JOURNAL OF APPLIED
COMPUTER SCIENCE
Vol. 19 No. 1 (2011), pp. 85–116

# On Properties of a Lattice Structure for a Wavelet Filter Bank Implementation: Part I

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**Abstract.** *This paper presents concept of a lattice structure for parametrization and implementation of a Discrete Wavelet Transform. Theoretical properties of the lattice structure are discussed in detail. An algorithm for converting the lattice structure to a wavelet filter bank coefficients is constructed. A theoretical proof demonstrating that filters implemented by the lattice structure fulfil conditions imposed on an orthogonal wavelet filter bank is conducted.*
**Keywords:** *wavelet transform, filter parametrization, lattice structure.*

## 1. Introduction

During the last two decades Discrete Wavelet Transform (DWT) became one of the most popular tools in the area of signal processing. Unlike other linear transform – Fourier, Cosine or Hartley – the wavelet transform isn't based on one strictly defined set of basis functions. Instead, a set of conditions that must be fulfilled by a wavelet filter bank is given. These conditions bind some degrees of freedom. The remaining degrees of freedom allow to adapt wavelet in order to achieve some desired properties (e.g. maximal smoothness). To perform wavelet adaptation in an effective manner, a wavelet filter bank parametrization must be defined. Many such parametrizations have been proposed so far [1, 2, 3, 4, 5, 6]. Among them are the parametrizations based on lattice filters [7, 8] that allow both fast implementation of the wavelet transform as well as effective manipulation of the wavelet by adjustment of parameters. In [9] a new approach to wavelet filter parametrization based on a lattice structure was introduced. In this paper, theoretical properties of that parametrization are discussed in detail.

This paper is organized as follows. Section 2 presents theory behind the discrete wavelet transform. Traditional approach to wavelet filtration is discussed and the conditions imposed on an orthogonal wavelet filter bank are given. Section 3 introduces the concept of a two-point base operation and defines the lattice structure used for wavelet parametrization. In section 4 an algorithm for converting lattice structure's parameters to wavelet filter coefficients is constructed. In section 5 this algorithm is used to conduct a theoretical proof, demonstrating that filter bank implemented by the lattice structure fulfils conditions defined in section 2.

## 2. Orthogonal wavelet filter banks

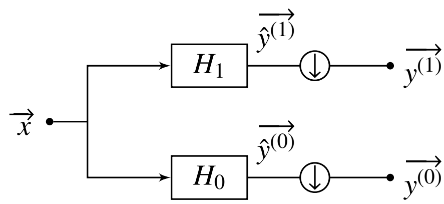



Figure 1. Filter bank used for signal analysis

Wavelet filter bank used for signal analysis is composed of a pair of filters $H_0$ and $H_1$, as shown in Figure 1. Both of these are Finite Impulse Response (FIR) filters. Upper index will be used to denote the length of the impulse response, e.g. $H_0^{(L)}$, $H_1^{(L)}$. Impulse response coefficients of the low-pass analysis filter $H_0^{(L)}$ are denoted as $(h_0(0), h_0(1), \ldots, h_0(L))$, while impulse response coefficients of the high-pass analysis filter $H_1^{(L)}$ are denoted as $(h_1(0), h_1(1), \ldots, h_1(L))$. Output signals from the filters ($\overrightarrow{\hat{y}^{(0)}}$ and $\overrightarrow{\hat{y}^{(1)}}$ in Figure 1) have the same length as the input signal $\overrightarrow{x}$. In order to maintain the number of samples required to represent the signal, every other sample of $\overrightarrow{\hat{y}^{(0)}}$ and $\overrightarrow{\hat{y}^{(1)}}$ is removed. This is known as the decimation and is represented using the ⓓ symbol in Figure 1. To reconstruct the signal, synthesis filter bank is created as shown in Figure 2. $F_0$ is the low-pass reconstruction filter and $F_1$ is the high-pass reconstruction filter. ⓤ symbol represents signal expansion, i.e. inserting zero between each two samples of the input signal. Filters $H_0$, $H_1$, $F_0$ and $F_1$ must be designed in such a way, that they ensure perfect reconstruction of a signal, i.e. signals $\overrightarrow{x}$ and $\overrightarrow{\hat{x}}$ are identical with respect to signal delay.

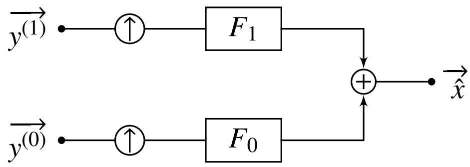



Figure 2. Filter bank used for signal synthesis

To allow perfect reconstruction of the input signal, filter $H_0$ must fulfil the following condition[1]:

$$|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 2 \ , \tag{1}$$

where $H_0(\omega)$ is the Fourier transform of the $H_0$ filter. If the filters $H_0$ and $H_1$ satisfy

$$|H_0(\omega)|^2 + |H_1(\omega)|^2 = 2 \ , \tag{2}$$

then together they form an orthogonal filter bank. It can be demonstrated that Equation (1) holds true if:

$$\sum_{n=0}^{L-1} h_0(n) \cdot h_0(n + 2m) = \delta(m) \ . \tag{3}$$

By substituting $m = 0$ to Equation (3) we get:

$$\sum_{n=0}^{L-1} h_0(n)^2 = 1 \ , \tag{4}$$

which implies that the low-pass filter must have unit norm. For $m \neq 0$ we get:

[1] All the theory in the remaining part of this section is taken from [10, 11, 12].

$$\sum_{n=0}^{L-1} h_0(n) \cdot h_0(n+2m) = 0 \quad , \tag{5}$$

which means that filter shifts by 2 must be orthogonal to the original filter. Using Equation (5) it can also be proved, that $H_0$ impulse response must have even length. Using the dilation equation:

$$\phi(x) = \sqrt{2} \sum_{n=0}^{L-1} h_0(n) \cdot \phi(2x-n) \quad , \tag{6}$$

where $\phi(x)$ is the scaling function with unit norm, it can demonstrated that equation

$$\sum_{n=0}^{L-1} h_0(n) = \sqrt{2} \tag{7}$$

holds true. If the filter fulfils the above equation, then it passes the constant component of the signal and is therefore a low-pass filter. If, in addition, such filter is to attenuate the component with the pulsation $\pi$, the following condition must hold true:

$$\sum_{n=0}^{L-1} (-1)^n h_0(n) = 0 \quad . \tag{8}$$

Based on the wavelet equation

$$\psi(x) = \sqrt{2} \sum_{n=0}^{L-1} h_1(n) \cdot \phi(2x-n) \quad , \tag{9}$$

where $\psi(x)$ is the wavelet function, it can be demonstrated, that the high-pass filter coefficients must fulfil:

$$\sum_{n=0}^{L-1} h_1(n) = 0 \quad . \tag{10}$$

This equation always holds true for the high-pass filters that have been created by inverting the order of low-pass filter coefficients and modulating them with signal $(-1)^n = e^{\hat{\imath}\omega\pi}$:

$$h_1(n) = \pm(-1)^n h_0(L-n-1) \quad , \tag{11}$$

where $n = 0, \ldots, L-1$ and $L$ is the length of the filter's impulse response. Modulating the coefficients of the low-pass filter using signal $(-1)^n$ shifts the middle pulsation of the filter from 0 to $\pi$, which makes the modulated filter a high-pass one.

It is required, that low-pass and high-pass filters are orthogonal to each other:

$$\sum_{n=0}^{L-1} h_0(n) h_1(n+2m) = 0 \ . \tag{12}$$

If the $H_0$ filter fulfils condition 3 and $H_1$ filter is constructed using Equation (11), then conditions 10 and 12 automatically hold true. This will be demonstrated in section 5.

To sum up, in the literature it has been shown that designing an orthogonal wavelet filter bank can be reduced to determining the coefficients of the impulse response $h_0(n)$, that fulfil the conditions:

$$\begin{cases} \sum_{n=0}^{L-1} h_0(n) = \sqrt{2} \\ \sum_{n=0}^{L-1} h_0(n) h_0(n+2m) = \delta(m) \quad \text{for } m = 0, 1, 2, \ldots, \frac{L}{2} - 1 \end{cases} . \tag{13}$$

Above is the system of $\frac{L}{2} + 1$ equations with $L$ variables. This implies, that for $L > 2$, where $L$ is even, there are more variables than equations, leading to $\frac{L}{2} - 1$ degrees of freedom. These degrees of freedom can be used to adapt filter's properties. In the most widely spread Daubechies wavelets [13] they are used to ensure maximal smoothness of the wavelet, but other adaptation criteria can be given as well [14, 15].

## 3. The lattice structure for wavelet filter bank implementation

In this section wavelet parametrization based on the lattice structure is presented. This parametrization allows to effectively manipulate the degrees of freedom that are available in the filter synthesis process.

