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Lattice Structures for Synthesis and Implementation of Wavelet Transforms

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Abstract. *In this paper the novel lattice structure composed of homogeneous invertible two-point operations which are connected in regular and simple structure is proposed. Further on the pipeline scheme for implementation of such a structure is presented. It is proved with the orthogonal variant of the presented scheme that with respect to the computational complexity it is equivalent to the lifting technique. It means that the proposed scheme belongs to the class of the most effective algorithms for calculation of orthogonal wavelet transforms. The variant of the lattice structure with simplified two-point operations is also proposed. Finally the fundamentals of the synthesis of lattice structure coefficients with the aid of artificial neural networks and some aspects of lattice structures implementation on basic computational architectures are discussed.*

1. Introduction

The wavelet transforms satisfy the perfect reconstruction condition which in matrix notation can be described by the formula

$$\mathbf{P_N}^{-1}\mathbf{P_N} = \mathbf{I} \tag{1}$$

where $\mathbf{P_N}$ and $\mathbf{P_N}^{-1}$ are $N \times N$ element matrices representing forward and inverse transforms respectively and $\mathbf{I}$ is the identity matrix. The first two rows of the forward transform matrix $\mathbf{P_N}$ are composed of impulse responses of $H = [h_0, h_1, ..., h_{K-1}]$ and $G = [g_0, g_1, ..., g_{K-1}]$ filters complemented with zeros. The successive pairs of rows are obtained by the cyclic N - element shift of the previous pair of rows right by two elements. The effectiveness of the transformation is determined by the sparsity of its matrix. It means that only the pair of sparse $\mathbf{P_N}$ and $\mathbf{P_N}^{-1}$ matrices forms fast wavelet transform. The synthesis of $\mathbf{P_N}$ and $\mathbf{P_N}^{-1}$ matrices utilizes the orthogonality, biorthogonality and wavelet type describing equations [1].

Such an approach has several disadvantages. In the first place there is no possibility to adapt wavelet transformation to the requirements of performed task. There are known only specific cases of orthogonal wavelet transforms (for $K = 4$ and $K = 6$) that has the capability of adaptation [2-4]. Next the well-known effective implementation schemes of orthogonal wavelet transforms are relatively complicated in practice and/or have the limited character [5-6]. In particular the lifting scheme requires K dependent factorizations [5, 6] what make its software and hardware implementations difficult. The reduction techniques that replace the impulse response with the linear combination of vectors with trivial elements depend on the values of filter coefficients [7]. Therefore intensive research efforts were concentrated recently upon the issues of improvement of wavelet transform implementations [7-10].

Hence it is required to develop methods of synthesis of wavelet transforms with sparse structures, simple for implementation on basic computational architectures, that would also have the capability to adapt to the demands of specific applications.

2. The lattice structure

The well known approach to the design of data processing methods and algorithms consists in development of a method that meets the assumed requirements and its further effective implementation with respect to the specific computational architecture. In this paper we exploit entirely opposite concept that is applicable in construction of artificial neural networks [11-14]. At first the computational structure that is simple for implementation on basic computational architectures is selected. Then it is shown that the selected structure is suitable for the specific task of data processing.

In order to do so let us consider the following two-point base operation

$$\mathbf{D_k} = \begin{bmatrix} d_{11}^k & d_{12}^k \\ d_{21}^k & d_{22}^k \end{bmatrix},$$

where k stands for the index of operation. Let us assume that $\mathbf{D_k}$ is invertible, it means that $d_{11}^k d_{22}^k - d_{21}^k d_{12}^k \neq 0$ condition is satisfied. Hence there exists inverse operation $\mathbf{D_k}^{-1}$ such that $\mathbf{D_k}^{-1}\mathbf{D_k} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix.

