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SELECTED METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

LODZ UNIVERSITY OF TECHNOLOGY
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Preface

The following book deals with various boundary value problems for differential equations. As Juliusz Schauder, one of the pioneers and unsurpassed masters (at least for the author), used to say, the most important thing is to know the methods, not the theorems. Thus, we are interested in a set of methods of Nonlinear Analysis applied to such boundary value problems. Since we want to avoid the difficulties associated with partial equations (already the theory of linear partial differential equations requires the use of subtle concepts and tools of Functional Analysis), we choose examples showing applications of the above-mentioned methods among ordinary differential equations. We are interested in nonlinear equations, but the boundary conditions we discuss are usually linear. By boundary conditions, we mean here any additional equations that the solutions of the differential equation are expected to satisfy. Such additional conditions are necessary if we want to have one (or more) solutions - after all, a given differential equation has infinitely many solutions. These additional conditions may be initial conditions, conditions to be satisfied by the function at the extremes of the domain (boundary conditions), but they may also be multipoint or, more broadly, nonlocal e.g. when there is an integral of the solution in the equation.

On the other hand, the wealth of methods of nonlinear analysis is so great that we emphasize a certain set of methods (mainly topological) preferred by the author. The examples on which we present applications of these methods are in majority taken from the work of the research team that consists of the author and his former PhD students. Thus, this survey is not a monograph of the subject in a strict sense, which is reflected in the first word of the title "Selected". The author is responsible for any errors that appear in this work.

Contents

Chapter 1

Introduction

Boundary value problems for ordinary differential equations (ODE) are less natural, however the question if you can arrive at given time $t = T$ to an end point $x = B$ starting from a point $x = A$ at the time $t = 0$ leads to the boundary value problem (BVP):

$$x'' = F(t, x, x'), \qquad x(0) = A, \quad x(T) = B,$$

F stands for a force, mass of the moving point $m = 1$.

Another questions:

- $x' = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and T-periodic w.r.t. t. If one looks for T-periodic solutions, one should solve this equation with boundary conditions $x(0) = x(T)$ on $[0, T]$.

- similar problem for second-order equations $x'' = f(t, x, x')$ – the boundary conditions are $x(0) = x(T)$, $x'(0) = x'(T)$.

- if you seek for *radial solutions* of the Poisson equation $\Delta u = f(\|x\|)$, $x \in \mathbb{R}^n$, that satisfies the Dirichlet condition $u|\partial B(0, R) = 0$, you should solve

$$v''(r) + \frac{n-1}{r} v'(r) = f(r), \qquad v'(0) = 0, \quad v(R) = 0.$$

- For partial differential equations (PDE), especially nonlinear, one looks for the so-called *travelling waves.* For example, $u_t = u_{xx} + f(u)$, Kolmogorov-Fisher equation has a solution beeing a travelling wave, if $u(t,x) = v(x - ct)$, where $c > 0$ is a fixed real – the speed of this wave. Usually, one needs

$$\lim_{s\to-\infty} v(s) = u_-, \qquad \lim_{s\to+\infty} v(s) = u_+.$$

It gives BVP:

$$-cv'(s) = v''(s) + f(v), \qquad \lim_{s\to-\infty} v(s) = u_-, \qquad \lim_{s\to+\infty} v(s) = u_+.$$

Here, $u_\pm$ have to be zeros of f.

Boundary value problems appear directly for PDEs as well. However, in most cases, equations are linear. If u is a function describing concentration of some compound depending on the time t and the position $x \in \mathbb{R}^3$, then it is governed by the *diffusion equation*

$$u_t - a^2\Delta u = f(t,x),$$

where $a > 0$ is a diffusion coefficient, and f introduces an external influence: a source $f \geq 0$ or a sink $f \leq 0$ of the compound. If we know the initial values $u(0,\cdot)$ and the compound does not exit a domain $\Omega \subset \mathbb{R}^3$, then the resulting BVP is linear:

$$u_t - a^2\Delta u = f(t,x), \qquad u(0,x) = \varphi(x), \qquad \frac{\partial}{\partial\nu}u(t,x) = 0, \quad \text{for } x \in \partial\Omega.$$

Both the differential equation and boundary conditions are linear. But in many cases the f depends on u, too and $f(t,x,u)$ is not a linear function of u. Then the BVP is nonlinear. This situation is typical if we have two (or more) compounds and they interact (chemotaxis). Similarly, we can consider electric field U produced by a charge with density f in a space domain Ω. If the boundary $\partial\Omega$ is grounded, then U satisfies

$$\Delta U = 4\pi f(x), \quad x \in \Omega, \qquad U|_{\partial\Omega} = 0.$$

The BVP is linear again but f can depend nonlinearly on U, too and the BVP will be nonlinear. If we look at a physical introduction of the string (one dimensional wave) equation, then the tension at the point x and the moment t is proportional to $\sin\alpha = \frac{u_x}{\sqrt{1+u_x^2}}$ which equals approximately to

u_x for small deviations. This approximation lead to the linear equation – the string equation. However, without it we get a nonlinear equation

$$u_{tt} = c^2 \frac{\partial}{\partial x} \frac{u_x}{\sqrt{1 + u_x^2}}.$$

In most cases, a differential equation is linear but additional conditions - initial and/or boundary - are very simple and linear. However a nonlinear dependence on u in boundary conditions is also possible. In differential equations considered above derivatives of unknown function were computed at the same point (time t or space x) as other functions appearing in the equation. If these points are different, we have the so-called functional-differential equations. The most natural are ordinary *differential equations with delay*:

$$u'(t) = f(t, u(t - \tau))$$

where τ is a given positive constant (or a given positive function). Initial conditions for such equations have the form: u is a given function on the interval $[-\tau, 0]$ and in the second order equation one can study the Dirichlet BVP:

$$u''(t) = f(t, u(t - \tau)), \qquad u(t) = \varphi(t), \;\; t \in [-\tau, 0], \qquad u(T) = 0,$$

where $T > 0$ is given. The derivative of the unknown function can depend on the whole behaviour of u in the past as we have in *integral-differential equations*:

$$u'(t) = \int_{-\infty}^{t} K(t, s) f(s, u(s))\, ds.$$

Here, K and f are given and the initial condition has the form:

$$u|_{[-\tau,0]} = \varphi.$$

The topological methods that we prefer below can be applied for all kinds of problems since they have the form of a nonlinear equation in some Banach spaces (sometimes more general spaces are needed: Fréchet or locally convex spaces or manifolds if some constraints appear).

The main question we shall answer is the existence of a solution. The problem is crucial not only from theoretical point of view. Almost all numerical methods approximating such solutions can be applied to equations without solutions, hence they approximate nothing. Most of engineers believe equations they work on have a solution since "the real world" exists, but it is not true. The equations are only mathematical model of some real items and sometimes circumstances of experiments need changes in the model. If the existence of a solution is proved then numerical methods can be applied but the numerics is beyond this short monograph. Obviously, if the existence is obtained by the Contraction Principle then iterations tend to the unique solution and even the error can be controlled effectively. Similarly, if a solution is a minimum of a functional (variational methods), then many gradient methods are applicable, the steepest descent method, for example.

Linear equations corresponding linear BVPs have only one solution or infinitely many solutions composing a hyperplane. There is no such an alternative for nonlinear BVPs. We shall show some methods giving the existence of at least two or more solutions. The set of solutions (even if it is infinite) is discrete usually and we can approximate their points if we localize them. Obviously, for real world problems, the existence of many solutions means we should indicate which one corresponds the studied case.

Chapter 2

Linear BVPs for ODEs

Consider linear equation of the first-order:

$$x' - A(t)x = r(t) \tag{2.1}$$

where $A : [\alpha, \beta] \to L(\mathbb{R}^n)$, $r : [\alpha, \beta] \to \mathbb{R}^n$ are continuous, with the ,,boundary" condition:

$$Bx = b_0 \tag{2.2}$$

where $B \in L(C([\alpha, \beta], \mathbb{R}^n), \mathbb{R}^n)$, $b_0 \in \mathbb{R}^n$. This condition contains typical: initial one $Bx := x(\alpha)$, periodic one $Bx := x(\beta) - x(\alpha)$, a nonlocal condition $Bx := \int_\alpha^\beta x(s)\, ds$, Nicoletti BC $Bx := (x_i(t_i))_{i=1}^n$. In most cases (except the last two), $Bx := B_1 x(\alpha) + B_2 x(\beta)$, where $B_{1,2}$ are linear operators on $\mathbb{R}^n$.