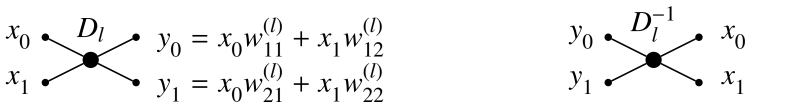
## 3.1. Two-point base operation

The lattice structure is based on two-point base operations, meaning that such operations have two inputs and two outputs. Parameters of the base operation can be represented using a matrix:

$$D_l = \begin{bmatrix} w_{11}^{(l)} & w_{12}^{(l)} \\ w_{21}^{(l)} & w_{22}^{(l)} \end{bmatrix} , \tag{14}$$

where $l$ represents the index of the operation, $w_{11}^{(l)}$, $w_{12}^{(l)}$, $w_{21}^{(l)}$ and $w_{22}^{(l)}$ represent the parameters (weights). The following matrix equation represents the concept of a two-point base operation:

$$Y = D_l \cdot X \ , \quad \text{where } X = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \quad Y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \ . \tag{15}$$




a) The forward base operation b) The inverse base operation

Figure 3. Two-point base operations

Figure 3.a shows the forward base operation. To reconstruct the original matrix $X$ using the transformed matrix $Y$, the base operation $D_l$ must be invertible, which means that the determinant of the matrix given by Equation (14) must be different than zero:

$$w_{11}^{(l)} w_{22}^{(l)} - w_{12}^{(l)} w_{21}^{(l)} \neq 0 \ . \tag{16}$$

This implies, that there exists inverse matrix $D_l^{-1}$, such that equation

$$D_l^{-1} D_l = I \tag{17}$$

holds true, where $I$ is the identity matrix. The inverse two-point base operation is shown in Figure 3.b.

### 3.2. Orthogonal two-point base operation

Let us consider a case, in which the matrix $D_l$, given by Equation (14), is an orthogonal matrix. This implies, that the inverse of the $D_l$ matrix is equal to its transpose $D_l^T$:

$$D_l^T D_l = I \ . \tag{18}$$

The following conditions must hold true, in order to satisfy the Equation (18):

$$w_{11}^{(l)} w_{21}^{(l)} + w_{12}^{(l)} w_{22}^{(l)} = 0 \ , \tag{19}$$

$$(w_{11}^{(l)})^2 + (w_{12}^{(l)})^2 = 1 \ . \tag{20}$$

Using one of the substitutions below is the sufficient condition to satisfy the Equation (19):

- $w_{21}^{(l)} = w_{12}^{(l)}$ and $w_{22}^{(l)} = -w_{11}^{(l)}$. This implies, that the base operation is symmetric:

$$P_l = P_l^T = \begin{bmatrix} w_{11}^{(l)} & w_{12}^{(l)} \\ w_{12}^{(l)} & -w_{11}^{(l)} \end{bmatrix} \ . \tag{21}$$

  If, in addition, the Equation (20) is also satisfied, then the forward base operation is identical to the inverse base operation:

$$P_l = P_l^T = P_l^{-1} \ . \tag{22}$$

- $w_{21}^{(l)} = -w_{12}^{(l)}$ and $w_{22}^{(l)} = w_{11}^{(l)}$. This substitution leads to asymmetric base operation:

$$O_l = \begin{bmatrix} w_{11}^{(l)} & w_{12}^{(l)} \\ -w_{12}^{(l)} & w_{11}^{(l)} \end{bmatrix} \ , \tag{23}$$

  If the Equation (20) is also satisfied, then the following equation holds true:

$$O_l^T = O_l^{-1} \ . \tag{24}$$

If one of the above substitutions is assumed, then the sufficient condition to fulfil Equation (20) is using the following substitution [16]:

$$\begin{aligned} w_{11}^{(l)} &= cos(\alpha_l) \ , \\ w_{12}^{(l)} &= sin(\alpha_l) \ . \end{aligned} \tag{25}$$

Equation 25, combined with earlier substitutions, leads to two basic forms of the orthogonal two-point base operation: symmetric and asymmetric. Forward and inverse symmetric orthogonal base operation is given by equation:

$$S_l = S_l^{-1} = \begin{bmatrix} cos(\alpha_l) & sin(\alpha_l) \\ sin(\alpha_l) & -cos(\alpha_l) \end{bmatrix} \ . \tag{26}$$

Forward and inverse asymmetric orthogonal base operation is given by equations:

$$F_l = \begin{bmatrix} cos(\alpha_l) & sin(\alpha_l) \\ -sin(\alpha_l) & cos(\alpha_l) \end{bmatrix} \ , \tag{27}$$

$$F_l^{-1} = \begin{bmatrix} cos(\alpha_l) & -sin(\alpha_l) \\ sin(\alpha_l) & cos(\alpha_l) \end{bmatrix} \ . \tag{28}$$

## 3.3. The lattice structure

The lattice structure is a computational scheme, that can be used for implementation of the forward and inverse discrete wavelet transform. Lattice structure for implementation of the forward wavelet transform is composed of $\frac{L}{2}$ layers, each layer containing $\frac{N}{2}$ two-point base operations[2], where $L$ and $N$ represent the length of the impulse response of the filter implemented by the structure and the length of a processed signal, respectively. A signal to be transformed is passed as an input for the first layer of the lattice structure and then processed by the subsequent layers. In each of the layers elements of the signal are processed in pairs by the $D_l$ base operations. After every layer, a cyclic shift of the output signal from that layer is performed, i.e. the lower input of the last base operation in the current layer is connected to the upper output of the first base operation in the previous layer (arrows

[2]Lattice structure based only on orthogonal base operations will be called an orthogonal lattice structure.

$s_1$ and $s_2$ in Figure 4.a). Upper outputs of the $D_l$ base operations in the last layer are called the low-pass filter outputs and the values of these outputs correspond to the low-pass wavelet coefficients. Lower outputs of the $D_l$ operations in the last layer are called the high-pass outputs, since their values correspond to high-pass wavelet coefficients. This will be proved in section 4.

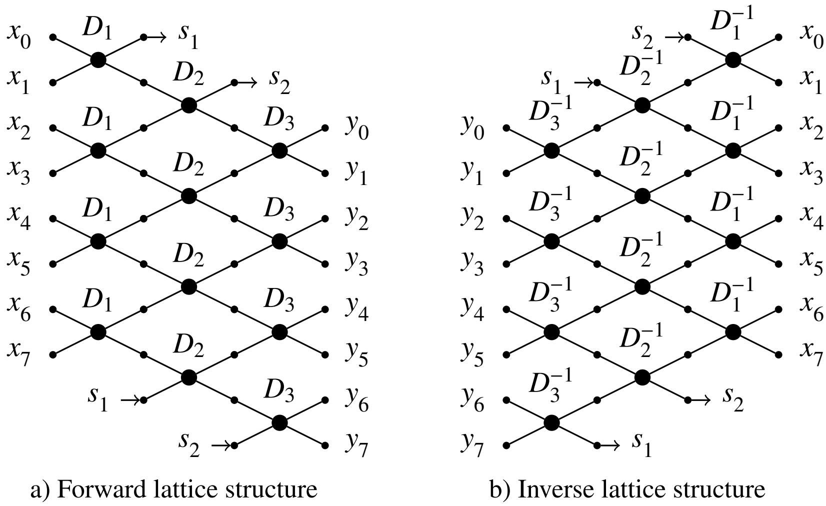



Figure 4. Forward and inverse lattice structure for implementation of a 6-tap transform of an 8-element signal

It must be noted, that implementation of a wavelet transform using the lattice structure causes cyclic shift of the output signal by $\frac{L}{2}-1$ elements. In the case of example lattice structure shown in Figure 4.a this shift equals 2, which means that input signal $(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ is transformed to $(y_6, y_7, y_0, y_1, y_2, y_3, y_4, y_5)$, where $(y_0, y_2, y_4, y_6)$ are the low-pass wavelet coefficients and $(y_1, y_3, y_5, y_7)$ are the high-pass wavelet coefficients.

Lattice structure for implementation of the inverse wavelet transform contains the same number of layers and base operations in each layer as the forward lattice structure. Base operations in the layer $l$ are constructed by inverting the base operations in the layer $L - l$ of the forward lattice structure. The inverse lattice structure is constructed as a mirror reflection of the forward lattice structure. After each layer a cyclic shift of the signal in the opposite direction is performed: upper input of the first base operation in the layer is connected to the lower output of the last base operation in the previous layer (arrows $s_1$ and $s_2$ on Figure 4.b).

# 4. Conversion of the lattice structure to a filter bank coefficients

The lattice structure presented in section 3 can be used for implementing the signal filtration using an orthogonal filter bank. In the lattice structure containing $L/2$ layers, each output depends exactly on $L$ inputs. In this section it will be demonstrated that this dependency corresponds to filtration of the input signal with a pair of filters with impulse response of length $L$. Below an algorithm for calculating coefficients of the impulse responses of the filters implemented by the lattice structure is constructed.