We introduce *the forward lattice structure* that is composed of $K/2$ stages, each containing the number of $N/2$ operations $\mathbf{D_k}$, where K and N are the lengths of the filter impulse response and the processed signal respectively, see Figure 1a. In the first stage the pairs of x_{2i} and x_{2i+1}, $i = 0,1,...,N/2-1$, samples are assigned to the inputs of each $\mathbf{D_k}$ operation. Base operations in successive stages are shifted down by one position and the lower input of the last base operation in the current stage is connected to the upper output of the first base operation in the preceding stage. In other words we perform the cyclic N - element shift of the mentioned outputs left (upward). The outputs of the last stage are the outputs y_i of the whole structure where $i = 0,1,...,N-1$. Each of $\mathbf{D_k}$ operations is invertible.

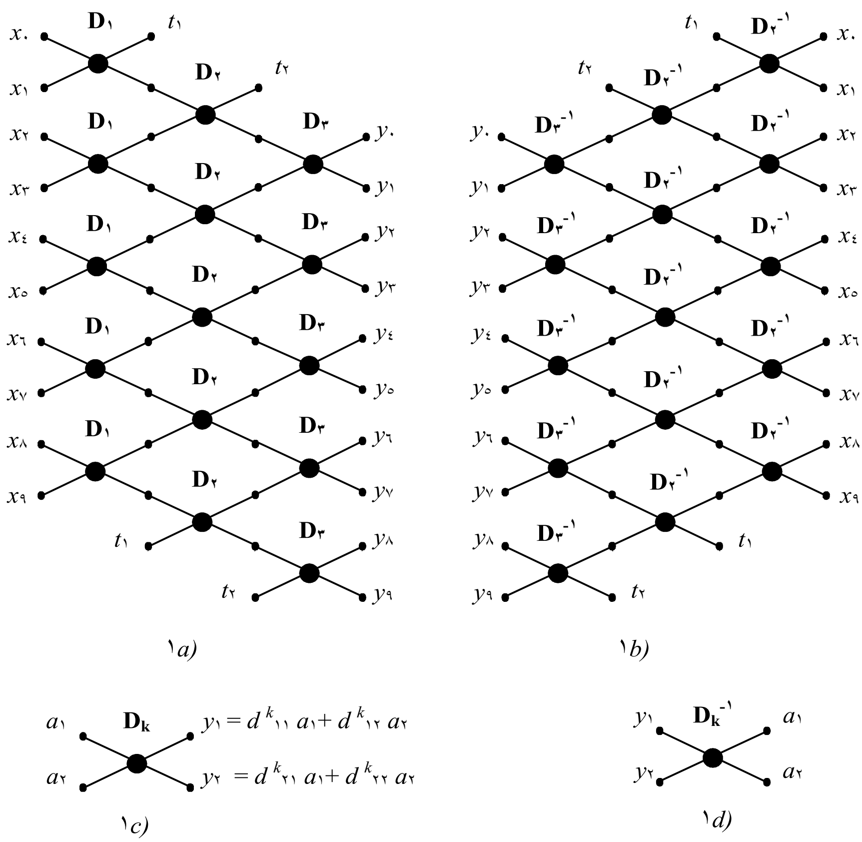

Fig. 1. The lattice structure for $K = 6$, $N = 10$: 1a) forward, 1b) inverse.
Base operations: 1c) forward, 1d) inverse

The inverse lattice structure emerges as the reversed forward structure where base $\mathbf{D_k}$ operations are substituted by inverse operations $\mathbf{D_k}^{-1}$ and the cyclic shift is performed in the right direction. An example of forward and inverse structures is depicted in Figure 1. The cyclic left (Figure 1a) and right (Figure 1b) shift operations are represented by t_1 and t_2 symbols.