Theorem 1. *Let U denotes the resolvent operator for A, i.e. $U : [\alpha, \beta]^2 \to L(\mathbb{R}^n)$ satisfies $\frac{\partial}{\partial t} U(t, s) = A(t)U(t, s)$, $U(t, t) = I$. If $B\,(U(\cdot, \alpha)x_0) = 0$ implies $x_0 = 0$, then for any r and any b_0 BVP* (2.1)-(2.2) *has the unique solution. The assumption is equivalent to the claim that $x' + A(t)x = 0$, $Bx = 0$ has only trivial solution. For $Bx = B_1 x(\alpha) + B_2 x(\beta)$, the assumption means $B_1 + B_2 U(\beta, \alpha)$ is an isomorphism.*

If the above condition is satisfied, we call the problem (also for nonlinear equations) nonresonant, if not – resonant. How can you solve the general linear BVP which is nonresonant? First, you should find a solution to $x' = A(t)x$, $Bx = b_0$ by means of the resolvent. $U(\cdot, \alpha)x_0$ is a general solution to the differential equations, hence one needs to solve linear algebraic system:

$$B\,(U(\cdot, \alpha)x_0) = b_0.$$

The solution is the sum of this function and the solution of

$$x' = A(t)x + r(t), \qquad Bx = 0. \tag{2.3}$$

The last one can be described due to the *Variation of Constants Formula*:

$$x(t) = U(t,\alpha)x_0 + \int_\alpha^t U(t,s)r(s)\,ds$$

is a solution of (2.1) with initial value $x(\alpha) = x_0$. Hence the problem is again algebraic:

$$B\left(U(\cdot,\alpha)x_0\right) = -B\left(\int_\alpha^{\cdot} U(\cdot,s)r(s)\,ds\right).$$

In the case $Bx = B_1x(\alpha) + B_2x(\beta)$, you have the explicit formula.

Theorem 2. *The unique solution of nonresonant BVP* (2.3) *with* $Bx = B_1x(\alpha) + B_2x(\beta)$ *is*

$$x(t) = \int_\alpha^\beta G(t,s)r(s)\,ds$$

where $G : [\alpha,\beta]^2 \to L(\mathbb{R}^n)$ *is the so-called* Green function

$$G(t,s) := \begin{cases} -U(t,\alpha)[B_1 + B_2U(\beta,\alpha)]^{-1}B_2U(\beta,s) + U(t,s) & \textit{for} \quad s < t, \\ -U(t,\alpha)[B_1 + B_2U(\beta,\alpha)]^{-1}B_2U(\beta,s) & \textit{for} \quad s > t. \end{cases} \tag{2.4}$$

Notice that the Green function is defined on the square without its diagonal, $G(\cdot,s)$ satisfies linear homogeneous equation $x' = A(t)x$ for any s and $Bx = 0$ and

$$\lim_{t\to s+} G(t,s) - \lim_{t\to s-} G(t,s) = I.$$

These conditions uniquely define the Green function. (Check it.)

Similarly one can consider higher order equations

$$x^{(m)} + A_{m-1}(t)x^{(m-1)} + \ldots + A_1(t)x' + A_0(t)x = r(t)$$

with boundary conditions $Bx = b_0$, where B is a bounded linear operator on the space $C^{m-1}([\alpha,\beta],\mathbb{R}^n)$ with values in $\mathbb{R}^{nm}$. To avoid technicalities, we restrict ourselves to the second-order equations

$$x'' + A_1(t)x' + A_0(t)x = r(t) \tag{2.5}$$

with BC

$$\begin{aligned} B_{11}x(\alpha) + B_{12}x(\beta) + B_{13}x'(\alpha) + B_{14}x'(\beta) = b_1, \\ B_{21}x(\alpha) + B_{22}x(\beta) + B_{23}x'(\alpha) + B_{24}x'(\beta) = b_2, \end{aligned} \tag{2.6}$$

where $A_{0,1} : [\alpha, \beta] \to L(\mathbb{R}^n)$, $r : [\alpha, \beta] \to \mathbb{R}^n$ are continuous, $B_{ij} \in L(\mathbb{R}^n)$, $b_i \in \mathbb{R}^n$. This BC contain most important cases: Dirichlet, Neumann, periodic and Sturm-Liouville conditions. We have the similar dichotomy: nonresonant and resonant BVPs.

Theorem 3. *If homogeneous BVP* (2.5) *with* $r = 0$, (2.6) *with* $b_1 = b_2 = 0$ *has only trivial solution, then* (2.5),(2.6) *has the unique solution for each* r, b_1 *and* b_2.

This solution is the sum of:

(1) the solution of homogeneous (2.5) *with initial conditions* $x(\alpha) = x_0$, $x'(\alpha) = x_1 : x(t) = U(t, \alpha)(x_0, x_1)$ (U *– the resolvent of the second-order equation) with* $(x_0, x_1) \in R^{2n}$ *chosen such that* (2.6) *is satisfied;*

(2) the solution of (2.5) *with* $Bx = 0$ *given in the form*

$$x(t) = \int_\alpha^\beta G(t, s)r(s)\, ds,$$

where G *is again called the Green function of the problem.*

Now, the Green function is continuous on the square, satisfies homogeneous differential equation and $Bx = 0$ as the function of the first variable and

$$\lim_{t\to s+} \frac{\partial}{\partial t} G(t, s) - \lim_{t\to s-} \frac{\partial}{\partial t} G(t, s) = I.$$

These conditions uniquely define the Green function. (Check it.)

The above results enables us to reduce the most important nonlinear BVPs to some fixed point problems. For example, in a nonresonant case

$$x' - A(t)x = f(t, x), \qquad B_1x(\alpha) + B_2x(\beta) = 0$$

or

$$\begin{aligned} x'' + A_1(t)x' + A_0(t)x = f(t, x), \\ B_1(x(\alpha), x(\beta), x'(\alpha), x'(\beta)) = 0, \\ B_2(x(\alpha), x(\beta), x'(\alpha), x'(\beta)) = 0, \end{aligned}$$

is equivalent to finding continuous solution to the following integral equation:

$$x(t) = \int_\alpha^\beta G(t, s)f(s, x(s))\, ds.$$

One can treat the right-hand side of this equation as definition of a nonlinear operator acting on the Banach space $C([\alpha,\beta],\mathbb{R}^n)$ and the question is to know if it has a fixed point. Perhaps you know some fixed point theorems but we need (and develop) deeper methods.

Exercise 1. *Find Green functions for x'' with Dirichlet's boundary conditions (BC). Find a condition that guarantees $x' - A(t)x$ with periodic condition is nonresonant and, similarly for $x'' - A(t)x$ and Neumann's conditions. Next, find Green functions for both cases.*

Exercise 2. *Consider symmetric $A \in L(\mathbb{R}^n)$ being invertible. Then there are two orthogonal subspaces X_+ and X_- such that $\mathbb{R}^n = X_+ \oplus X_-$ and $A(X_+) \subset X_+$ and $A(X_-) \subset X_-$ and $\sigma(A|X_+) \subset (0,\infty)$, $\sigma(A|X_-) \subset (-\infty, 0)$. $\sigma(A)$ stands for the spectrum of A. Find all solutions of $x'' = Ax$. (Look first at the case $n = 1$ and $A > 0$ and $A < 0$.) Here $\sigma(B)$ stands for the spectrum of operator B (only eigenvalues in finite dimension).*

One can consider differential equations defined on the whole real line with an additional condition – the boundedness of a solution – treated as BVP. First, look at the problem of the existence of bounded on $\mathbb{R}$ solutions to

$$x' - Ax = r(t),$$

where $r : \mathbb{R} \to \mathbb{R}^n$ is continuous and bounded and $\sigma(A)$ does not meet the imaginary axis, thus it is divided into two sets $\sigma_-(A)$ – eigenvalues with negative imaginary part and $\sigma_+(A)$ – eigenvalues with positive this part. Then $\mathbb{R}^n$ is the direct sum of two invariant subspaces

$$\mathbb{R}^n = X_- \oplus X_+$$

and let $P_\pm$ denote projectors onto one of these subspaces along the second one. Define the *main Green function*:

$$G_A(t) := \begin{cases} \exp(tA)P_- & \text{for} \quad t > 0, \\ -\exp(tA)P_+ & \text{for} \quad t < 0. \end{cases}$$

Notice that

$$\|G_A(t)\| \le N \exp(-\nu|t|),$$

where $N > 0$ and $\nu \in (0, d(\sigma(A), i\mathbb{R}))$ – $d(\cdot,\cdot)$ distance between sets, for any $t \in \mathbb{R}$ and $t \mapsto G_A(t)x_0$ is a solution of $x' = Ax$ for $t > 0$ and for $t < 0$. Moreover, $G_A(0+) - G_A(0-) = I$ as in the definition of the Green function for usual boundary conditions.