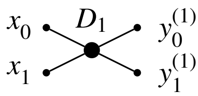



Figure 5. A fragment of a one-layer lattice structure

Let us begin by analyzing the simplest case of a one-layer lattice structure implementing a 2-tap signal transform. A fragment of such structure is presented in Figure 5. By analyzing this figure, it can be seen that dependency between outputs $y_0^{(1)}$, $y_1^{(1)}$ and inputs $x_0$, $x_1$ can be written as:

$$y_0^{(1)} = x_0 w_{11}^{(1)} + x_1 w_{12}^{(1)} \quad , \tag{29}$$

$$y_1^{(1)} = x_0 w_{21}^{(1)} + x_1 w_{22}^{(1)} \quad . \tag{30}$$

This is based directly on the definition of the two-point base operation (equation 15 and Figure 3.a). In this most elementary case, the weights of the $D_l$ base operation directly correspond to coefficients of the low-pass and high-pass filter:

$$H_0^{(2)} = [w_{11}^{(1)}, w_{12}^{(1)}] \quad , \tag{31}$$

$$H_1^{(2)} = [w_{21}^{(1)}, w_{22}^{(1)}] \quad . \tag{32}$$

Let us now analyze two-layer lattice structure implementing a 4-tap signal transform. In Figure 6 a fragment of such structure is presented, demonstrating

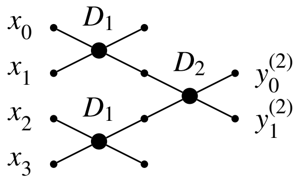



Figure 6. A fragment of a two-layer lattice structure

that values of outputs $y_0^{(2)}$ and $y_1^{(2)}$ depend on the values of four inputs: $x_0$, $x_1$, $x_2$ and $x_3$. These dependencies can be written as:

$$\begin{aligned} y_0^{(2)} &= y_1^{(1)} w_{11}^{(2)} + y_0^{(1)} w_{12}^{(2)} = \\ &= [x_0 w_{21}^{(1)} + x_1 w_{22}^{(1)}] w_{11}^{(2)} + [x_2 w_{11}^{(1)} + x_3 w_{12}^{(1)}] w_{12}^{(2)} = \\ &= x_0 w_{21}^{(1)} w_{11}^{(2)} + x_1 w_{22}^{(1)} w_{11}^{(2)} + x_2 w_{11}^{(1)} w_{12}^{(2)} + x_3 w_{12}^{(1)} w_{12}^{(2)} \ , \end{aligned} \tag{33}$$

$$\begin{aligned} y_1^{(2)} &= y_1^{(1)} w_{21}^{(2)} + y_0^{(1)} w_{22}^{(2)} = \\ &= [x_0 w_{21}^{(1)} + x_1 w_{22}^{(1)}] w_{21}^{(2)} + [x_2 w_{11}^{(1)} + x_3 w_{12}^{(1)}] w_{22}^{(2)} = \\ &= x_0 w_{21}^{(1)} w_{21}^{(2)} + x_1 w_{22}^{(1)} w_{21}^{(2)} + x_2 w_{11}^{(1)} w_{22}^{(2)} + x_3 w_{12}^{(1)} w_{22}^{(2)} \ . \end{aligned} \tag{34}$$

It must be noted that variable $y_1^{(1)}$ is substituted with exactly the same value as in Equation (30), while the substitution of variable $y_0^{(1)}$ requires change in the values of the input signal $\vec{x}$. This signal must be shifted by 2: $x_0$ is changed to $x_2$, $x_1$ is changed to $x_3$ and so on. This results from the fact, that the lower input of $D_2$ base operation in Figure 6 is connected to the upper output of $D_1$ base operation that received as an input the values of $\vec{x}$ shifted by 2. By grouping the terms in equations 33 and 34, the following low-pass and high-pass filters are obtained:

$$H_0^{(4)} = [w_{21}^{(1)} w_{11}^{(2)}, w_{22}^{(1)} w_{11}^{(2)}, w_{11}^{(1)} w_{12}^{(2)}, w_{12}^{(1)} w_{12}^{(2)}] \ , \tag{35}$$

$$H_1^{(4)} = [w_{21}^{(1)} w_{21}^{(2)}, w_{22}^{(1)} w_{21}^{(2)}, w_{11}^{(1)} w_{22}^{(2)}, w_{12}^{(1)} w_{22}^{(2)}] \ . \tag{36}$$

Let us now consider a fragment of a three-layer lattice structure, shown in Figure 7. Such a structure implements 6-tap transform of a signal. The dependencies between inputs and outputs can be written as:

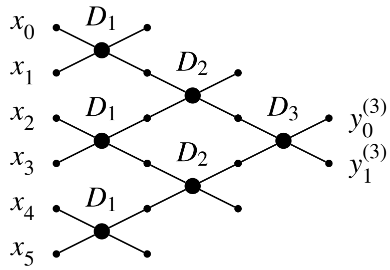



Figure 7. A fragment of a three-layer lattice structure

$$
\begin{aligned}
y_0^{(3)} &= y_1^{(2)} w_{11}^{(3)} + y_0^{(2)} w_{12}^{(3)} = \\
&= [x_0 w_{21}^{(1)} w_{21}^{(2)} + x_1 w_{22}^{(1)} w_{21}^{(2)} + x_2 w_{11}^{(1)} w_{22}^{(2)} + x_3 w_{12}^{(1)} w_{22}^{(2)}] w_{11}^{(3)} + \\
&+ [x_2 w_{21}^{(1)} w_{11}^{(2)} + x_3 w_{22}^{(1)} w_{11}^{(2)} + x_4 w_{11}^{(1)} w_{12}^{(2)} + x_5 w_{12}^{(1)} w_{12}^{(2)}] w_{12}^{(3)} = \\
&= x_0 w_{21}^{(1)} w_{21}^{(2)} w_{11}^{(3)} + x_1 w_{22}^{(1)} w_{21}^{(2)} w_{11}^{(3)} + \\
&+ x_2 [w_{11}^{(1)} w_{22}^{(2)} w_{11}^{(3)} + w_{21}^{(1)} w_{11}^{(2)} w_{12}^{(3)}] + \\
&+ x_3 [w_{12}^{(1)} w_{22}^{(2)} w_{11}^{(3)} + w_{22}^{(1)} w_{11}^{(2)} w_{12}^{(3)}] + \\
&+ x_4 w_{11}^{(1)} w_{12}^{(2)} w_{12}^{(3)} + x_5 w_{12}^{(1)} w_{12}^{(2)} w_{12}^{(3)} \quad ,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
y_1^{(3)} &= y_1^{(2)} w_{21}^{(3)} + y_0^{(2)} w_{22}^{(3)} = \\
&= [x_0 w_{21}^{(1)} w_{21}^{(2)} + x_1 w_{22}^{(1)} w_{21}^{(2)} + x_2 w_{11}^{(1)} w_{22}^{(2)} + x_3 w_{12}^{(1)} w_{22}^{(2)}] w_{21}^{(3)} + \\
&+ [x_2 w_{21}^{(1)} w_{11}^{(2)} + x_3 w_{22}^{(1)} w_{11}^{(2)} + x_4 w_{11}^{(1)} w_{12}^{(2)} + x_5 w_{12}^{(1)} w_{12}^{(2)}] w_{22}^{(3)} = \\
&= x_0 w_{21}^{(1)} w_{21}^{(2)} w_{21}^{(3)} + x_1 w_{22}^{(1)} w_{21}^{(2)} w_{21}^{(3)} + \\
&+ x_2 [w_{11}^{(1)} w_{22}^{(2)} w_{21}^{(3)} + w_{21}^{(1)} w_{11}^{(2)} w_{22}^{(3)}] + \\
&+ x_3 [w_{12}^{(1)} w_{22}^{(2)} w_{21}^{(3)} + w_{22}^{(1)} w_{11}^{(2)} w_{22}^{(3)}] + \\
&+ x_4 w_{11}^{(1)} w_{12}^{(2)} w_{22}^{(3)} + x_5 w_{12}^{(1)} w_{12}^{(2)} w_{22}^{(3)} \quad .
\end{aligned} \tag{38}
$$

When substituting value of variable $y_0^{(2)}$ into the above equations, the elements of input signal $\overrightarrow{x}$ are shifted by 2, compared to values given in Equation (33).

