \\ &-h_0(L-4)\sin(\alpha) + h_0(1)\cos(\alpha),\ \ldots,\ h_0(1)\sin(\alpha) + h_0(L-4)\cos(\alpha), \\ &-h_0(0)\sin(\alpha) + h_0(L-3)\cos(\alpha),\ h_0(L-2)\cos(\alpha),\ h_0(L-1)\cos(\alpha)] \quad . \end{aligned} \tag{35}$$

[2]To shorten the notation, in the following equations angle α_{L+1} is denoted as α.

while coefficients of $H_0^{(L+2)}$ take the following form:

$$\begin{aligned}
H_0^{(L+2)} = [&h_1(0)\cos(\alpha)\,,\ h_1(1)\cos(\alpha)\,,\ h_1(2)\cos(\alpha) + h_0(0)\sin(\alpha)\,,\\
&h_1(3)\cos(\alpha) + h_0(1)\sin(\alpha)\,,\ \ldots\ ,\ h_1(L-2)\cos(\alpha) + h_0(L-4)\sin(\alpha)\,,\\
&h_1(L-1)\cos(\alpha) + h_0(L-3)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ h_0(L-1)\sin(\alpha)] \stackrel{((34))}{=}\\
= [&-h_0(L-1)\cos(\alpha)\,,\ h_0(L-2)\cos(\alpha)\,,\ -h_0(L-3)\cos(\alpha) + h_0(0)\sin(\alpha)\,,\\
&h_0(L-4)\cos(\alpha) + h_0(1)\sin(\alpha)\,,\ \ldots\ ,\ -h_0(1)\cos(\alpha) + h_0(L-4)\sin(\alpha)\,,\\
&h_0(0)\cos(\alpha) + h_0(L-3)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ h_0(L-1)\sin(\alpha)]\ .
\end{aligned} \tag{36}$$

Let us reverse order of $H_0^{(L+2)}$ coefficients, given by Equation (36):

$$\begin{aligned}
[&h_0(L-1)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ h_0(0)\cos(\alpha) + h_0(L-3)\sin(\alpha)\,,\\
&-h_0(1)\cos(\alpha) + h_0(L-4)\sin(\alpha)\,,\ \ldots\ ,\ h_0(L-4)\cos(\alpha) + h_0(1)\sin(\alpha)\,,\\
&-h_0(L-3)\cos(\alpha) + h_0(0)\sin(\alpha)\,,\ h_0(L-2)\cos(\alpha)\,,\ -h_0(L-1)\cos(\alpha)]\ .
\end{aligned} \tag{37}$$

Let us change the sign of even coefficients[3] of the filter given by the Equation (37):

$$\begin{aligned}
[&h_0(L-1)\sin(\alpha)\,,\ -h_0(L-2)\sin(\alpha)\,,\ h_0(0)\cos(\alpha) + h_0(L-3)\sin(\alpha)\,,\\
&h_0(1)\cos(\alpha) - h_0(L-4)\sin(\alpha)\,,\ \ldots\ ,\ h_0(L-4)\cos(\alpha) + h_0(1)\sin(\alpha)\,,\\
&h_0(L-3)\cos(\alpha) - h_0(0)\sin(\alpha)\,,\ h_0(L-2)\cos(\alpha)\,,\ h_0(L-1)\cos(\alpha)]\ .
\end{aligned} \tag{38}$$

It can be seen, that coefficients given by the Equations (35) and (38) are identical. Therefore assumption that filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil condition (12) has led to a conclusion, that filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ fulfil condition (13). Thus, the second claim has been proved.

Let us now prove the third claim. We assume that Equation (13) holds true for filters of length L. This means, that there's a following dependency between the $h_0(n)$ and $h_1(n)$ coefficients:

[3]It must be noted, that even coefficients are indexed be odd numbers. The first filter coefficient has index 0, the second coefficient has index 1 and so on.

$$\begin{aligned}
h_1(0) &= h_0(L-1) \quad , \\
h_1(1) &= -h_0(L-2) \quad , \\
h_1(2) &= h_0(L-3) \quad , \\
&\vdots \\
h_1(L-3) &= -h_0(2) \quad , \\
h_1(L-2) &= h_0(1) \quad , \\
h_1(L-1) &= -h_0(0) \quad .
\end{aligned} \tag{39}$$