Exercise 3. *Prove that the unique bounded on $\mathbb{R}$ solution to $x' - Ax = r(t)$ is*

$$x(t) = \int_{-\infty}^{\infty} G_A(t-s)r(s)\,ds.$$

Exercise 4. *Find the Green function for $x'' = Ax$, x bounded on $\mathbb{R}$, where A is symmetric and positive, i.e. $\langle Ax, x\rangle > 0$ for $x \neq 0$. Prove that for any bounded and continuous function $f : \mathbb{R} \to \mathbb{R}^n$ there exists the unique bounded solution of $x'' = Ax + f(t)$. Find it.*

Similar result is valid for A symmetric and negative and f integrable on $\mathbb{R}$.

Remark. All problems considered in exercices can be generalized to equations with A being a linear bounded operator in a Banach space X. The assumption A is symmetric needs a scalar product in X so it is a Hilbert space and A should be selfadjoint. For differential equations in Banach spaces you can refer to [13].

Chapter 3

Topological methods of nonlinear analysis

3.1 Metric fixed points theorems

Theorem 4. *(Banach Contraction Principle [18]) If X is a complete metric space and $T : X \to X$ is a Lipschitz map with constant $q < 1$, i.e.*

$$d(Tx, Ty) \leq qd(x, y), \qquad x, y \in X,$$

then there exists the unique $x_\infty \in X$ such that $Tx_\infty = x_\infty$. This point is a limit of the sequence of successive aproximants starting with arbitrary x_0 : $x_{n+1} = Tx_n$. We can also estimate an error:

$$d(x_m, x_\infty) \leq \frac{q^m}{1-q} d(Tx_0, x_0).$$

T satisfying Lipschitz condition with constant < 1 is called a *contraction*. There are a lot of generalizations of this result for $q = 1$ – nonexpansive maps, however, they have small potential for applications. They need special structure of the space: X is a convex closed subset of a Hilbert space (or a Banach space of a special kind). We loose the uniqueness of a fixed point in the assertion as well.

An application of the Contraction Principle for $T : C([\alpha, \beta], \mathbb{R}^n) \to C([\alpha, \beta], \mathbb{R}^n)$ of the form

$$Tx(t) := \int_\alpha^\beta G(t, s) f(s, x(s))\, ds \tag{3.1}$$

is very simple. Let $\gamma := \sup_{t,s} |G(t,s)|$, $f : [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|f(t,x) - f(t,y)| \le q|x-y|, \qquad x, y \in \mathbb{R}^n, \ \ t \in [\alpha, \beta] \tag{3.2}$$

and $\gamma \cdot q \cdot (\beta - \alpha) < 1$, then T satisfies the conditions of Banach Principle.

More explicit example is BVP:

$$-x'' = f(t,x), \qquad x(0) = 0 = x(1).$$

The Green function equals

$$G(t,s) = \begin{cases} s(1-t) & s < t \\ t(1-s) & s > t, \end{cases}$$

$\gamma = 1/4$. Therefore, this Dirichlet problem has the unique solution if (3.2) holds for f with $q < 4$.

Exercise 5. *Improve the above result by enlarging condition $q < 4$ by $q < 8$ (use better estimates).*

Exercise 6. *Prove the existence of a* global *solution to initial problem*

$$x' = f(t,x), \qquad x(\alpha) = x_0,$$

where $f : [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies Lipschitz condition w.r.t. x. In the space $C([\alpha, \beta], \mathbb{R}^n)$ use the norm *introduced by Adam* Bielecki*:*

$$\|\varphi\| := \sup_t e^{-Lt} |\varphi(t)|,$$

where L is a Lipschitz constant.

Exercise 7. *Prove that equation*

$$(p(t)x')' + q(t)x = f(t)$$

has a solution converging to 0 as $t \to \infty$ provided that $p, q : [0, \infty) \to \mathbb{R}$ are continuous, $p(t) > 0$, $q(t) \ge 0$ for all t, functions $1/p$ and f are integrable on $[0, \infty)$, function $q \cdot P$ is also integrable, where $P(t) := \int_t^\infty 1/p$.

Replace the problem by an integral equation with integral operator acting on the space $C_0[0, \infty)$ – of all continuous real functions tending to 0 at ∞ with the supremum norm.

Exercise 8. *Let* $f : [\alpha, \beta] \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies*

$$|f(t,x,x') - f(t,y,y')| \leq K|x-y| + L|x'-y'|, \qquad t \in [\alpha, \beta], \;\; x, x', y, , y' \in \mathbb{R},$$

where K *and* L *are positive constants. Prove the existence of unique solution to*

$$x'' = f(t, x, x'), \qquad x(\alpha) = A, \quad x(\beta) = B$$

A, B *any constants, provided that*

$$\frac{K}{8}(\beta - \alpha)^2 + \frac{L}{2}(\beta - \alpha) < 1.$$

Use the space of C^1*-function with the norm*

$$\|\varphi\| := \sup_t \left(K|\varphi(t)| + L|\varphi'(t)| \right).$$

Exercise 9. *Try to get similar results for Neumann and periodic problems for* $x'' = f(t, x)$.

Exercise 10. *Prove the existence of a bounded on* $\mathbb{R}$ *solution to equation* $x'' = Ax + f(t, x)$, *where* A *is symmetric and positive and* f *is continuous and satisfies the Lipschitz condition w.r.t* x *with sufficiently small constant.*

3.2 Topological fixed points theorems in finite dimension

The most known result in this direction is:

Theorem 5. *(*Brouwer Fixed Point Theorem [15]) *Any continuous mapping* $F : B \to B$ *has a fixed point, where* B *is a closed ball in* $\mathbb{R}^n$.

We say that a topological space X has the fixed point property if all continuous maps of this space has a fixed point. It is easy to see that if X is homeomorphic to B, then it has this property and if X has it and $Y \subset X$ is its retract, i.e. there exists continuous $r : X \to Y$ being an extension of the identity of Y, then Y has the fixed point property.

There are many essentially different proofs of the Brouwer Theorem. We will prove it after a development of the degree theory due to Nagumo, however all finite dimensional degree theories are equivalent and usually they are called Brouwer's degree theory. Let Ω be an open bounded subset of $\mathbb{R}^n$, $f : \overline{\Omega} \to \mathbb{R}^n$ be continuous and $p \in \mathbb{R}^n \setminus f(\partial\Omega)$. We'll define an integer denoted by $\deg(f, \Omega, p)$ which counts an *algebraic* number of solutions to $f(x) = p$ in Ω.

First, suppose $f \in C^1(\Omega)$ and p is the so called regular value of f : for each $x \in f^{-1}(p)$, $\det f'(x) \neq 0$. Points of $f^{-1}(p)$ are isolated by the Inverse Function Theorem thus this set is finite and we define

$$\deg(f, \Omega, p) := \sum_{f(x)=p} \operatorname{sgn} \det f'(x).$$

One can prove that this number is the same for all p' in a neighborhood of p and we use the deep analytic result Sard's Theorem:

The set of critical values of f (the value is critical if it is not regular) has the Lebesgue measure in $\mathbb{R}^n$ equal to 0. Thus the set of regular values are dense.

Hence, we can drop the assumption that p is regular value of f. Similarly, the degree does not change if one perturbs slightly f. Thus, we take the approximation of any continuous $f : \overline{\Omega} \to \mathbb{R}^n$ by smooth g (the Weierstrass Theorem) and define

$$\deg(f, \Omega, p) := \deg(g, \Omega, p).$$

The *Brouwer degree* has several properties:

1. (additivity) if $f : \overline{\Omega} \to \mathbb{R}^n$, Ω_1, Ω_2 are disjoint open subsets of Ω such that $p \notin f(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then

$$\deg(f, \Omega, p) = \deg(f, \Omega_1, p) + \deg(f, \Omega_2, p).$$

2. (Kronecker's property) If $\deg(f, \Omega, p) \neq 0$, then equation $f(x) = p$ has a solution in Ω.

3. (*homotopy invariance*) If $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is continuous and $p \notin h(\partial\Omega \times [0, 1])$, then

$$\deg(h(\cdot, 0), \Omega, p) = \deg(h(\cdot, 1), \Omega, p).$$

4. (normalization) $\deg(I, \Omega, p) = 1$ for any $p \in \Omega$.

These properties defines the Brouwer degree uniquely.

Remarks.

- The additivity holds for any finite number of open subsets.
- If $\Omega' \subset \Omega$ is open and $p \notin f(\overline{\Omega} \setminus \Omega')$, then $\deg(f, \Omega', p) = \deg(f, \Omega, p)$.
- If $f|\partial\Omega = g|\partial\Omega$, then $\deg(f, \Omega, p) = \deg(g, \Omega, p)$.
- $\deg(f, \Omega, p) = \deg(f - p, \Omega, 0)$.