Equations 37 and 38 show that for a three-layer lattice structure implementing a 6-tap transform, the following low-pass and high-pass filter coefficients are obtained:

$$\begin{aligned} H_0^{(6)} &= [w_{21}^{(1)}w_{21}^{(2)}w_{11}^{(3)}, w_{22}^{(1)}w_{21}^{(2)}w_{11}^{(3)}, (w_{11}^{(1)}w_{22}^{(2)}w_{11}^{(3)} + w_{21}^{(1)}w_{11}^{(2)}w_{12}^{(3)}), \\ &\quad (w_{12}^{(1)}w_{22}^{(2)}w_{11}^{(3)} + w_{22}^{(1)}w_{11}^{(2)}w_{12}^{(3)}), w_{11}^{(1)}w_{12}^{(2)}w_{12}^{(3)}, w_{12}^{(1)}w_{12}^{(2)}w_{12}^{(3)}] \end{aligned}, \tag{39}$$

$$\begin{aligned} H_1^{(6)} &= [w_{21}^{(1)}w_{21}^{(2)}w_{21}^{(3)}, w_{22}^{(1)}w_{21}^{(2)}w_{21}^{(3)}, (w_{11}^{(1)}w_{22}^{(2)}w_{21}^{(3)} + w_{21}^{(1)}w_{11}^{(2)}w_{22}^{(3)}), \\ &\quad (w_{12}^{(1)}w_{22}^{(2)}w_{21}^{(3)} + w_{22}^{(1)}w_{11}^{(2)}w_{22}^{(3)}), w_{11}^{(1)}w_{12}^{(2)}w_{22}^{(3)}, w_{12}^{(1)}w_{12}^{(2)}w_{22}^{(3)} \end{aligned}. \tag{40}$$

Analysis of the above equations leads to a recursive dependency:

$$\begin{cases} \begin{aligned} y_0^{(1)} &= x_0 w_{11}^{(1)} + x_1 w_{12}^{(1)} \\ y_1^{(1)} &= x_0 w_{21}^{(1)} + x_1 w_{22}^{(1)} \end{aligned} & \text{for layer } l = 1 \\ \\ \begin{aligned} y_0^{(l)} &= y_1^{(l-1)} w_{11}^{(l)} + y_0^{(l-1)} w_{12}^{(l)} \\ y_1^{(l)} &= y_1^{(l-1)} w_{21}^{(l)} + y_0^{(l-1)} w_{22}^{(l)} \end{aligned} & \text{for layer } l = 2, ..., \frac{L}{2} \end{cases}. \tag{41}$$

When substituting value of variable $y_0^{(l-1)}$ into the above equations, the elements of input signal $\overrightarrow{x}$ must be shifted accordingly by 2. Moreover, the following recursive dependency can be used to calculate the coefficients of the low-pass and high-pass filters implemented by the lattice structure:

$$\begin{cases} \begin{aligned} H_0^{(2)} &= [w_{11}^{(1)}, w_{12}^{(1)}] \\ H_1^{(2)} &= [w_{21}^{(1)}, w_{22}^{(1)}] \end{aligned} & \text{for layer } l = 1 \\ \\ \begin{aligned} H_0^{(2l)} &= [H_1^{(2l-2)} w_{11}^{(l)}, 0, 0] + [0, 0, H_0^{(2l-2)} w_{12}^{(l)}] \\ H_1^{(2l)} &= [H_1^{(2l-2)} w_{21}^{(l)}, 0, 0] + [0, 0, H_0^{(2l-2)} w_{22}^{(l)}] \end{aligned} & \text{for layer } l = 2, ..., \frac{L}{2} \end{cases}. \tag{42}$$

**Algorithm 4**: Conversion of lattice structure parameters to filter coefficients

$H_0^{(2)} = [w_{11}^{(1)}, w_{12}^{(1)}]$
$H_1^{(2)} = [w_{21}^{(1)}, w_{22}^{(1)}]$
**for** $l = 2$ **to** $\frac{L}{2}$ **do**
    $H_0^{(2l)} = [H_1^{(2l-2)} w_{11}^{(l)}, 0, 0] + [0, 0, H_0^{(2l-2)} w_{12}^{(l)}]$
    $H_1^{(2l)} = [H_1^{(2l-2)} w_{21}^{(l)}, 0, 0] + [0, 0, H_0^{(2l-2)} w_{22}^{(l)}]$
**end**

Equation 42 describes a recursive algorithm for converting the lattice structure parameters to coefficients of the low-pass and high-pass filters. This algorithm can be easily converted to an iterative one – its outline is presented as Algorithm 4.

# 5. Implementation of the orthogonal filter bank using the lattice structure - symmetric base operations

In section 3.2 two types of orthogonal base operations have been defined: symmetric $S_l$ (equation 26) and asymmetric $F_l$ (equation 28). In this section it will be proved that filters implemented by the orthogonal lattice structure based on symmetric base operations fulfil conditions for an orthogonal wavelet filter bank defined in section 2. The following dependencies between the trigonometric functions will be used in the proof:

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \quad , \tag{43}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad , \tag{44}$$

$$\sin(\alpha) + \cos(\beta) = \sqrt{2}\cos\left(\frac{\pi}{4} - \alpha\right) \quad , \tag{45}$$

$$\cos(\alpha) - \sin(\beta) = \sqrt{2}\cos\left(\frac{\pi}{4} + \alpha\right) \quad . \tag{46}$$

For the symmetric base operation, the following substitutions are used:

$$\begin{array}{llll} w_{11}^{(l)} = \cos(\alpha_l) & , & w_{12}^{(l)} = \sin(\alpha_l) & , \\ w_{21}^{(l)} = \sin(\alpha_l) & , & w_{22}^{(l)} = -\cos(\alpha_l) & . \end{array} \tag{47}$$

The following notation will be used:

$$\beta_j = \sum_{l=1}^{j} \alpha_l \quad , \tag{48}$$

where $\alpha_l$ are the angles of the lattice structure, as defined in Equation (47).

Let $H_0$ and $H_1$ be a pair of filters, created according to Equation (42). For such filters it will be proved, that:

1. Equation (7) holds true if the following condition is fulfilled:

$$\exists_{n\in\mathbb{Z}} \quad \sum_{l=1}^{L/2} \alpha_l = \frac{\pi}{4} + \left[ \left( \frac{L}{2} - 1 \right) \bmod 4 \right] \cdot \frac{\pi}{2} + 2n\pi \quad . \tag{49}$$

2. Equation (10) holds true, if Equation (49) holds true.
3. Coefficients of $H_1$ satisfy

$$h_1(n) = (-1)^n h_0(L - n - 1) \quad , \tag{50}$$

   which is a precise form of Equation (11).
4. $H_0$ filter is orthogonal to its shifts by 2:

$$\sum_{n=0}^{L-1} h_0(n) h_0(n + 2m) = 0 \text{ for } m \in \mathbb{Z} \setminus \{0\} \quad . \tag{51}$$

5. $H_1$ filter is orthogonal to its shifts by 2:

$$\sum_{n=0}^{L-1} h_1(n) h_1(n + 2m) = 0 \text{ for } m \in \mathbb{Z} \setminus \{0\} \quad . \tag{52}$$

6. Equation (12) holds true, i.e. $H_0$ and $H_1$ filters are orthogonal to each other.
7. $H_0$ and $H_1$ filters have unit norm:

$$\sum_{n=0}^{L-1} h_0(n)^2 = 1 \quad , \tag{53}$$

$$\sum_{n} h_1(n)^2 = 1 \quad . \tag{54}$$

Let us begin by proving equations 7 and 10. We will analyze how does the sum of angles in the lattice structure depend on the number of layers, assuming that Equation (7) is to be fulfilled. According to Equation (42), for a one-layer lattice structure:

$$\begin{aligned} H_0^{(2)} &= [\cos(\alpha_1), \quad \sin(\alpha_1)] \ , \\ H_1^{(2)} &= [\sin(\alpha_1), -\cos(\alpha_1)] \ , \end{aligned} \tag{55}$$

Let us calculate what angle $\alpha_1$ allows to fulfil Equation (7):

$$\begin{aligned} \cos(\alpha_1) + \sin(\alpha_1) &= \sqrt{2} \\ \sqrt{2}\cos\left(\frac{\pi}{4} - \alpha_1\right) &= \sqrt{2} \\ \cos\left(\frac{\pi}{4} - \alpha_1\right) &= 1 \\ \frac{\pi}{4} - \alpha_1 &= 0 \\ \alpha_1 &= \frac{\pi}{4} \end{aligned} \tag{56}$$

Let us check, if the above angle fulfils Equation (10):

$$\sin\left(\frac{\pi}{4}\right) + \left[-\cos\left(\frac{\pi}{4}\right)\right] = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 \ . \tag{57}$$

Therefore, in order for equations 7 and 10 to hold true for a one-layer lattice structure, angle $\alpha_1$ in the matrix of base operation $S_1$ must have a value of $\frac{\pi}{4}$, which is in accordance with Equation (49).