Let us estimate the number of arithmetic operations in proposed structures. In a general case in order to execute $K/2$ stages while operating on N sample signals it is needed to perform the number of $(N/2)(K/2)$ base operations $\mathbf{D_k}$ where each operation performs two additions and four multiplications. Hence the forward and the inverse structures require $\alpha(K,N)$ additions and $\mu(K,N)$ multiplications separately where $\alpha(K,N)=KN/2$ and $\mu(K,N)=KN$. In other words for one pair of the processed elements each structure requires K additions and $2K$ multiplications. However while processing one pair of elements directly with $\mathbf{P_N}$ matrix it is required to perform $2K-2$ additions and $2K$ multiplications. Hence almost twofold reduction in the number of additions is obtained without the increase of the number of multiplications. In order to obtain further reduction in arithmetic operations it is required to consider the particular base operations, see Sections 3 and 4 of this paper.

The main advantage of the proposed lattice structure is the ease of its realization in the pipeline scheme. In a general case the pipeline scheme contains $K/2$ blocks of base operations and $K/2-1$ blocks z^{-1} with delays by one element, see Figure 2.

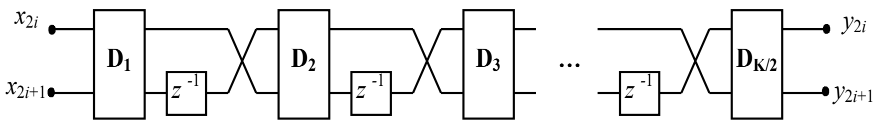

Fig. 2. The pipeline scheme for *K*/2 – stage lattice structure

In the scheme from Figure 2 there is no cyclic shift which must be taken under consideration with hardware or software implementation, e.g. it can be performed by cyclic repetition of the first samples of input signal. The proposed pipeline scheme is simpler than the well-known lifting scheme [5, 6]. At first it reduces the number of required two-point base operations, e.g. by one for $K=4$ and $K=6$. In the second place it utilizes base operations of one type.

3. The orthogonal lattice structure

Let us assume that two-point operation $\mathbf{D_k}$ is orthogonal, i.e. it satisfies condition: $d_{11}^k d_{21}^k - d_{12}^k d_{22}^k = 0$. Then lattice structure is also the orthogonal transformation. We consider two cases of such operations: 1) symmetric, when $\mathbf{S_k} = \mathbf{S_k}^{-1}$; 2) asymmetric, when $\mathbf{F_k} \neq \mathbf{F_k}^{-1}$

$$\mathbf{S_k} = \mathbf{S_k}^{-1} = \begin{bmatrix} p_k & q_k \\ q_k & -p_k \end{bmatrix}, \quad \mathbf{F_k} = \begin{bmatrix} p_k & q_k \\ -q_k & p_k \end{bmatrix}, \tag{2}$$

$$\mathbf{F_k}^{-1} = \begin{bmatrix} p_k & -q_k \\ q_k & p_k \end{bmatrix},$$

where p_k and q_k are any non-zero numbers. Each of operations (2) still requires two additions and four multiplications. We will demonstrate with an example of $\mathbf{S_k}$ operation the technique of reducing the number of multiplications in structures from Figure 1 and 2. In order to do it the following factorization is introduced

$$\mathbf{S_k} = \mathbf{E_k}\,\mathbf{T_k} \tag{3}$$

where

$$\mathbf{T_k} = \begin{bmatrix} 1 & t_k \\ t_k & -1 \end{bmatrix}, \; t_k = q_k / p_k; \; \mathbf{E_k} = \begin{bmatrix} p_k & 0 \\ 0 & p_k \end{bmatrix}.$$

Now let us modify the pipeline scheme for the orthogonal lattice structure from Figure 2. The factorization (3) is applied to each stage of the lattice structure. However we perform only $\mathbf{T_k}$ operations while operations $\mathbf{E_k}$ are grouped in the additional stage, see Figure 3.