Now, we are able to prove three fixed points theorems:

Theorem 6. *If $f : \bar{B}(0, R) \to \mathbb{R}^n$ is continuous and one of the following conditions holds:*

- *(Rothe)* $\|f(x)\| \leq \|x\|$;
- *(Altman)* $\|f(x) - x\|^2 \geq \|f(x)\|^2 - \|x\|^2$;
- *(Krasnosielski)* $\langle f(x), x\rangle \leq \|x\|^2$;

for $x \in \partial B(0, R)$, then f has a fixed point.

Rothe's theorem is a stronger version of the Brouwer one.

There are many other application of the degree theory (there is no retraction of the ball on its boundary; any homeomorphic image of a sphere cuts $\mathbb{R}^n$ into two connected components – *Jordan Theorem* for $n = 2$; if $n > 1$ is odd, then on $\partial B(0, R)$ there is no tangent nonvanishing vector field –no hairing of spheres.

The following *Borsuk's theorem* is very strong:

Theorem 7. *If $\Omega \subset \mathbb{R}^n$ is open, bounded and symmetric w.r.t. the origin and$f : \overline{\Omega} \to \mathbb{R}^n$ is odd on the boundary, then deg$(f, \Omega, 0)$ is odd integer. Thus f has a fixed point.*

Exercise 11. *Prove the* Borsuk Antipodal Theorem*: there is no $f : \bar{B}(0, R) \to \mathbb{R}^n \setminus \{0\}$ odd on the boundary.*

Exercise 12. *Prove the* Borsuk-Ulam Theorem*: for each continuous map $f : \partial B(0, R) \to \mathbb{R}^{n-1}$, $B(0, R)$ is a ball in $\mathbb{R}^n$, there exists a pair of antipodal points such that $f(x) = f(-x)$.*

We will apply the theory to a problem of the existence of a T-periodic solution to the first order equation $x' = g(t, x)$, where $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and T-periodic w.r.t. t. We assume additionally g is Lipschitz continuous with respect to x and that, for some R,

$$\langle g(t, x), x\rangle < 0, \qquad t \in [0, T], \;\; |x| = R. \tag{3.3}$$

Then all initial value problems for this equation have local solutions that are unique. Define $f(x_0) := \varphi(T)$, where φ is a solution to the initial problem $x' = g(t, x)$, $x(0) = x_0$. If $x_0 \in \bar{B}(0, R)$, then the local solution cannot exit the ball since on the boundary vectors of the fields are directed inside the ball and the solution is global for $t > 0$. In particular, it is defined for $t = T$ and $f(x_0) \in \bar{B}(0, R)$. The continuity of f follows from the continuous dependence of a solution with respect to the initial data. The Brouwer Fixed Point Theorem implies there exists x_0 such that $f(x_0) = x_0$, i.e. $\varphi(0) = \varphi(T)$, hence φ is T-periodic.

Exercise 13. *Try to prove the same under weaker condition – weak inequality in* (3.3) *(perturbe g by $-\frac{1}{n}x$, $n \in \mathbb{N}$ and use the Ascoli-Arzelá Theorem).*

Remark. One can omit the assumption of Lipschitz continuity, since the existence of a local solution will be obtained below by using Schauder's Fixed Point Theorem. Since the solution is not uniquely determined, the function f defined as above is multivalued $f : \bar{B} \to 2^{\bar{B}}$. Due to the work of Aronszajn [8] it has values being R_δ sets (countable intersection of contractible sets – see any book on topology). The generalization of the Brouwer Theorem for such mappings introduced by Kakutani gives $x_0 \in f(x_0)$ and this solution ends the proof. The continuity of f is replaced by the lower continuity:
for any x and $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $f(x) \subset B(f(x_0), \varepsilon)$.

For the fixed point theory of multivalued maps refer to [28].

3.3 Leray-Schauder theory

Balls in infinite dimensional Banach spaces are noncompact and they have no fixed point property as the following example shows:

Let $E = l^2$ be the space of real sequences such that for $x = (t_n)$,

$$\|x\| := \left(\sum_{n=0}^{\infty} |t_n|^2\right)^{1/2} < \infty,$$

$f : B \to B$ is a map acting on the closed unit ball $B \subset E$ defined by the formula

$$f(x = (t_n)) := (s_n), \qquad s_0 := \sqrt{1 - \|x\|^2}, \qquad s_n := t_{n-1}, \;\; n \geq 1.$$

f is continuous and maps B on its boundary, hence if it has a fixed point x, then $\|x\| = 1$, but this implies $x = f(x) = 0$ – a contradiction. More complicated arguments show that continuous maps without fixed points can be defined in any infinite dimensional Banach space.

Since the existence of a degree having four properties of the Brouwer degree in a Banach space E implies the fixed point property for the ball in E, one cannot define such a degree in infinite dimensional spaces. The crucial point is the homotopy property: if it holds for **every** continuous maps, then all maps will be homotopical nad thus they have degree 0. Juliusz Schauder and Jean Leray had an idea to restrict the family of maps to *compact* perturbations of the identity: $F = I - f$, where f is continuous and maps bounded sets into relatively compact ones. The homotopy property for such mappings says that if one compact map f can be continuously deformed to the other g within the family, then $I - f$ and $I - g$ have the same degree. The idea was suggested by the Schauder Fixed Point Theorem proved in 1930 : all compact mappings acting on a closed ball in E have fixed points.

The construction of the Leray-Schauder degree is based on several lemmas.

Lemma 1. If $K \subset E$ is compact, then, for any $\varepsilon > 0$, there exist a finite dimensional space $E_\varepsilon \subset E$ and a continuous mapping $P_\varepsilon : K \to E_\varepsilon$ such that $\|P_\varepsilon(x) - x\| < \varepsilon$ for $x \in K$.

If we take a cover of K by balls $B(x_j, \varepsilon)$, $j = 1, \ldots, n$, and a continuous partition of identity φ_j, $j = 1, \ldots, n$, subbordinated to the cover (support of φ_j is included in $B(x_j, \varepsilon)$), then $E_\varepsilon := \mathrm{Lin}(x_j,\ j = 1, \ldots, n)$,

$$P_\varepsilon(x) := \sum_{j=1}^{n} \varphi_j(x) x_j.$$

Lemma 2. Mappings of the form $F = I - f$ with f compact are *proper*, i.e. $F^{-1}(K)$ is compact for any K compact, and they map bounded closed sets into closed ones.

Let Ω be an open bounded set in E, $f : \overline{\Omega} \to E$ – a compact mapping, $F := I - f$, and $y \notin F(\partial\Omega)$. By Lemma 2, the distance $d(y, F(\partial\Omega)) > 0$. Let

$\varepsilon > 0$ be less than this distance. Due to Lemma 1 there exist a continuous mapping $f_\varepsilon := P_\varepsilon f$ such that $|f_\varepsilon(x) - f(x)| < \varepsilon$ for any x from the bounded domain of f. Notice that f_ε takes values in a finite dimensional space E_ε and without loss of generality $y \in E_\varepsilon$. Moreover, $\Omega_\varepsilon := \Omega \cap E_\varepsilon$ is open and bounded in E_ε and $y \notin F_\varepsilon(\partial\Omega)$, where $F_\varepsilon := I - f_\varepsilon$, thanks to the choice of ε. Thus, one can define

$$\deg_{LS}(F, \Omega, y) := \deg(F_\varepsilon|_{\Omega_\varepsilon}, \Omega_\varepsilon, y)$$

and one should prove this definition does not depend on the choice of E_ε (the independence on the choice of approximate function follows from the homotopy property for the Brouwer degree).

Lemma 3. If $f : \overline{\Omega} \to E'$ is continuous $\dim E' < \infty$ and $E' \subset E''$ with $\dim E'' < \infty$, then

$$\deg(F|_{E'}, \Omega \cap E', y) = \deg(F|_{E''}, \Omega \cap E'', y).$$

The proof for the first step of the Brouwer degree definition relies on calculations of jacobians $F'(x)$ at $x \in F^{-1}(y)$ – these points and determinants are the same for both subspaces. The proof for critical values y and $f \notin C^1$ follows form the first step.

Remark. The only question we have omitted is the definition of the Brouwer degree in any finite dimensional linear space E. For such a space, there is an isomorphic homeomorphism $h : E \to \mathbb{R}^n$. If $\Omega \subset E$ is open and bounded, $f : \overline{\Omega} \to E$ is continuous and $p \in E \setminus f(\partial\Omega)$, then the definition

$$\deg(f, \Omega, p) := \deg(h \circ f \circ h^{-1}, h(\Omega), h(p))$$

does not depend on h. The proofs of it and of all properties of the degree are straightforward.