It will be demonstrated, that assuming the inductive hypothesis (39) and using the Equation (8) leads to obtaining filter coefficients that fulfil Equation (12). The coefficients of $H_1^{(L+2)}$ take form[4]:

$$\begin{aligned}
H_1^{(L+2)} &= [-h_1(0)\sin(\alpha)\,,\ -h_1(1)\sin(\alpha)\,,\ -h_1(2)\sin(\alpha) + h_0(0)\cos(\alpha)\,, \\
&-h_1(3)\sin(\alpha) + h_0(1)\cos(\alpha)\,,\ \ldots\ ,\ -h_1(L-2)\sin(\alpha) + h_0(L-4)\cos(\alpha)\,, \\
&-h_1(L-1)\sin(\alpha) + h_0(L-3)\cos(\alpha)\,,\ h_0(L-2)\cos(\alpha)\,,\ h_0(L-1)\cos(\alpha)] \overset{((39))}{=} \\
&= [-h_0(L-1)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ -h_0(L-3)\sin(\alpha) + h_0(0)\cos(\alpha)\,, \\
&h_0(L-4)\sin(\alpha) + h_0(1)\cos(\alpha)\,,\ \ldots\ ,\ -h_0(1)\sin(\alpha) + h_0(L-4)\cos(\alpha)\,, \\
&h_0(0)\sin(\alpha) + h_0(L-3)\cos(\alpha)\,,\ h_0(L-2)\cos(\alpha)\,,\ h_0(L-1)\cos(\alpha)] \quad .
\end{aligned} \tag{40}$$

while coefficients of $H_0^{(L+2)}$ take the following form:

$$\begin{aligned}
H_0^{(L+2)} &= [h_1(0)\cos(\alpha)\,,\ h_1(1)\cos(\alpha)\,,\ h_1(2)\cos(\alpha) + h_0(0)\sin(\alpha)\,, \\
&h_1(3)\cos(\alpha) + h_0(1)\sin(\alpha)\,,\ \ldots\ ,\ h_1(L-2)\cos(\alpha) + h_0(L-4)\sin(\alpha)\,, \\
&h_1(L-1)\cos(\alpha) + h_0(L-3)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ h_0(L-1)\sin(\alpha)] \overset{((39))}{=} \\
&= [h_0(L-1)\cos(\alpha)\,,\ -h_0(L-2)\cos(\alpha)\,,\ h_0(L-3)\cos(\alpha) + h_0(0)\sin(\alpha)\,, \\
&-h_0(L-4)\cos(\alpha) + h_0(1)\sin(\alpha)\,,\ \ldots\ ,\ h_0(1)\cos(\alpha) + h_0(L-4)\sin(\alpha)\,, \\
&-h_0(0)\cos(\alpha) + h_0(L-3)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ h_0(L-1)\sin(\alpha)] \quad .
\end{aligned} \tag{41}$$

Let us reverse order of $H_0^{(L+2)}$ coefficients, given by Equation (41):

[4] To shorten the notation, in the following equations angle α_{L+1} is denoted as α.

$$\begin{gathered}
[h_0(L-1)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ -h_0(0)\cos(\alpha)+h_0(L-3)\sin(\alpha)\,,\\
h_0(1)\cos(\alpha)+h_0(L-4)\sin(\alpha)\,,\ \ldots\ ,\ -h_0(L-4)\cos(\alpha)+h_0(1)\sin(\alpha)\,,\\
h_0(L-3)\cos(\alpha)+h_0(0)\sin(\alpha)\,,\ -h_0(L-2)\cos(\alpha)\,,\ h_0(L-1)\cos(\alpha)]\ .
\end{gathered} \tag{42}$$

Let us change the sign of odd coefficients[5] of the filter given by the Equation (42):

$$\begin{gathered}
[-h_0(L-1)\sin(\alpha)\,,\ h_0(L-2)\sin(\alpha)\,,\ h_0(0)\cos(\alpha)-h_0(L-3)\sin(\alpha)\,,\\
h_0(1)\cos(\alpha)+h_0(L-4)\sin(\alpha)\,,\ \ldots\ ,\ h_0(L-4)\cos(\alpha)-h_0(1)\sin(\alpha)\,,\\
h_0(L-3)\cos(\alpha)+h_0(0)\sin(\alpha)\,,\ h_0(L-2)\cos(\alpha)\,,\ h_0(L-1)\cos(\alpha)]\ .
\end{gathered} \tag{43}$$

It can be seen, that coefficients given by the Equations (40) and (43) are identical. Therefore assumption that filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil condition (13) has led to a conclusion, that filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ fulfil condition (12). Earlier it has been proved that if filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil condition (12) then filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ fulfil condition (13). It has also been shown that filters $H_0^{(2)}$ and $H_1^{(2)}$ fulfil Equation (12). Thus, by the power of mathematical induction, it has been proved that filters $H_0^{(L)}$ and $H_1^{(L)}$, created according to algorithm given by Equation (8), fulfil either Equation (12) (when $\frac{L}{2}$ is odd) or Equation (13) (when $\frac{L}{2}$ is even).