Let us calculate the sum of angles in a two-layer lattice structure required to fulfil Equation (49). Based on Equation (42), we can write:

$$\begin{aligned} \sum_{n=0}^{3} h_0(n) &= [\sin(\alpha_1) - \cos(\alpha_1)]\cos(\alpha_2) + [\sin(\alpha_1) + \cos(\alpha_1)]\sin(\alpha_2) = \\ &= \sin(\alpha_1)\cos(\alpha_2) - \cos(\alpha_1)\cos(\alpha_2) + \sin(\alpha_1)\sin(\alpha_2) + \cos(\alpha_1)\sin(\alpha_2) = \\ &= \sin(\alpha_1 + \alpha_2) - \cos(\alpha_1 + \alpha_2) = \\ &= \sin(\beta_2) - \cos(\beta_2) \ . \end{aligned} \tag{58}$$

Let us calculate what angle $\beta_2$ allows to fulfil Equation (7):

$$\begin{aligned} \sin(\beta_2) - \cos(\beta_2) &= \sqrt{2} \\ -\sqrt{2}\cos\left(\frac{\pi}{4} + \beta_2\right) &= \sqrt{2} \\ \cos\left(\frac{\pi}{4} + \beta_2\right) &= -1 \\ \frac{\pi}{4} + \beta_2 &= \pi \\ \beta_2 &= \frac{3}{4}\pi \end{aligned} \tag{59}$$

The sum of coefficients of a 4-tap high-pass filter $H_1^{(4)}$ can be written as:

$$\begin{aligned} \sum_{n=0}^{3} h_1(n) &= [\sin(\alpha_1) - \cos(\alpha_1)]\sin(\alpha_2) - [\sin(\alpha_1) + \cos(\alpha_1)]\cos(\alpha_2) = \\ &= \sin(\alpha_1)\sin(\alpha_2) - \cos(\alpha_1)\sin(\alpha_2) - \sin(\alpha_1)\cos(\alpha_2) - \cos(\alpha_1)\cos(\alpha_2) = \\ &= -\cos(\alpha_1 + \alpha_2) - \sin(\alpha_1 + \alpha_2) = \\ &= -\cos(\beta_2) - \sin(\beta_2) \quad . \end{aligned} \tag{60}$$

By substituting $\beta_2 = \frac{3}{4}\pi$ in Equation (60) we get:

$$-\cos\left(\frac{3}{4}\pi\right) - \sin\left(\frac{3}{4}\pi\right) = -\left(-\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 \quad . \tag{61}$$

Therefore, for a two-layer lattice structure, equations 7 and 10 hold true if and only if the sum of angles is equal to $\frac{3}{4}\pi$, which stays in accordance with equation 49.

Let us analyze a three-layer lattice structure. Equations 58 and 60 show, that sum of $H_0^{(4)}$ and $H_1^{(4)}$ coefficients can be shortly written as:

$$\pm\cos(\beta_2) \pm \sin(\beta_2) \quad . \tag{62}$$

Since the multiplication distributes over addition, the above results can be used to calculate the sum of $H_0^{(6)}$ and $H_1^{(6)}$ coefficients:

$$\begin{aligned}\sum_{n=0}^{5} h_0(n) &= [-\cos(\beta_2) - \sin(\beta_2)]\cos(\alpha_3) + [\sin(\beta_2) - \cos(\beta_2)]\sin(\alpha_3) = \\ &= -\cos(\beta_2)\cos(\alpha_3) - \sin(\beta_2)\cos(\alpha_3) + \sin(\beta_2)\sin(\alpha_3) - \cos(\beta_2)\sin(\alpha_3) = \\ &= -\cos(\beta_2 + \alpha_3) - \sin(\beta_2 + \alpha_3) = \\ &= -\cos(\beta_3) - \sin(\beta_3) \quad .\end{aligned} \tag{63}$$

Let us determine the angle $\beta_3$ that allows to fulfil Equation (7):

$$\begin{aligned}-\cos(\beta_3) - \sin(\beta_3) &= \sqrt{2} \\ -\sqrt{2}\cos\left(\frac{\pi}{4} - \beta_3\right) &= \sqrt{2} \\ \cos\left(\frac{\pi}{4} - \beta_3\right) &= -1 \\ \frac{\pi}{4} - \beta_3 &= \pi \\ \beta_3 &= -\frac{3}{4}\pi\end{aligned} \tag{64}$$

Due to periodicity of cos(·) function, we can write:

$$\beta_3 = \frac{5}{4}\pi \quad . \tag{65}$$

The sum of $H_1^{(6)}$ high-pass filter is given as:

$$\begin{aligned}\sum_{n=0}^{5} h_1(n) &= [-\cos(\beta_2) - \sin(\beta_2)]\sin(\alpha_3) - [\sin(\beta_2) - \cos(\beta_2)]\cos(\alpha_3) = \\ &= -\cos(\beta_2)\sin(\alpha_3) - \sin(\beta_2)\sin(\alpha_3) - \sin(\beta_2)\cos(\alpha_3) + \cos(\beta_2)\cos(\alpha_3) = \\ &= -\sin(\beta_2 + \alpha_3) + \cos(\beta_2 + \alpha_3) = \\ &= -\sin(\beta_3) + \cos(\beta_3) \quad .\end{aligned} \tag{66}$$

By substituting $\beta_3 = \frac{5}{4}\pi$ into Equation (66), we obtain:

$$-\sin\left(\frac{5}{4}\pi\right) + \cos\left(\frac{5}{4}\pi\right) = -\left(-\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 \quad . \tag{67}$$

Therefore, for a three-layer lattice structure, equations 7 and 10 hold true if and only if the sum of angles is equal to $\frac{5}{4}\pi$, which stays in accordance with equation 49.

Let us analyze a four-layer lattice structure. Using the same method as earlier, we substitute the sum of $H_0^{(6)}$ and $H_1^{(6)}$ coefficients, given by equations 63 and 66, into the Equation (42), which allows to calculate the sum of $H_0^{(8)}$ filter coefficients:

$$\begin{aligned}\sum_{n=0}^{7} h_0(n) = [-\sin(\beta_3) + \cos(\beta_3)]\cos(\alpha_4) + [-\cos(\beta_3) - \sin(\beta_3)]\sin(\alpha_4) = \\ = -\sin(\beta_3)\cos(\alpha_4) + \cos(\beta_3)\cos(\alpha_4) - \cos(\beta_3)\sin(\alpha_4) - \sin(\beta_3)\sin(\alpha_4) = \\ = -\sin(\beta_3 + \alpha_4) + \cos(\beta_3 + \alpha_4) = \\ = -\sin(\beta_4) + \cos(\beta_4) \quad .\end{aligned} \tag{68}$$

Let us determine the angle $\beta_4$ that allows to fulfil Equation (7):

$$\begin{aligned}-\sin(\beta_4) + \cos(\beta_4) &= \sqrt{2} \\ \sqrt{2}\cos\left(\frac{\pi}{4} + \beta_4\right) &= \sqrt{2} \\ \cos\left(\frac{\pi}{4} + \beta_4\right) &= 1 \\ \frac{\pi}{4} + \beta_4 &= 0 \\ \beta_4 &= -\frac{\pi}{4}\end{aligned} \tag{69}$$

Due to periodicity of cos(·) function, we can write:

$$\beta_4 = \frac{7}{4}\pi \quad . \tag{70}$$

The sum of $H_1^{(8)}$ high-pass filter coefficients is given as:

$$\begin{aligned}\sum_{n=0}^{7} h_1(n) = [-\sin(\beta_3) + \cos(\beta_3)]\sin(\alpha_4) - [-\cos(\beta_3) - \sin(\beta_3)]\cos(\alpha_4) = \\ = -\sin(\beta_3)\sin(\alpha_4) + \cos(\beta_3)\sin(\alpha_4) + \cos(\beta_3)\cos(\alpha_4) + \sin(\beta_3)\cos(\alpha_4) = \\ = \cos(\beta_3 + \alpha_4) + \sin(\beta_3 + \alpha_4) = \\ = \cos(\beta_4) + \sin(\beta_4) \quad .\end{aligned} \tag{71}$$

By substituting $\beta_4 = \frac{7}{4}\pi$ into the Equation (71), we obtain:

$$\cos\left(\frac{7}{4}\pi\right) + \sin\left(\frac{7}{4}\pi\right) = \left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0 \ . \tag{72}$$

Therefore, for a four-layer lattice structure, equations 7 and 10 hold true if and only if the sum of angles is equal to $\frac{7}{4}\pi$, which stays in accordance with equation 49.