All properties of the degree are satisfied for the Leray-Schauder degree. The only attention is needed for the homotopy property that has the form now:

If $h : [0,1] \times \overline{\Omega} \to E$ is compact, $H(\lambda, x) = x - h(\lambda, x)$, $y \notin H([0,1] \times \partial\Omega)$, then

$$\deg_{LS}(H(0,\cdot), \Omega, y) = \deg_{LS}(H(1,\cdot), \Omega, y).$$

Fixed points theorems of Rothe, Altman and Krasnosielski are satisfied for compact mapppings on balls. In particular,

Theorem 8. *(*Schauder's Fixed Point Theorem.*) If $f : K \to K$ is compact on a closed bounded and convex set K in a Banach space, then f has a fixed point.*

We are working in the space of continuous functions on a compact metric space, thus we need a compactness criterion for subsets of such spaces.

Theorem 9. *(*Ascoli-Arzelá*) A set $K \subset C(X)$ (X – compact, $C(X)$ – space of continuous functions $X \to \mathbb{R}$ with the supremum norm) is relatively compact if and only if it is equibounded $\sup_{f\in K}\sup_{x\in X}|f(x)| < \infty$ and equicontinuous, i.e. for every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, x_0) < \delta$, then $|f(x) - f(x_0)| < \varepsilon$ for all $f \in K$.*

Proof of sufficiency. We shall prove the equivalent condition: one ca choose from any sequence $f_n \in K$, $n \in \mathbb{N}$, a uniformly convergent subsequence. Take a dense sequence of points in $X : (x_k)_{k\in\mathbb{N}}$. Since $\{f_n(x_1) : \ n \in N\}$ is bounded in $\mathbb{R}$, there exists a convergent subsequence $(f_n^{(1)}(x_1))$. We repeat the same arguments for $\{f_n^{(1)}(x_2) : n \in \mathbb{N}\}$ and get convergent $(f_n^{(2)}(x_2))$. Induction. Observe that $(f_n^{(m)}(x_p))$ is convergent not only for $p = m$ but also for $p < m$. Take the diagonal sequence $f_n^{(n)}$; it is convergent for all points of $\{x_k : \ k \in \mathbb{N}\}$.

Take $\varepsilon > 0$. By the equicontinuity there is $\delta > 0$ such that

$$d(x, x') < \delta; \Rightarrow \ |f_n^{(n)}(x) - f_n^{(n)}(x')| \le \frac{\varepsilon}{3}.$$

A covering of X by balls centered at x_k, $k \in \mathbb{N}$, with radius δ, has a finite subcovering

$$X \subset B(x_{k_1}, \delta) \cup \dots B(x_{k_p}, \delta)$$

and we can find N such that, for $n, m \ge N$ and $j = 1, \dots, p$,

$$|f_n^{(n)}(x_{k_j}) - f_m^{(m)}(x_{k_j})| \le \frac{\varepsilon}{3}.$$

Hence, for any x and $n, m \ge N$,

$$|f_n^{(n)}(x) - f_m^{(m)}(x)| \le |f_n^{(n)}(x) - f_n^{(n)}(x_{k_j})| + |f_n^{(n)}(x_{k_j}) - f_m^{(m)}(x_{k_j})|$$

$$+|f_m^{(m)}(x_{k_j}) - f_m^{(m)}(x)| \le \varepsilon$$

where we took x_{k_j} such that $x \in B(x_{k_j}, \delta)$.

Exercise 14. *Prove the necessity.*

Remark. The simplest method to prove equicontinuity in applications is showing that all functions in K satisfy the Lipschitz condition with a common constant. The last holds if we prove equiboundedness of the set $\{f' : f \in K\}$.
The Ascoli-Arzelá Theorem has the same form in the case $C(X, E)$, where E is a finite dimensional space (prove it). The case of $\dim E = \infty$ needs an additional assumption:

$\{f(x) : f \in K\}$ is relatively compact in E for each $x \in X$. The proof does not change.

Exercise 15. *Prove that the* Urysohn operator $T : C([\alpha, \beta], \mathbb{R}^n) \to C([\alpha, \beta], \mathbb{R}^n)$ *defined by the formula:*

$$T(x)(t) := \int_\alpha^\beta K(t, s, x(s))\, ds,$$

where $K : [\alpha, \beta]^2 \times \mathbb{R}^n \to \mathbb{R}$ *is continuous, is completely continuous. Try to improve the result in such a way that it will contain cases* $K(t, s, x) := G(t, s)f(s, x)$ *with jumps in* g.

Exercise 16. *Prove that a family of continuous functions* $K\ X \to \mathbb{R}$, X *– compact, which is equicontinuous at every* $x_0 \in X$, *satisfies*

$$\forall_{\varepsilon>0}\exists_{\delta>0}\forall_{x,x'\in X} d(x, x') < \delta \Rightarrow \forall_{f\in K}|f(x') - f(x)| < \varepsilon$$

(it is uniformly equicontinuous).

If we are working in any other Banach space, we need sufficient condition for the compactness (relative compactness) of subsets in such spaces. For example, in the space of continuous functions vanishing at infinity $C_0[0, \infty)$ with the topology of the uniform convergence we need an additional condition:
for any $\varepsilon > 0$, there exists T such that $|f(t)| < \varepsilon$ for $f \in K$ and $t > T$.
For $E = BC[0, \infty)$ with the uniform convergence topology, the additional condition is as follows: for $\varepsilon > 0$, there is $T > 0$ and $\delta > 0$ such that if $|f(t) - g(t)| < \delta$ for $t \in [0, T]$, then $|f(t) - g(t)| < \varepsilon$ for $t > T$ and every $f, g \in K$. These results generate appropriate compactness criteria for spaces of C^p-functions. We refer to [45, 46] for compactness criteria in general spaces of bounded continuous functions with the topology of uniform convergence.

The problem is quiet different in Hilbert and, more general, Sobolev spaces.

Theorem 10. *If e_n, $n \in \mathbb{N}$, is an orthonormal basis of a Hilbert space H, then $K \subset H$ is relatively compact iff K is bounded and the convergence of Fourier series of $f \in K$ are uniform, i.e.*
for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|x - \sum_{n \leq n_0} \langle x, e_n \rangle e_n\| \leq \varepsilon$$

for $x \in K$, or equivalently

$$\sum_{n=n_0+1}^{\infty} |\langle x, e_n \rangle|^2 \leq \varepsilon^2$$

for $x \in K$.

For $L^p(\mathbb{R}^n)$ (with respect to the Lebesgue measure), $p \in [1, \infty)$, the *compactness criterion of Kolmogorov-Riesz* is useful, see [81]. For Sobolev spaces, the compact inclusions in spaces of continuous functions or L^p are applied. The problem is large but we restrict in applications to the case of spaces of continuous functions, hence we omit the details.

Chapter 4

Nonlinear BVPs for ODEs

4.1 Using topological degree to simple BVPs

Theorem 11. *(*Peano*) Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and bounded. The initial value problem*

$$x' = f(t,x), \qquad x(0) = 0,$$

has a global solution.

Proof. Let X be the subspace of C^1 functions $[0,T] \to \mathbb{R}^n$ which vanish at $t = 0$, the operator of differentiation L is a bijection of X onto $C := C([0,T], \mathbb{R}^n)$ with the inverse

$$L^{-1}(x)(t) = \int_0^t x(s)\, ds.$$

Obviously, L is a homeomorphism of Banach spaces. Denote by $J : X \to C$ the natural embedding and by $F : C \to C$ the *superposition (Nemytski) operator*

$$F(x)(t) := f(t, x(t)).$$

Then our problem is equivalent to finding a fixed point to $JL^{-1}F : C \to C$. This mapping is compact since F is continuous and maps bounded sets into bounded ones and JL^{-1} is compact by Ascoli-Arzelá Theorem. Moreover the range of F is a bounded set, hence $JL^{-1}F$ maps the whole space into a ball. A fixed point exists due to Schauder's Fixed Point Theorem.

Exercise 17. *Prove the existence of a solution to the Dirichlet BVP:*

$$x'' = f(t, x, x'), \qquad x(0) = 0 = x(T),$$

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded.