Let us now prove the orthogonality of the filter H_0 and shifts by 2 of the filter H_1 (Equation (16)). For a lattice structure with even number of layers, i.e. when filters H_0 i H_1 satisfy Equation (12), the proof is identical as for the lattice structure with symmetric base operations (see [1], equations 81 through 85) and will not be repeated here. For a lattice structure with odd number of layers we expand the Equation (16):

$$\begin{gathered}
\sum_n h_0(n)h_1(n+2m) \stackrel{((12))}{=} \sum_n h_0(n)(-1)^{n+2m+1}h_0(L-(n+2m)-1) = \\
\sum_{n=1,3,\ldots}^{L-1} h_0(n)h_0(L-n-2m-1) - \sum_{n=0,2,\ldots}^{L-2} h_0(n)h_0(L-n-2m-1)\ .
\end{gathered} \tag{44}$$

Since the above formula must be equal to 0, the sufficient condition for its fulfilment is:

[5] See footnote 3 on page 133.

$$\sum_{n=0,2,\ldots}^{L-2} h_0(n)h_0(L-n-2m-1) = \sum_{n=1,3,\ldots}^{L-1} h_0(n)h_0(L-n-2m-1) \ . \tag{45}$$

This gives us Equation identical to Equation 82 in [1]. Further calculations are identical and therefore will not be repeated here. Proofs of Equations (14), (15), (17) and (18) are also the same as for a lattice structure with symmetric base operations, hence they will not be repeated.

□

3. Lattice structure representation ensuring required sum of angles

As was demonstrated in Section 2 and in [1], the sum of angles in the lattice structure must be equal to a value dependant on the number of layers. Otherwise, the structure implements a Quadrature Mirror Filter bank, which ensures perfect reconstruction of the signal but is not a wavelet transform [2, 3]. In this section we will discuss representing the lattice structure in a form that ensures fulfilment of the condition imposed on the angles. Let us recall that for a lattice structure with symmetric base operations (see Equation (5)) the required condition for the sum of angles is given as:

$$\exists_{n\in\mathbb{Z}} \quad \sum_{l=1}^{L/2} \alpha_l = \frac{\pi}{4} + \left[\left(\frac{L}{2}-1\right) \bmod 4\right] \cdot \frac{\pi}{2} + 2n\pi \ , \tag{46}$$

while for a lattice structure with asymmetric base operations (see Equation (6)) the condition is given by Equation (10). Let us denote the required sum of angles as ϑ. Instead of representing the lattice structure with angles α_i, where $i = 1, \ldots, \frac{L}{2}$, we will use representation depending on angles φ_i, where $i = 1, \ldots, \frac{L}{2} - 1$. The dependency between the angles is defined as [2]:

$$\begin{aligned} &\alpha_1 = \vartheta - \varphi_1 \ , \\ &\alpha_i = (-1)^i(\varphi_{i-1} + \varphi_i) \ , \text{ for } i = 2, \ldots, \frac{L}{2} - 1 \ , \\ &\alpha_{\frac{L}{2}} = \varphi_{\frac{L}{2}-1} \ . \end{aligned} \tag{47}$$

Such a substitution ensures that the sum of angles α_i always equals ϑ:

$$\sum_{i=1}^{L/2} \alpha_i = \vartheta - \varphi_1 + \varphi_1 + \varphi_2 - \varphi_2 + \ldots - \varphi_{\frac{L}{2}-1} + \varphi_{\frac{L}{2}-1} = \vartheta \quad . \tag{48}$$

This parametrization reduces the number of available degrees of freedom by one. Let us analyze influence of introducing substitution (47) on the class of wavelets possible to synthesize using the lattice structure. Let us assume that one of the α_i angles is increased by π:

$$\alpha'_i = \alpha_i + \pi \quad . \tag{49}$$

It is obvious that:

$$\begin{aligned} \sin\left(\alpha'_i\right) &= -\sin\left(\alpha_i\right) \quad , \\ \cos\left(\alpha'_i\right) &= -\cos\left(\alpha_i\right) \quad . \end{aligned} \tag{50}$$