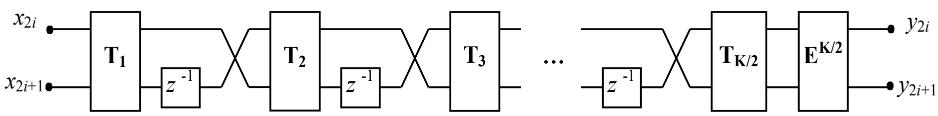

Fig. 3. The pipeline scheme for *K*/2 – stage lattice orthogonal structure

It can be seen that we obtain $K/2$ blocks of simplified base operations $\mathbf{T_k}$, $K/2-1$ blocks with z^{-1} delays and one block $\mathbf{E^{K/2}} = \mathbf{E_1 E_2 ... E_{K/2}}$. Each of $\mathbf{T_k}$ operation requires two multiplications and two addition. Within $\mathbf{E^{K/2}}$ block two multiplications are performed. Hence the scheme from Figure 3 requires K additions and $K+2$ multiplications for one pair of output elements. As a result almost twofold reduction in the number of multiplications with the same number of additions in comparison to a general lattice structure is obtained. It should be noted that $\mathbf{E^{K/2}}$ block performs multiplications of its inputs by a constant value and therefore it can be omitted by grouping the constants at successive factorizations of lengths $N/2, N/4,\ldots$, what results in reduction of two multiplications.

In the following table we present the numbers of arithmetic operations required to calculate two output elements with one factorization stage for the selected values of K. The last column contains results obtained with the variant omitting $\mathbf{E^{K/2}}$ block.

Table. The number of arithmetic operations for orthogonal transforms algorithms

K	Standard	Lifting[5]	Lattice	Lattice 1
4	14	9	10	8
6	22	14	14	12

It should be noted that the orthogonal transformation can be calculated with the general scheme from Figure 1a or 1b with no increase in the number of arithmetic calculations. In order to do so $\mathbf{D_k}$ operations should be replaced with $\mathbf{T_k}$ for $k=1,2,\ldots,K/2-1$ and transforms $\mathbf{E^{K/2}T_{K/2}}$ must be joined into one operation $\mathbf{F_{K/2}}$. As an example we consider the parameterized variant of Daubechies 4 transform with filter

$$H=\left[\frac{1+m}{\sqrt{2}(m^2+1)},\frac{m(m+1)}{\sqrt{2}(m^2+1)},\frac{m(m-1)}{\sqrt{2}(m^2+1)},\frac{1-m}{\sqrt{2}(m^2+1)}\right], \tag{4}$$

where $m=\sqrt{3}$ is the arbitrary non-zero number, $m\neq\pm1$ [3,4]. With simple verification we ascertain that the transformation with filter (4) corresponds with orthogonal lattice structure with base operations $\mathbf{T_1}$, $\mathbf{T_2}$ and $\mathbf{E^2}$ or the lattice structure with base operations $\mathbf{T_1}$ and $\mathbf{F_2}$:

$$\mathbf{T_1}=\begin{bmatrix} m & -1 \\ 1 & m \end{bmatrix},\quad \mathbf{T_2}=\begin{bmatrix} 1 & t_2 \\ -t_2 & 1 \end{bmatrix},\quad \mathbf{E^2}=\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix},$$

$$\mathbf{F_2}=\mathbf{E^2T_2}=\begin{bmatrix} e & et_2 \\ -et_2 & e \end{bmatrix},$$

where $t_2=(m-1)/(m+1)$, $e=(m+1)/(\sqrt{2}(m^2+1))$.

Similarly, it is possible to prove that Daubechies 6 filter can be realised with the aid of three stage lattice structure with two $\mathbf{T}$ operations (two multiplications and two additions) and one orthogonal $\mathbf{F}$ operation (four multiplications and two additions) including previously described technique of reduction of two multiplications.