Let us analyze a five-layer lattice structure. We substitute the sum of $H_0^{(8)}$ and $H_1^{(8)}$ coefficients, given by equations 68 and 71, into the Equation (42), which allows to calculate the sum of $H_0^{(10)}$ filter coefficients:

$$\begin{aligned} \sum_{n=0}^{9} h_0(n) &= [\cos(\beta_4) + \sin(\beta_4)]\cos(\alpha_5) + [\cos(\beta_4) - \sin(\beta_4)]\sin(\alpha_5) = \\ &= \cos(\beta_4)\cos(\alpha 5) + \sin(\beta_4)\cos(\alpha_5) + \cos(\beta_4)\sin(\alpha_5) - \sin(\beta_4)\sin(\alpha_5) = \\ &= \cos(\beta_4 + \alpha_5) + \sin(\beta_4 + \alpha_5) = \\ &= \cos(\beta_5) + \sin(\beta_5) \ . \end{aligned} \tag{73}$$

We notice, that the sum of coefficients of a $H_0^{(10)}$ low-pass filter is given by the same formula as for a one-layer lattice structure. This implies, that for a five-layer lattice structure Equation (7) is fulfilled for angle $\beta_5 = \beta_1 = \frac{\pi}{4}$. Let us now calculate the sum of coefficients of a high-pass filter $H_1^{(10)}$:

$$\begin{aligned} \sum_{n=0}^{9} h_1(n) &= [\cos(\beta_4) + \sin(\beta_4)]\sin(\alpha_5) - [\cos(\beta_4) - \sin(\beta_4)]\cos(\alpha_5) = \\ &= \cos(\beta_4)\sin(\alpha_5) + \sin(\beta_4)\sin(\alpha_5) - \cos(\beta_4)\cos(\alpha_5) + \sin(\beta_4)\cos(\alpha_5) = \\ &= \sin(\beta_4 + \alpha_5) - \cos(\beta_4 + \alpha_5) = \\ &= \sin(\beta_5) - \cos(\beta_5) \ . \end{aligned} \tag{74}$$

We notice, that the sum of coefficients of a $H_1^{(10)}$ high-pass filter is given by the same formula as for a one-layer lattice structure. This implies, that for a five-layer lattice structure Equation (10) is fulfilled for angle $\beta_5 = \beta_1 = \frac{\pi}{4}$.

For a five-layer lattice structure, the sum of filter coefficients is given by the same formula as for a one-layer lattice structure. Therefore adding more layers to the structure will produce the same results as the ones obtained above for a one, two, three and four-layer structure. Hence it has been proved, that the low-pass and high-pass filters implemented by the orthogonal lattice structure based on the symmetric base operations fulfil Equation (7) and 10, assuming that condition 49 holds true.

The Equation (50) will be proved using the mathematical induction method. This requires proving two claims:

1. filters $H_0^{(2)}$ and $H_1^{(2)}$ fulfil Equation (50),
2. the fact that filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil Equation (50) leads to conclusions, that filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ also fulfil that equation.

Let us begin by proving first claim. According to equations 42 and 47, filters $H_0^{(2)}$ and $H_1^{(2)}$ take form:

$$H_0^{(2)} = [\cos(\alpha_1), \quad \sin(\alpha_1)] \ , \tag{75}$$

$$H_1^{(2)} = [\sin(\alpha_1), -\cos(\alpha_1)] \ . \tag{76}$$

It can be easily noticed, that coefficients of $H_1^{(2)}$ filter are created by reversing the order of $H_0^{(2)}$ filter coefficients and changing the sign of every even coefficient:

$$\begin{aligned} h_1(0) &= (-1)^0 h_0(2-0-1) = \phantom{-}h_0(1) = \phantom{-}\sin(\alpha_1) \ , \\ h_1(1) &= (-1)^1 h_0(2-1-1) = -h_0(0) = -\cos(\alpha_1) \ , \end{aligned} \tag{77}$$

which proves the first claim.

Let us now prove the second claim. We assume that Equation (50) holds true for filters of length $L$. This means, that there's a following dependency between the $h_0(n)$ and $h_1(n)$ coefficients:

$$\begin{aligned} h_1(0) &= \phantom{-}h_0(L-1) \ , \\ h_1(1) &= -h_0(L-2) \ , \\ h_1(2) &= \phantom{-}h_0(L-3) \ , \\ &\vdots \\ h_1(L-3) &= -h_0(2) \ , \\ h_1(L-2) &= \phantom{-}h_0(1) \ , \\ h_1(L-1) &= -h_0(0) \ . \end{aligned} \tag{78}$$

It will be demonstrated, that assuming the inductive hypothesis 78 and using the Equation (42) leads to obtaining filter coefficients that fulfil Equation (50). The coefficients of $H_1^{(L+2)}$ take form[3]:

$$\begin{aligned} H_1^{(L+2)} &= [h_1(0)\sin(\alpha),\ h_1(1)\sin(\alpha),\ h_1(2)\sin(\alpha) - h_0(0)\cos(\alpha), \\ &h_1(3)\sin(\alpha) - h_0(1)\cos(\alpha),\ \ldots,\ h_1(L-2)\sin(\alpha) - h_0(L-4)\cos(\alpha), \\ &h_1(L-1)\sin(\alpha) - h_0(L-3)\cos(\alpha),\ -h_0(L-2)\cos(\alpha),\ -h_0(L-1)\cos(\alpha)] \overset{(78)}{=} \\ &= [h_0(L-1)\sin(\alpha),\ -h_0(L-2)\sin(\alpha),\ h_0(L-3)\sin(\alpha) - h_0(0)\cos(\alpha), \\ &-h_0(L-4)\sin(\alpha) - h_0(1)\cos(\alpha),\ \ldots,\ h_0(1)\sin(\alpha) - h_0(L-4)\cos(\alpha), \\ &-h_0(0)\sin(\alpha) - h_0(L-3)\cos(\alpha),\ -h_0(L-2)\cos(\alpha),\ -h_0(L-1)\cos(\alpha)]\ , \end{aligned} \tag{79}$$

while coefficients of $H_0^{(L+2)}$ take the following form:

$$\begin{aligned} H_0^{(L+2)} &= [h_1(0)\cos(\alpha),\ h_1(1)\cos(\alpha),\ h_1(2)\cos(\alpha) + h_0(0)\sin(\alpha), \\ &h_1(3)\cos(\alpha) + h_0(1)\sin(\alpha),\ \ldots,\ h_1(L-2)\cos(\alpha) + h_0(L-4)\sin(\alpha), \\ &h_1(L-1)\cos(\alpha) + h_0(L-3)\sin(\alpha),\ h_0(L-2)\sin(\alpha),\ h_0(L-1)\sin(\alpha)] \overset{(78)}{=} \\ &= [h_0(L-1)\cos(\alpha),\ -h_0(L-2)\cos(\alpha),\ h_0(L-3)\cos(\alpha) + h_0(0)\sin(\alpha), \\ &-h_0(L-4)\cos(\alpha) + h_0(1)\sin(\alpha),\ \ldots,\ h_0(1)\cos(\alpha) + h_0(L-4)\sin(\alpha), \\ &-h_0(0)\cos(\alpha) + h_0(L-3)\sin(\alpha),\ h_0(L-2)\sin(\alpha),\ h_0(L-1)\sin(\alpha)]\ . \end{aligned} \tag{80}$$

Let us reverse order of $H_0^{(L+2)}$ coefficients, given by equation 80:

$$\begin{aligned} &[h_0(L-1)\sin(\alpha),\ h_0(L-2)\sin(\alpha),\ h_0(L-3)\sin(\alpha) - h_0(0)\cos(\alpha), \\ &h_0(L-4)\sin(\alpha) + h_0(1)\cos(\alpha), \ldots,\ h_0(1)\sin(\alpha) - h_0(L-4)\cos(\alpha), \\ &h_0(0)\sin(\alpha) + h_0(L-3)\cos(\alpha),\ -h_0(L-2)\cos(\alpha),\ h_0(L-1)\cos(\alpha)]\ . \end{aligned} \tag{81}$$

Let us change the sign of even coefficients[4] of the filter given by the Equation (81):

$$\begin{aligned} &[h_0(L-1)\sin(\alpha),\ -h_0(L-2)\sin(\alpha),\ h_0(L-3)\sin(\alpha) - h_0(0)\cos(\alpha), \\ &-h_0(L-4)\sin(\alpha) - h_0(1)\cos(\alpha),\ \ldots,\ h_0(1)\sin(\alpha) - h_0(L-4)\cos(\alpha), \\ &-h_0(0)\sin(\alpha) - h_0(L-3)\cos(\alpha),\ -h_0(L-2)\cos(\alpha),\ -h_0(L-1)\cos(\alpha)]\ . \end{aligned} \tag{82}$$

[3]To shorten the notation, in the following equations angle $\alpha_{L+1}$ is denoted as $\alpha$.

[4]It must be noted, that even coefficients are indexed be odd numbers. The first filter coefficient has index 0, the second coefficient has index 1 and so on.

It can be seen, that coefficients given by the equations 79 and 82 are identical. Therefore assumption that filters $H_0^{(L)}$ and $H_1^{(L)}$ fulfil condition 50 has led to a conclusion, that filters $H_0^{(L+2)}$ and $H_1^{(L+2)}$ also fulfil that condition. Therefore, by the power of mathematical induction, it has been proved that filters $H_0$ and $H_1$ fulfil Equation (50).