Therefore, increasing the value of any angle in the lattice structure by π results in changing the sign of a base operation corresponding to that angle[6]:

$$\begin{aligned} \begin{bmatrix} cos(\alpha'_i) & sin(\alpha'_i) \\ sin(\alpha'_i) & -cos(\alpha'_i) \end{bmatrix} &= \begin{bmatrix} -cos(\alpha_i) & -sin(\alpha_i) \\ -sin(\alpha_i) & cos(\alpha_i) \end{bmatrix} = \\ &= -1 \cdot \begin{bmatrix} cos(\alpha_i) & sin(\alpha_i) \\ sin(\alpha_i) & -cos(\alpha_i) \end{bmatrix} \quad . \end{aligned} \tag{51}$$

By analyzing Equation (8) it follows that changing the sign of one base operation leads to reversing the sign of all coefficients of both filters implemented by the structure. This in turn leads to reversing the sign of wavelet coefficients calculated by the lattice structure. From a practical point of view this is irrelevant, because in all the signal processing applications in the DWT domain, e.g. compression or watermark embedding, only the absolute value of the coefficients is relevant.

We will now analyze the same situation in case of angles φ_i, where $i = 1, \ldots, \frac{L}{2} - 1$. Let us assume that one of the angles is increased by π:

$$\varphi'_i = \varphi_i + \pi \quad . \tag{52}$$

[6]The same is true when the value of an angle is decreased by π.

According to Equation (47), every φ_i angle influences the value of two α_i angles in the adjacent layers. Therefore changing φ_i by π either increases or decreases two α_i angles, which leads to changing the sign of base operations in two layers. As was already said, changing the sign of one base operations in one layer changes the sign of the filters implemented by the structure. Hence, changing the sign of base operations in two layers has no effect, since the two negations cancel each other. Therefore substitution (52) influences neither the values nor the sign of filters implemented by the structure. This leads to two important conclusions:

1. Since replacing the value of angle φ_i by $\varphi_i + \pi$ leads to exactly the same filter coefficients, then the range of φ_i angles can be limited to $[0, \pi)$.

2. Structure defined by $\frac{L}{2}$ angles $\alpha_i \in [0, 2\pi)$ is able to implement a class of filters with the same absolute coefficient values but with opposite signs. By introducing lattice structure representation with angles $\varphi_i \in [0, \pi)$, possibility of implementing such filters by the structure has been eliminated. From the practical point of view this does not limit the amount of filters possible to synthesize, since only the absolute value of the wavelet coefficients is important.

4. Summary

This paper concludes the analysis of properties of the orthogonal lattice structure. The first part of the article introduced the concept of two-point base operations, symmetric and asymmetric orthogonal base operations. Definition of lattice structure was given. Algorithm for converting parameters of such structure to wavelet filter bank coefficients was constructed in both iterative and recursive form. Conditions imposed on an orthogonal wavelet filter bank have been defined. A proof has been carried out, showing that parametrization based on an orthogonal lattice structure with symmetric base operations fulfils conditions imposed on the orthogonal wavelet filter bank, assuming that additional condition is fulfilled.

In the second part of this paper orthogonal lattice structure with asymmetric base operations was discussed. A proof was carried out, demonstrating that such a structure also fulfils conditions imposed on an orthogonal wavelet filter bank. It was noted, that such structure differs slightly from the structure with symmetric base operations: condition on the sum of angles in the structure is different as well as the dependency between low-pass and high-pass filter coefficients. Representation of the lattice structure ensuring fulfilment of conditions imposed on the angles was discussed. It was demonstrated that such representation allows to reduce the range of angles from $[0, 2\pi)$ to $[0, \pi)$ and that it eliminates the class of filters with opposite signs of the coefficients.

References

[1] Stolarek, J., *On properties of a lattice structure for a wavelet filter bank implementation: Part I*, Journal of Applied Computer Science, Vol. 19, No. 1, 2011.

[2] Rieder, P., Götze, J., Nossek, J. S., and Burrus, C. S., *Parameterization of orthogonal wavelet transforms and their implementation*, IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing, Vol. 45, No. 2, Feb. 1998, pp. 217–226.

[3] Villasenor, J., Belzer, B., and Liao, J., *Wavelet filter evaluation for image compression*, IEEE Transactions on Image Processing, Vol. 4, No. 8, Aug. 1995, pp. 1053–1060.