4. The lattice structure with simplified base operation

The reduction in the number of computations can be obtained with two additional conditions on the triviality of some coefficients in base operations $\mathbf{D_k}$. In particular we assume two trivial coefficients with values 0 and 1. Let us consider the specific case of operation, i.e. the operation that does not modify the second output

$$\mathbf{L_k} = \begin{bmatrix} p_k & q_k \\ 0 & 1 \end{bmatrix}, \tag{5}$$

where p_k and q_k are any numbers, and $q_k \neq 0$. Then the lattice structure exploiting $\mathbf{L_k}$ operations, $k = 1,2,\ldots,K/2$, corresponds with application of two filters *H* and *G* of different lengths *K* and *K*-1. It is obvious that with the aid of base operation that does not modify the first output it is possible to synthesise the lattice structure where filters *H* and *G* are of lengths *K*-1 and *K* respectively.

It can be proved with direct check that the inverse operation to $\mathbf{L_k}$ is

$$\mathbf{L_k}^{-1} = \begin{bmatrix} p_k^1 & q_k^1 \\ 0 & 1 \end{bmatrix}, \tag{6}$$

where $p_k^1 = 1/p_k$, $q_k^1 = -q_k/p_k$.

It should be noted that simplified operations (5), (6) are also applied to the construction of lifting scheme for biorthogonal wavelet transforms [5, 6]. Each of (5), (6) operations requires one addition and two multiplications. Hence *K*/2 stage lattice structure with simplified operations requires *K*/2 additions and *K* multiplications for one pair of processed input elements.

5. Determination of the values of coefficients for lattice structures

The values of a lattice structure coefficients can be determined by the means of two methods: the standard one and the one taking advantage of artificial neural networks. A standard method consists in determining the relations between impulse responses of H and G filters and coefficients of base operations. In this method we rewrite the impulse responses in the form of algebraic expressions that are structure coefficients dependent. Next we follow the standard scheme that requires to dissolve the system of equations including the orthogonality and the zero moments equations [1]. This method is not simple (it requires to solve the system of K nonlinear equations) and in general produces coefficients that are irrelevant to specific requirements imposed on wavelet transform.

More interesting method utilizes the technique of artificial neural networks. This method was developed for the synthesis of fast algorithms for Fourier-like transformations [11-14]. In accordance with this method each stage of the lattice structure is replaced with one hidden layer of linear artificial neural network. Each $\mathbf{D_k}$ base operation is replaced with two neurons having two inputs and one output which guarantees the straightforward relation between the weights of a neural network and the coefficients of base operations. Here it is possible to perform simplifications resulting from usage of $\mathbf{T_k}$ [12] and $\mathbf{L_k}$. Hence by training such neural network we determine the values of lattice structure coefficients and it is also possible to take under consideration specific demands of wavelet transform (a lattice structure) applications. Such way of determining lattice structure coefficients can be utilised in any field of application of wavelet transform including: signal compression, adaptive filtering and time-frequency analysis of signals [15,16].

6. The aspects of lattice structure implementation

Let us consider principal aspects of lattice structures implementation on typical computational architectures.

For *iterative processors* a lattice structure can be implemented with the aid of a nested loop. Here all calculations can be realised in place with additional $K/2$-1 memory cells required for storage of t_k values.

In Sections 3 and 4 of this paper the possibility of effective implementation of lattice structures on *pipeline processors* was indicated [17]. Due to the feature of local data processing in accord with simple and regular structure the lattice structure is suitable for implementation on *systolic processors* and in general on parallel architectures.

During the design of *VLSI architecture* [9] and utilisation of *microprocessors* the structure coefficients should be replaced by “simplified” numbers, i.e. the powers of two, in a way enabling the substitution of multiplications by additions and shifting operations [3, 4, 7].

7. Conclusions

The lattice structure proposed in this paper can be characterized by the simplicity of its framework and the effectiveness of calculations. It is constructed on the basis of iterative repetition of simple two-point base operations and can be implemented as simple pipeline scheme. As a result it gives extensive possibilities of software and hardware implementations of wavelet transforms with special indication of integrated circuits.