More sophisticated arguments are used for proving the existence of a solution to:

$$x'' = f(t, x), \qquad x(0) = 0 = x(T), \tag{4.1}$$

where f is only continuous and, for some $M > 0$,

$$\langle f(t, x), x\rangle > 0 \qquad t \in [0, T], \ \ |x| = M. \tag{4.2}$$

Proof. We replace the homotopy family of BVPs:

$$x'' = \lambda f(t, x), \qquad x(0) = 0 = x(T),$$

$\lambda \in [0, 1]$ by integral equations:

$$x(t) = \lambda \int_0^T G(t, s) f(s, x(s))\, ds,$$

where G is the Green function of the main problem (you can compute it). The integral operator given by the right-hand side acts on the space $C := C([0, T], \mathbb{R}^n)$ and it is compact. We can use the Leray-Schauder degree of $I - \lambda K$, K is the integral operator, on the ball $B(0, M)$ in C at the point 0, if we shall show there are no fixed points on $\partial B(0, M)$ for any λ. Let $\lambda > 0$. If such a fixed point x exists, then $\varphi(t) := |x(t)|^2$ is a C^2 function with maximum value M^2 gained at some point $t_0 \in (0, T)$. It follows that $\varphi'(t_0) = 0$ and $\varphi''(t_0) \le 0$ but $\varphi'(t) = 2\langle x'(t), x(t)\rangle$ and

$$\varphi''(t) = 2|\varphi'(t)|^2 + 2\langle x''(t), x(t)\rangle = 2|x'(t)|^2 + 2\lambda\langle f(t, x(t)), x(t)\rangle$$

that is positive by (4.2).
Therefore $\deg_{LS}(I - K, B, 0) = \deg_{LS}(I, B, 0) = 1$.

Exercise 18. *You can slightly weaken assumption* (4.2) *replacing the sharp inequality by* $\geq$. *For any* $n \in \mathbb{N}$, *consider* $x'' = f(t, x) + x/n$ *with the same BCs. The Ascoli-Arzelá Theorem gives the result.*

4.2 A priori bounds for derivatives

If the right-hand side of the differential equation depends also on the derivative x', one should work in the space $C^1([0,T],\mathbb{R}^n)$ and need a priori bounds not only for $\sup|x(t)|$ but $\sup|x'(t)|$, as well. We shall prove the well-known result of this kind (comp. [29])

Theorem 12. *(*Bernstein*) Let $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ be continuous function such that:*
(i) there is $M>0$ such that $x\cdot f(t,x,0)>0$ for any t and $|x|\geq M$,
(ii) there exist continuous functions $A,B:[0,T]\times\mathbb{R}\to\mathbb{R}_+$ such that

$$|f(t,x,y)|\leq A(t,x)y^2+B(t,x),\qquad t\in[0,T],\ \ (x,y)\in\mathbb{R}^2. \tag{4.3}$$

Then the problem

$$x''=f(t,x,x'),\qquad x(0)=0=x(T)$$

has a solution.

The proof is based on similar arguments as the above to prove a priori bounds on $\sup|x(t)|<M$. In order to get similar estimates for the derivative, we divide $[0,T]$ on intervals $[a,b]$ with constant sign of x' and x' vanishes at at least one end point. For example, let $x'(a)=0$ and $x'(t)\geq 0$ for $t\in[a,b]$. Then denote by A and B, respectively, the maximum value of functions A and B on $[0,T]\times[-M,M]$ and we have

$$\frac{d}{dt}\big(\ln(Ax'(t)^2+B)\big)=\frac{2Ax'(t)x''(t)}{Ax'(t)^2+B}\leq 2Ax'(t).$$

The integration of this inequality from a to t gives

$$\ln(Ax'(t)^2+B)-\ln B\leq 4AM$$

and thus

$$|x'(t)|\leq\sqrt{\frac{B}{A}(\exp(4AM)-1)}.$$

Check similar bounds for other possibilities.

4.3 Lower and upper solutions

There is a pretty general method for solving BVPs – the method od sub- and super-solutions. We shall present it through an example: a result for Dirichlet BVP for second order ODEs.

Let $f : [\alpha, \beta] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. The C^2-function $u : [\alpha, \beta] \to \mathbb{R}$ is called a *sub-solution* for $x'' = f(t, x, x')$, if

$$u''(t) \geq f(t, u(t), u'(t)), \qquad t \in [\alpha, \beta].$$

Similarly one defines a *super-solution* v by inequality

$$v''(t) \leq f(t, v(t), v'(t)), \qquad t \in [\alpha, \beta].$$

Theorem 13. *Suppose that equation $x'' = f(t, x)$ has a sub-solution u and super-solution v such that $u(t) \leq v(t)$ for every t. Then, for each constants $A \in [u(\alpha), v(\alpha)]$ and $B \in [u(\beta, v(\beta)]$, the Dirichlet BVP:*

$$x'' = f(t, x), \qquad x(\alpha) = A, \quad x(\beta) = B$$

has a solution φ such that $u(t) \leq \varphi(t) \leq v(t)$ for $t \in [\alpha, \beta]$.

Proof. Define $g : [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ by the formulae:

$$g(t, x) := \begin{cases} f(t, v(t)) + \frac{x - v(t)}{1 + |x|} & \text{for} \quad x > v(t), \\ f(t, x) & \text{for} \quad u(t) \leq x \leq v(t), \\ f(t, u(t)) + \frac{x - u(t)}{1 + |x|} & \text{for} \quad x < u(t). \end{cases}$$

The function is a modification of f beyond a strip between graphs of u and v which preserves continuity but is bounded on $[\alpha, \beta] \times \mathbb{R}$. The BVP

$$x'' = g(t, x), \qquad x(\alpha) = A, \quad x(\beta) = B,$$

has a solution φ by the Schauder Fixed Point Theorem and we have to show that $\varphi(t) \in [u(t), v(t)]$ for $t \in (\alpha, \beta)$. If $\varphi(t) > v(t)$ for some t, then function $w(t) := \varphi(t) - v(t)$ takes its maximum at $t_0 \in (\alpha, \beta)$, hence $w(t_0) > 0$, $w'(t_0) = 0$ and

$$0 \geq w''(t_0) = \varphi''(t_0) - v''(t_0) \geq f(t_0, v(t_0)) + \frac{\varphi(t_0) - v(t_0)}{1 + |\varphi(t_0)|} - f(t_0, v(t_0)) > 0$$

–a contradiction. Similarly, if $\varphi(t) < u(t)$, then function $w(t) = \varphi(t) - u(t)$ takes its minimum at t_0, where $w(t_0) < 0$, $w'(t_0) = 0$ and

$$0 \le w''(t_0) = \varphi''(t_0) - u''(t_0) \le f(t_0, u(t_0)) + \frac{\varphi(t_0) - u(t_0)}{1 + |\varphi(t_0)|} - f(t_0, u(t_0)) < 0$$

– again a contradiction. Therefore, $u(t) \le \varphi(t) \le v(t)$ for any t and φ satisfies the differential equation with f on the right-hand side.

We only quote after [42] a similar result for f depending also on x'.

Theorem 14. *Let equation $x'' = f(t, x, x')$ has a sub- and super-solutions u and v, respectively, with $u \le v$. Moreover, let there exists function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t, x, y)| \le h(|y|) \qquad \textit{for} \quad t \in (\alpha, \beta), \quad u(t) \le x \le v(t), \quad y \in \mathbb{R}$$

and

$$\int_0^\infty \frac{s\,ds}{h(s)} = \infty.$$

Then the assertion of the previous theorem holds for $x'' = f(t, x, x')$, $x(\alpha) = A$, $x(\beta) = B$ provided that $u(\alpha) \le A \le v(\alpha)$, $u(\beta) \le B \le v(\beta)$.

The crucial point in the proof is based on an a priori bound for $|\varphi'|$ obtained similarly as in Bernstein's theorem for $h(s) = As^2 + B$. [42] gives another proof.

Exercise 19. *Show the* Knobloch Theorem*: if $x'' = f(t, x, x')$ with $f : \mathbb{R}^3 \to \mathbb{R}$ continuous and T-periodic w.r.t. t has a T-periodic sub- and super-solutions u and v, resp. with $u \le v$, then it has a T-periodic solution. Try to use similar arguments or consult with [62].*

You can refer to [14] for more information on the method of sub- and supersolutions (some authors call them lower and upper solutions).

Chapter 5

Resonant problems

5.1 Physical and mathematical notions of resonance

The origin of the notion "resonance" can be explained as follows. A pendulum moving near stable stationary point is described by equation: $x''+\omega^2 x = f(t)$, where function f is an external force (different with the gravitation which is hidden in $\omega^2 x$ term). You know that solutions of the pendulum equation with $f = 0$ are all $2\pi/\omega$-periodic $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Then all solutions of nonhomogeneous equation with $2\pi/\omega$-periodic function f have the form:

$$x(t) = \left(c_1 - \frac{1}{\omega}\int_0^t f(s)\sin\omega s\,ds\right)\cos\omega t + \left(c_2 + \frac{1}{\omega}\int_0^t f(s)\cos\omega s\,ds\right)\sin\omega t$$

and they are bounded if and only if (iff) f satisfies

$$\int_0^{2\pi/\omega} f(t)\cos\omega s\,ds = 0 = \int_0^{2\pi/\omega} f(t)\sin\omega s\,ds.$$

If at least one of these integrals do not vanish, then equation has only unbounded solutions. The physical meaning of the resonance is increasing unbounded vibrations (that can demage the modelled device after a finite time) and this is exactly the case of this equation.