Let us now prove the orthogonality of:

1. the filter $H_0$ and its shifts by 2 (equation 5),
2. the filter $H_0$ and shifts by 2 of the filter $H_1$ (equation 12),
3. the filter $H_1$ and its shifts by 2 (equation 52).

We will begin by showing that if the Equation (50) holds true, than Equation (12) also holds true. Next we will show, that if Equation (5) holds true, than Equation (52) also holds true. Finally it will be demonstrated that filter $H_0$ fulfils Equation (5).

Let us prove Equation (12):

$$\begin{aligned} \sum_{n=0}^{L-1} h_0(n)h_1(n+2m) &\overset{(50)}{=} \sum_{n=0}^{L-1} h_0(n)(-1)^{n+2m}h_0(L-(n+2m)-1) = \\ \sum_{n=0}^{L-2} h_0(n)h_0(L-n-2m-1) &- \sum_{n=1}^{L-1} h_0(n)h_0(L-n-2m-1) \quad . \end{aligned} \tag{83}$$

Since the above formula must be equal to 0, we can rewrite it as:

$$\sum_{n=0}^{L-2} h_0(n)h_0(L-n-2m-1) = \sum_{n=1}^{L-1} h_0(n)h_0(L-n-2m-1) \quad . \tag{84}$$

By expanding the sums we obtain:

$$\begin{aligned} h_0(0)h_0(L-1-2m) + h_0(2)h_0(L-3-2m) + \ldots + \\ +h_0(L-4)h_0(3-2m) + h_0(L-2)h_0(1-2m) = \\ = h_0(1)h_0(L-2-2m) + h_0(3)h_0(L-4-2m) + \ldots + \\ +h_0(L-3)h_0(2-2m) + h_0(L-1)h_0(-2m) \end{aligned} \quad . \tag{85}$$

We notice, that in Equation (85) the number of elements on both sides of the equality sign is identical. Moreover, indices of some of the coefficients may exceed the range of the filter (less than 0 or more than $L-1$). Such coefficients have value of 0. Thus, for $m \geq 0$, the $m$ last addends on both sides of the equation are equal to 0. For $m > \frac{L}{2} - 1$ all the addends on both sides of the equation are equal to 0 (in this case equation of course holds true). Similarly, for $m < 0$ the $m$ first addends on both sides of the equation are equal to 0. For $m < -\frac{L}{2} + 1$ all the addends on both sides of the equation are equal to 0 (in this case equation also holds true). Therefore, for $m \in \left\{-\frac{L}{2} + 1, \ldots, \frac{L}{2} - 1\right\}$ on the both sides of the equation there are $\frac{L}{2} - |m|$ addends. Thus, removing the $m$ last addends for $m \in \left\{0, \ldots, \frac{L}{2} - 1\right\}$ leads to:

$$\begin{aligned} & h_0(0)h_0(L-1-2m) + h_0(2)h_0(L-3-2m) + \ldots + \\ & \quad + h_0(L-4-2m)h_0(3) + h_0(L-2-2m)h_0(1) = \\ & = h_0(1)h_0(L-2-2m) + h_0(3)h_0(L-4-2m) + \ldots + \\ & \quad + h_0(L-3-2m)h_0(2) + h_0(L-1-2m)h_0(0) \quad , \end{aligned} \tag{86}$$

which is an identity equation: first addend on the left side is identical with the last addend on the right side, the second addend on the left is equal to the last but one addend on the right and so on. Similarly, for $m \in \left\{-\frac{L}{2} + 1, \ldots, -1\right\}$, we remove the $m$ first addends on both sides of Equation (85):

$$\begin{aligned} & h_0(-2m)h_0(L-1) + h_0(2-2m)h_0(L-3) + \ldots + \\ & \quad + h_0(L-4)h_0(3-2m) + h_0(L-2)h_0(1-2m) = \\ & = h_0(1-2m)h_0(L-2) + h_0(3-2m)h_0(L-4) + \ldots + \\ & \quad + h_0(L-3)h_0(2-2m) + h_0(L-1)h_0(-2m) \end{aligned} \quad . \tag{87}$$

Equation 87 is also an identity equation. Therefore it has been demonstrated that Equation (12) holds true for filters $H_0$ and $H_1$ fulfilling condition 50.

Let us now prove Equation (52):

$$
\begin{aligned}
&\sum_{n=0}^{L-1} h_1(n)h_1(n+2m) \overset{(50)}{=} \\
&= \sum_{n=0}^{L-1}(-1)^n h_0(L-n-1)(-1)^{n+2m}h_0(L-(n+2m)-1) = \\
&= \sum_{n=0}^{L-1}(-1)^{2(n+m)}h_0(L-n-1)h_0(L-n-2m-1) = \\
&= \sum_{n=0}^{L-1} h_0(L-n-1)h_0(L-n-2m-1) = 0 \quad .
\end{aligned} \tag{88}
$$

Substituting

$$k = L - n - 1$$

to Equation (88) transforms it into Equation (5):

$$\sum_{n=0}^{L} h_0(L-n-1)h_0(L-n-2m-1) = \sum_{k=0}^{L-1} h_0(k)h_0(k-2m) = 0 \quad . \tag{89}$$

Therefore the filter $H_1$, fulfilling condition 50, is orthogonal to its shifts by 2 if the Equation (5) holds true.

Let us then prove, that filter $H_0$ fulfils Equation (5). The proof will be carried out using the mathematical induction method. Two claims will be proved:

1. the filter $H_0^{(2)}$ fulfils Equation (5),
2. the fact that filter $H_0^{(L)}$ fulfils the Equation (5) leads to conclusion that filter $H_0^{(L+2)}$ also fulfils that equation.

Let us begin by proving the first claim. The filter $H_0^{(2)}$ has length 2 and therefore after shifting it by 2 (or multiple of 2) no non-zero coefficients overlap with the original filter. Thus, the first claim is automatically true. Let us now prove the second claim. Let us remind that the coefficients of $H_0^{(L+2)}$ filter are given by Equation (80). Let us expand the Equation (5) for filter $H_0^{(L+2)}$, assuming that $m \in \mathbb{C}_+$:

$$\begin{aligned}\sum_{n=0}^{L-1} h_0(n)h_0(n+2m) \overset{(80)}{=} & [h_1(0)\cos(\alpha)]\,[h_1(2m)\cos(\alpha) + h_0(2m-2)\sin(\alpha)] + \\ & + [h_1(1)\cos(\alpha)]\,[h_1(2m+1)\cos(\alpha) + h_0(2m-1)\sin(\alpha)] + \\ & + [h_1(2)\cos(\alpha) + h_0(0)\sin(\alpha)]\,[h_1(2m+2)\cos(\alpha) + h_0(2m)\sin(\alpha)] + \ldots + \\ & + [h_1(L-2m-1)\cos(\alpha) + h_0(L-2m-3)\sin(\alpha)] \cdot \\ & \cdot [h_1(L-1)\cos(\alpha) + h_0(L-3)\sin(\alpha)] + \\ & + [h_1(L-2m)\cos(\alpha) + h_0(L-2m-2)\sin(\alpha)]\,[h_0(L-2)\sin(\alpha)] + \\ & + [h_1(L-2m+1)\cos(\alpha) + h_0(L-2m-1)\sin(\alpha)]\,[h_0(L-1)\sin(\alpha)] \quad . \end{aligned} \tag{90}$$

Performing the multiplication leads to:

$$\begin{aligned} & h_1(0)h_1(2m)\cos^2(\alpha) + h_1(0)h_0(2m-2)\cos(\alpha)\sin(\alpha) + \\ & + h_1(1)h_1(2m+1)\cos^2(\alpha) + h_1(1)h_0(2m-1)\cos(\alpha)\sin(\alpha) + \\ & + h_1(2)h_1(2m+2)\cos^2(\alpha) + h_0(0)h_0(2m)\sin^2(\alpha) + \\ & + [h_1(2)h_0(2m) + h_0(0)h_1(2m+2)]\cos(\alpha)\sin(\alpha) + \ldots + \\ & + h_1(L-2m-1)h_1(L-1)\cos^2(\alpha) + h_0(L-2m-3)h_1(L-3)\sin^2(\alpha) + \\ & + [h_1(L-2m-1)h_0(L-3) + h_0(L-2m-3)h_1(L-1)]\cos(\alpha)\sin(\alpha) + \\ & h_1(L-2m)h_0(L-2)\cos(\alpha)\sin(\alpha) + h_0(L-2m-2)h_0(L-2)\sin^2(\alpha) + \\ & + h_1(L-2m+1)h_0(L-1)\cos(\alpha)\sin(\alpha) + h_0(L-2m-1)h_0(L-1)\sin^2(\alpha) \quad . \end{aligned} \tag{91}$$