Within the further development of the proposed lattice structures the relation between the lattice orthogonal structure and the orthogonal wavelet transform should be determined. It means that the class of wavelet transforms that can be implemented with the orthogonal lattice structures should be defined. From the practical point of view it is crucial to develop novel problem-oriented training techniques of neural networks with topologies based on the proposed lattice structure. It is also very important to elaborate exemplary implementations of lattice structures for various hardware architectures.

References

[1]Daubechies I.: *Ten Lectures on Wavelets*, SIAM, 1992.

[2]Chen Y., Oraintara S.: *Multiplierless Approximation of Transform With Adder Constraint*, IEEE Signal Processing Letters, vol. 9, nr 11, pp. 344-347, 2002.

[3]Yatsymirskyyy M.: *Synteza przekształceń falkowych dla mobilnych systemów e-gospodarki*; [in:] Wybrane problemy gospodarki elektronicznej, Łódź, pp. 273-282, 2008.

[4]Lipinski P., Yatsymirskyy M.: *On Synthesis of 4-tap and 6-tap Reversible wavelet Filters,* Przeglad elektrotechniczny, nr 12, pp.284-286, 2008.

[5]Daubechies I., Sweldens W.: *Factoring Wavelet Transform into Lifting Steps*, The Journal of Fourier Analysis and Applications, vol. 4, nr 3, pp. 245-267, 1998.

[6]Sweldens W.: *The lifting scheme: A construction of second generation wavelets*, SIAM Journal on Mathematical Analysis, vol. 29, nr. 2, pp. 511-546, 1998.

[7]Lipiński P., Yatsymirskyy M.: *Efficient 1D and 2D Daubechies Wavelet Transforms with Application to Signal Processing*, Lecture Notes for Computer Science, 4432, Berlin Heidelberg, pp. 391-398, 2007.

[8]Cooklev T.: *An Efficient Architecture for Orthogonal Wavelet Transforms,* IEEE Signal Processing Letters, vol. 13, nr. 2, pp.77-79, 2006.

[9]Huang C., Tseng P., Chen L.: *Analysis VLSI Architecture for 1-D and 2-D Discrete Wavelet Transform*, IEEE Trans. Signal Process., vol. 53, nr 4, pp.1575-1586, 2005.

[10]Olkkonen H., Olkkonen J.T., Pesola T., *Efficient Lifting Wavelet transform for Microprocessor and VLSI Applications*, IEEE Signal Processing Letters, vol. 12, nr 2, pp.110-112, 2005.

[11]Jacymirski M., Szczepaniak P. S.: *Neural Realization of Fast Linear Filters*; [in:] Proc. of 4th EURASIP-IEEE Region\&International Symposium on Video/Image Processing and Multimedia Communications, Zadar, Croatia, pp. 153-157, 2002.

[12]Stasiak B., Yatsymirskyy M., *Fast Orthogonal Neural Networks*, Artificial Intelligence and Soft Computing - ICAISC`06, Springer, pp. 142-149, 2006.

[13]Stokfiszewski K., Szczepaniak P.: *An Adaptive Fast Transform Based Image Compression*, ICAISC 2008, LNAI 5097, Springer, pp. 874-885, 2008.

[14]Szczepaniak P.S.: *Obliczenia inteligentne, szybkie przekształcenia i klasyfikatory*, Akademicka Oficyna Wydawnicza EXIT, Warszawa, 2006.

[15]Gonzalez R.C.,Woods R.E.: *Digital Image Processing*, Pearson Education, Inc.-Prentice Hall, 2002.

[16]Lipiński P., Yatsymirski M.: *Application of Hybrid Fourier Wavelet Transform to Data Space Reduction in Clothing Sales Volume Time Series,* Journal of Applied Computer Science, vol. 16, nr 1, pp.81-88, 2008.

[17]Rabiner L., Gold B.: *Theory and Application of Digital Signal Processing,* Englewood Cliffs, N.J., Prentice-Hall, Inc., 1975.