Notice that the periodic problem:

$$x'' + \omega^2 x = f(t), \qquad x(0) = x\left(\frac{2\pi}{\omega}\right), \quad x'(0) = x'\left(\frac{2\pi}{\omega}\right),$$

has a solution iff the above orthogonality conditions hold. If f is $2\pi/\omega$-periodic and we take the Fourier expansion of it (we apply complex notation for simplicity)

$$f(t) \sim \sum_{n\in\mathbb{Z}} c_n \exp(in\omega t),$$

where $c_{-n} = \overline{c_n}$ for any n to ensure real values. Substitute a Fourier series into equation

$$x(t) = \sum_{n\in\mathbb{Z}} a_n \exp(in\omega t)$$

and compare coefficients:

$$c_n = \omega^2(1-n^2)a_n$$

for every n. Then all a_n can be found iff $c_1 = 0$ which means the orthogonality conditions.

We mean the orthogonality in the sense of Hilbert space L^2. One can look at the left-hand side of the equation as an operator acting in this space. $Lx := x'' + \omega^2 x$. It is linear unbounded operator defined on a subspace of C^2-functions $2\pi/\omega$-periodic. One can extend it to a larger space (the Sobolev space $H^2_{per}(0, 2\pi/\omega)$ of functions with weak first and second derivative square integrable and satisfying periodic conditions). Is it surprising that the range of this extension is orthogonal to its kernel?

It is known from Functional Analysis that for bounded linear operators $(\mathrm{Im} L) \perp \ker L^*$, where L^* is the adjoint operator. It suggests that, here, L is selfadjoint. Notions of an adjoint operator for unbounded one is rather complicated. See Milman V., Eidelman Y. Tsolomitis A. - Functional Analysis, for instance. For our purpose, all differential operators of the second-order $(p(t)x')' + q(t)x$ with Dirichlet, Neumann, periodic , Sturm-Liouville boundary conditions are selfadjoint (with appropriate choice of domain). For nonresonant problems, the selfadjointness can be seen as the symmetry of the Green function $G(t,s) = G(s,t)$ for any s,t. If a BVP is resonant, one can add λx to the linear differential operator such that it starts to be nonresonant and use the above remark.

5.2 Perturbation method

Consider a simple nonlinear resonant problem:

$$x'' + x = f(t,x), \qquad x(0) = 0 = x(\pi), \tag{5.1}$$

where $f : [0, \pi] \times \mathbb{R} \to \mathbb{R}$. We treat $x'' + x$ as the linear part, however, one can add the same linear operator S to both sides and $Sx(t) + f(t, x(t))$ is a new nonlinear term. Notice that if f is bounded then the nonlinearity after this change is unbounded. Though we can extract linear parts of the equation in many ways, there is exactly one natural if

$$\lim_{|x|\to\infty} \sup_t \frac{f(t,x)}{x} = \lambda,$$

then equation $x'' + x = f(t, x)$ should be write down as $x'' + (1-\lambda)x = g(t, x)$ where $g(t, x) := f(t, x) - \lambda x$ and g is sublinear, i.e.

$$\lim_{|x|\to\infty} \sup_t \frac{|g(t,x)|}{x} = 0.$$

Then, if $\lambda \neq 1 - m^2$ with $m \in \mathbb{N}$, the new problem has the Green function (which is bounded) and it is reduced to the fixed point problem for compact operator on $C[0, \pi]$ with a compact integral operator $T : C \to C$ with the property $\lim_{\|x\|\to\infty} \frac{\|Tx\|}{\|x\|} = 0$.

Exercise 20. *Show that there exists a ball B such that $T(B) \subset B$.*

By the Schauder Fixed Point Theorem we are done.

The situation is completely different if $\lambda = 1 - m^2$ and BVP is essentially resonant. Let us assume that $\lambda = 0$ (for simplicity, though similar arguments can be repeated for othe m.) Suppose that there exist uniform limits:

$$f_-(t) := \lim_{x\to-\infty} f(t,x), \qquad f_+(t) := \lim_{x\to+\infty} f(t,x) \tag{5.2}$$

and they are finite for all t. We shall prove that if

$$\int_0^\pi f_+(t) \sin t \, dt < 0 < \int_0^\pi f_-(t) \sin t \, dt \tag{5.3}$$

or both inequalities are reversed, then (5.1) has a solution.

Proof. Perturb the equation to $x_n'' + (1 + \varepsilon_n)x_n = f(t, x_n)$ with the same BC, where $\varepsilon_n \to 0^+$. Since BVP is nonresonant for any $n \in \mathbb{N}$, we have a sequence of their solutions x_n, $n \in \mathbb{N}$ due to the Schauder Fixed Point Theorem. If these functions are equibounded, then functions x_n'' are equibounded, as well. By the Taylor Formula

$$x(t + h) = x(t) + x'(t)h + \frac{1}{2}x''(\xi)h^2,$$

we take $t+h=\pi$ for $t\in[0,\pi/2]$ and $t+h=0$ for $t\in(\pi/2,\pi]$ and obtain $|h|\geq\pi/2$ which gives x_n' are equibounded. Hence, due to the Ascoli-Arzelá Theorem, there exists a subsequence of (x_n) which is uniformly convergent – we denote it also x_n. Thus its limit satisfies (5.1).

Now, suppose that x_n are not equibounded. Passing to a subsequence, we can assume that $\|x_n\|\to\infty$. Let

$$y_n(t):=\frac{x_n(t)}{\|x_n\|}$$

for all n. These functions satisfy equations

$$y_n''+(1+\varepsilon_n)y_n=\frac{f(t,\|x_n\|y_n)}{\|x_n\|} \tag{5.4}$$

and Dirichlet conditions and they are equibounded. The right-hand side and $\varepsilon_n y_n$ tend to 0 uniformly, hence by the above arguments, y_n tend to a solution of $y''+y=0$. It follows that $y_n\rightrightarrows\pm\sin t$. If we have the sign $+$ then $f(t,\|x_n\|y_n(t))\to f_+(t)$ for $t\in(0,\pi)$. Then, multiplying (5.4) by $\sin t$ and integrating over $[0,\pi]$, we get on the left

$$\varepsilon_n\int_0^\pi y_n(t)\sin t\,dt$$

and on the right

$$\frac{1}{\|x_n\|}\int_0^\pi f(t,\|x_n\|y_n(t))\sin t\,dt.$$

It is impossible by the first inequality (5.3). Similar considerations for $y_n\rightrightarrows-\sin t$ lead to the contradiction with the second inequality (5.3).

If both inequalities in the theorem are reversed, you need $\varepsilon_n\to0^-$.

Exercise 21. *Consider the Dirichlet BVP*

$$x''+m^2x=f(t,x),\qquad x(0)=0=x(\pi)$$

where $m\in\mathbb{N}$, $f:[0,\pi]\times\mathbb{R}\to\mathbb{R}$ *is continuous and there exist uniform finite limits*

$$\lim_{x\to-\infty}f(t,x)=:f_-(t),\qquad\lim_{x\to+\infty}f(t,x)=:f_+(t).$$

Prove that this BVP has a solution if the following numbers have different signs:

$$\int_{t\in(0,\pi),\,\sin mt>0} f_-(t)\sin mt\,dt + \int_{t\in(0,\pi),\,\sin mt<0} f_+(t)\sin mt\,dt, \tag{5.5}$$

$$\int_{t\in(0,\pi),\,\sin mt>0} f_+(t)\sin mt\,dt + \int_{t\in(0,\pi),\,\sin mt<0} f_-(t)\sin mt\,dt. \tag{5.6}$$

The first paper, where a resonant problem has been studied appeared in 1970 written by E.M Landesman and A.C. Lazer [51]. They studied the Dirichlet problem for an elliptic equation:

$$\Delta u - \lambda_1 u = f(x,u), \qquad u|\partial\Omega = 0,$$

where f satisfies our assumptions and λ_1 is the first eigenvalue for Laplacian. It is well-known that this eigenvalue is simple: the eigenspace is spanned by a function w which is positive on Ω. The sufficient condition for the existence of a solution was the form: numbers

$$\int_\Omega f_-(x)w(x)\,dx, \qquad \int_\Omega f_+(x)w(x)\,dx$$

have the opposite signs. They are known as the *Landesman-Lazer conditions* also for any other resonant problems. Notice that these conditions describes the asymptotic behaviour of the nonlinear part on the kernel of linear part of the equation.