By grouping the terms we get:

$$\begin{aligned} & \cos^2(\alpha)\,[h_1(0)h_1(2m) + h_1(1)h_1(2m+1) + h_1(2)h_1(2m+2) \ldots + \\ & + h_1(L-2m-1)h_1(L-1)] + \sin^2(\alpha)\,[h_0(0)h_0(2m) + \ldots + \\ & + h_0(L-2m-3)h_0(L-3) + h_0(L-2m-2)h_0(L-2) + h_0(L-2m-1)h_0(L-1)] + \\ & \cos(\alpha)\sin(\alpha)\,[h_1(0)h_0(2m-2) + h_1(1)h_0(2m-1) + h_1(2)h_0(2m) + \ldots + \\ & + h_1(L-2m-1)h_0(L-3) + h_1(L-2m)h_0(L-2) + h_1(L-2m+1)h_0(L-1)] + \\ & \cos(\alpha)\sin(\alpha)\,[h_0(0)h_1(2m+2) + \ldots + h_0(L-2m-3)h_1(L-1)] \quad , \end{aligned} \tag{92}$$

which can be rewritten as:

$$\begin{aligned}
\cos^2(\alpha)\left[\sum_{n=0}^{L-2m-1} h_1(n)h_1(n+2m)\right]+ \\
+\sin^2(\alpha)\left[\sum_{n=0}^{L-2m-1} h_0(n)h_0(n+2m)\right]+ \\
+\cos(\alpha)\sin(\alpha)\left[\sum_{n=0}^{L-2m+1} h_1(n)h_0(n+2m-2)\right]+ \\
+\cos(\alpha)\sin(\alpha)\left[\sum_{n=0}^{L-2m-3} h_1(n)h_0(n+2m+2)\right] .
\end{aligned} \tag{93}$$

We have made an inductive hypothesis, that Equation (5) holds true for $H_0^{(L)}$ filter and therefore expression

$$\sum_{n=0}^{L-2m-1} h_0(n)h_0(n+2m)$$

in the Equation (93) equals 0. As was shown earlier, if Equation (5) holds true, the the Equation (52) also holds true. Therefore, from the inductive hypothesis we obtain that Equation (52) holds true for $H_1^{(L)}$ filter. Thus, expression

$$\sum_{n=0}^{L-2m-1} h_1(n)h_1(n+2m)$$

also equals 0. In the earlier part of this proof we have demonstrated that filters $H_0^{(L)}$ and $H_1^{(L)}$ are orthogonal to each other with respect to shifts by 2 (equation 12). Therefore expressions

$$\sum_{n=0}^{L-2m+1} h_1(n)h_0(n+2m-2)$$

and

$$\sum_{n=0}^{L-2m-3} h_1(n)h_0(n+2m+2)$$

are equal to 0. This leads to a conclusion, that Equation (93) always equals 0. This demonstrates, that if the filter $H_0^{(L)}$ fulfils condition 5, then filter $H_0^{(L+2)}$ also fulfils that condition. Therefore, by the power of mathematical induction, it has been demonstrated that Equation (5) holds true for every low-pass filter created using the Equation (42). Earlier we have demonstrated that for filters fulfilling relation 50, the Equation (52) holds true if the Equation (5) is also true. Now we can conclude, that Equation (52) holds true.

Finally, we will demonstrate that $H_0$ and $H_1$ filters have unit norm. Let us use an already proved Equation (50) to determine when does the $H_1$ filter have a unit norm (equation 54):

$$\begin{gathered}\sum_{n=0}^{L} h_1(n)^2 \overset{(50)}{=} \sum_{n=0}^{L} [(-1)^n h_0(L-n-1)]^2 = \\ \sum_{n=0}^{L} \left[(-1)^{2n} h_0(L-n-1)^2\right] = \sum_{n=0}^{L} h_0(L-n-1)^2 \overset{(4)}{=} 1 \ .\end{gathered} \tag{94}$$

Therefore Equation (54) holds true if the Equation (4) holds true. Let us now demonstrate that $H_0$ filter fulfils Equation (4). The proof will be carried out using the mathematical induction rule. Two claims will be proved:

1. $H_0^{(2)}$ filter fulfils 4,
2. the fact that $H_0^{(L)}$ filter fulfils Equation (4) leads to conclusion, that $H_0^{(L+2)}$ filter also fulfils that equation.

Let us remind, that coefficients of the $H_0^{(2)}$ filter are given as:

$$H_0^{(2)} = [\cos(\alpha_1), \sin(\alpha_1)] \ .$$

Let us calculate the norm of the $H_0^{(2)}$ filter:

$$\sum_{n=0}^{1} h_0(n)^2 = \cos^2(\alpha_1) + \sin^2(\alpha_1) = 1 \ . \tag{95}$$

Therefore $H_0^{(2)}$ filter has unit norm. Let us now prove the second claim. We begin by calculating the norm of the $H_0^{(L+2)}$ filter, coefficients of which are given by

Equation (80):

$$
\begin{aligned}
\sum_{n=0}^{L+1} h_0(n)^2 &= h_1(0)^2\cos^2(\alpha) + h_1(1)^2\cos^2(\alpha) + \\
+h_1(2)^2\cos^2(\alpha) + 2h_1(2)h_0(0)\sin(\alpha)\cos(\alpha) &+ h_0(0)^2\sin^2(\alpha) + \ldots + \\
+h_1(L-1)^2\cos^2(\alpha) &+ 2h_1(L-1)h_0(L-3)\sin(\alpha)\cos(\alpha) + \\
+h_0(L-3)^2\sin^2(\alpha) + h_0(L-2)^2\sin^2(\alpha) &+ h_0(L-1)^2\sin^2(\alpha) \quad .
\end{aligned} \tag{96}
$$

By grouping the terms we get:

$$
\begin{aligned}
\cos^2(\alpha)\left[h_1(0)^2 + h_1(1)^2 + h_1(2)^2 + \ldots + h_1(L-1)^2\right] &+ \\
\sin^2(\alpha)\left[h_0(0)^2 + h_0(1)^2 + h_0(2)^2 + \ldots + h_0(L-1)^2\right] &+ \\
+2\sin(\alpha)\cos(\alpha)\left[h_0(0)h_1(2) + h_0(1)h_1(3) + \ldots + h_0(L-3)h_1(L-1)\right] &= \\
= \cos^2(\alpha)\left[\sum_{n=0}^{L-1} h_1(n)^2\right] + \sin^2(\alpha)\left[\sum_{n=0}^{L-1} h_0(n)^2\right] &+ \\
2\sin(\alpha)\cos(\alpha)\left[\sum_{n=0}^{L-3} h_0(n)h_1(n+2)\right] & \quad .
\end{aligned} \tag{97}
$$

By the power of inductive hypothesis, expression

$$\sum_{n=0}^{L-1} h_0(n)^2$$

equals 1. Earlier we have demonstrated that if the Equation (4) holds true, then the Equation (54) also holds true, and therefore – by the power of inductive hypothesis – the expression

$$\sum_{n=0}^{L-1} h_1(n)^2$$

also equals 1. As was demonstrated earlier, the $H_0$ i $H_1$ filters, created using Equation (42), are orthogonal to each other, and therefore expression

$$\sum_{n=0}^{L-3} h_0(n)h_1(n+2)$$

in the above equation equals 0. Therefore Equation (97) is reduced to Pythagorean trigonometric identity, which is always equal to 1. This demonstrates that if the filter $H_0^{(L)}$ has unit norm, then the filter $H_0^{(L+2)}$ also has unit norm. This proves that $H_0$ filter has unit norm. According to Equation (94), the $H_1$ filter also has unit norm.

We have therefore proved that $H_0$ and $H_1$ filters implemented by the lattice structure fulfil all the conditions imposed in section 2 on the filters forming an orthogonal wavelet filter bank.

## 6. Summary

In this paper properties of a wavelet parametrization method based on the lattice structure have been discussed in detail. Two-point symmetric and asymmetric orthogonal base operations based on the trigonometric functions have been introduced. An algorithm for converting the parameters of the lattice structure to the wavelet filter coefficients has been constructed, both in an iterative and recursive form. Conditions imposed on an orthogonal wavelet filter bank have been given. The proof has been carried out, showing that parametrization based on an orthogonal lattice structure with symmetric base operations fulfils conditions imposed on the orthogonal wavelet filter bank. In the proof it has been shown, that the angles used in the trigonometric functions to parametrize the symmetric orthogonal base operations must fulfil additional condition imposed on their sum.

This concludes the first part of this paper. In the second part an analogical proof for the lattice structure with asymmetric base operations will be conducted. We will also discuss representing the lattice structure in a form that guarantees fulfilment of conditions imposed on the sum of angles in the layers of the lattice structure.

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