Exercise 22. *Let us study:*

$$x'' + x = f(t,x), \qquad x(0) = 0 = x(\pi),$$

where $f : [0,\pi]\times\mathbb{R}\to\mathbb{R}$ is continuous, bounded and $f(t,\cdot)$ are monotonic for any t. Show that the Landesman-Lazer conditions (5.3) *are necessary for the existence, as well.*

5.3 Coincidence degree

There is a general theory to study abstract resonant problems due to Jean Mawhin. Let X and Z be Banach spaces, $L : X \supset Y \to Z$ be a linear operator such that $\dim\ker L = \operatorname{codim}\operatorname{im}L > 0$. (Such operators are called

Fredholm operators with index 0. More generally, Fredholm operators have finite dimensional kernels, closed images with finite dimensional $Z/\mathrm{im}L$. The index of the operator is the difference of these dimensions.) Let $N : X \to Z$ be continuous (and usually nonlinear). Let P be a linear projector on $\ker L$ and Q a linear projector in Z along $\mathrm{im}L$. Denote by K_P the inverse of

$$L|(\ker P \cap Y) : \ker P \cap Y \to \mathrm{im}L.$$

Operator N is called L-compact if QN and $K_P(I-Q)N$ are compact (usually K_P is compact and N maps bounded sets into bounded ones).

Theorem 15. *(*Mawhin*[25]) Let $\Omega \subset X$ be open and bounded, L be a Fredholm operator of index 0, N be L-compact operator and J is an arbitrary isomorphism of im$Q$ onto* $\ker L$. *Suppose that equations*

$$Lx = \lambda N(x)$$

have no solutions on $\partial\Omega \cap Y$ for any $\lambda \in (0,1]$ and the Brouwer degree is defined and

$$\deg(JQN|\ker L, \Omega \cap \ker L, 0) \neq 0.$$

Then equation $Lx = N(x)$ has a solution in Ω. The Brouwer degree in the main assumption is often called coincidence degree *of L and N.*

Proof. Denote $x = \bar{x} + \tilde{x}$, where $\bar{x} \in \ker L = \mathrm{im}P$, $\tilde{x} \in \ker P$. Equation $Lx = \lambda N(x)$ is equivalent to

$$x = Px + \lambda K_P N(x) \quad \text{and} \quad QN(x) = 0. \tag{5.7}$$

The last conjunction is equivalent to

$$x = Px + (JQ + \lambda K_P)N(x).$$

The right-hand side operator is compact and fixed point free on $\partial\Omega$ by the assumption for any $\lambda \in [0,1]$ (for $\lambda = 0$ it follows from the assumption the Brouwer degree exists. Thus

$$\deg_{LS}(I - P - (JQ + K_P)N, \Omega, 0) = \deg_{LS}(I - P - JQN, \Omega, 0)$$

but $P + JQN$ takes values in the finite dimensional space $\ker L$. Therefore the last degree is the Brouwer degree of $\bar{x} \mapsto JQN(\bar{x})$ on $\Omega \cap Y$ at 0 and a solution exists by Kronecker's property of degree.

For the BVP

$$x'' + x = f(t,x), \qquad x(0) = 0 = x(\pi),$$

the coincidence degree gives the same result as the perturbation method. If we solve linear BVP (f depending only on t), then

$$x(t) = \left(C + \int_0^t f(s)\cos s\,ds\right)\sin t - \int_0^t f(s)\sin s\,ds\,\cos t,$$

and the simplest choice of projectors is

$$Pf(t) = Qf(t) := \int_0^\pi f(s)\sin s\,ds \cdot \sin t, \qquad J = I$$

and then

$$K_P f(t) = \int_0^t (\sin t\cos s - \sin s\cos t) f(s)\,ds = \int_0^t \sin(t-s) f(s)\,ds.$$

For the nonlinear problem,

$$JQN(d\sin t) = \int_0^\pi f(s, d\sin s)\,ds \cdot \sin t$$

and we should calculate the Brouwer degree of the map defined on $\mathbb{R}$:

$$F := d \mapsto \int_0^\pi f(s, d\sin s)\,ds.$$

The Landesman-Lazer conditions guarantees this function does not vanish for large $|d|$ thus $\deg(F, (-R,R), 0)$ is defined for large $R > 0$ and this degree equals $+1$ if

$$\int_0^\pi f_-(s)\sin s\,ds < 0 < \int_0^\pi f_+(s)\sin s\,ds$$

and -1 if the inequalities are reversed.

Exercise 23. *Use Mawhin's Continuation Theorem to get the existence of a periodic solution to $x' = f(t,x)$, where $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and T-periodic w.r.t. t. Define*

$$g(x) := \frac{1}{T}\int_0^T f(t,x)\,dt$$

for $x \in \mathbb{R}^n$. Then a sufficient condition has the form: there exists a ball $B(0,R) \subset \mathbb{R}^n$ such that the following Brouwer degree is defined and is not 0 deg$(g, B(0,M), 0)$.

We can combine both methods in some problems. For example, in [77], we study a multipoint BVP:

$$x'' = f(t, x, x'), \qquad x(0) = 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i), \tag{5.8}$$

where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous function, $0 < \eta_1 < \ldots < \eta_k < 1$, ξ_i, $i = 1, \ldots, k$, are real numbers. The problem is resonant, iff

$$\sum_{i=1}^{k} \xi_i \eta_i = 1$$

since the linear functions $t \mapsto ta$ with $a \in \mathbb{R}^n$ are solutions of $x'' = 0$ satisfying boundary conditions.

Theorem 16. *([77] If all coordinates f_i of f satisfythe following growth condition*

$$|f_i(t, x, y)| \leq b_i(x) + c_i(x) y_i^2 + d_i(, x, y_1, \ldots, y_{i-1})$$

for every $t \in [0,1]$, $x, y \in \mathbb{R}^n$, where b_i, c_i, d_i are given continuous functions and there exists positive number a_0 such that

$$a_i f_i(t, ta, y) \geq 0 \tag{5.9}$$

for any i, $t \in [0,1]$, $a = (a_1, \ldots, a_n)$ and $y = (y_1, \ldots, y_n)$ such that $|a_i| = \max_j |a_j| \geq sgn(a_i) y_i \geq a_0$, then BVP (5.8) has a solution.

Proof. First assume all inequalities in (5.9) are strong. One can apply Mawhin's Continuation Theorem, if apriori bounds of the supremum of solutions to equation $x'' = \lambda f$ with the boundary conditions for $\lambda \in (0,1]$ will be found and the same for their derivatives. Let x be such a solution and

$$z(t) = x(t) - t(\max\{a_0, x_i(1)\})_{i=1}^n.$$

If one of the coordinates z_i of this function has the modulus greater than a_0 in a point, then it takes the maximum at a point t_i $z_i(t_i) > 0$, or $z_i(t) < 0$ for all $t \in (0,1)$. In the first case, $z_i'(t_i) = 0$ and $z_i''(t_i) \leq 0$. Hence $x_i'(t_i) = \max\{a_0, x_i(1)\} =: r_i$ and $x_i(t_i) > r_i t$. Moreover,

$$0 \geq z_i''(t_i) = x_i''(t_i) = \lambda f_i(t, x(t_i), r_i)$$

which contradicts (5.9).

In the second case, $x_i(1) > a_0$ and $x_i(t) \le x_i(1)t$ for $t \in (0,1)$. Therefore $x_i(\eta_j) \le x_i(1)\eta_j$ for any $j \in \{1,\dots,k\}$ and the second boundary condition can be satisfied only if $x_i(\eta_j) = x_i(1)\eta_j$. But then, for $t \in [0,1]$,

$$\frac{x_i(t) - x_i(\eta_j)}{t - \eta_j} \le \frac{x_i(1)t - x_i(1)\eta_j}{t - \eta_j} = x_i(1),$$

thus $x_i'(\eta_j) = x_i(1) > a_0$ and $x_i''(\eta_j) \le$ which contradicts (5.9). This gives the apriori bound for $\sup_{t\in[0,1]} |x_i(t)|$. The arguments leading to the estimates of $|x_i'(t)|$ are obtained similarly as in the proof of Bernstein's Theorem. We refer to [77] for more calculations. Due to Mawhin's Theorem we get the assertion in the case of strong inequalities. In the general case (weak inequalities in (5.9)). we perturb the BVP

$$x'' = f(t,x,x') + \frac{1}{m}x, \qquad x(0) = 0, \quad x(1) = \sum_{i=1}^{k} \xi_i x(\eta_i)$$

for $m \in \mathbb{N}$. From the first step of the proof, they have solutions x_m. By the Ascoli-Arzelá Theorem, there exists a subsequence which is convergent in $C^1([0,1],\mathbb{R}^n)$ and its limit is a solution for the main problem.

We can mention articles, where nonlinear terms appear also in boundary conditions. Then both considered methods can be applied but with some modifications. In [43], we consider the following nonlocal problem:

$$x'(t) = f(t,x(t)), \qquad h\left(\int_0^1 x(s)\,dg(s)\right),$$

where $f : [0,1] \times \mathbb{R}^k \to \mathbb{R}^k$, $h : \mathbb{R}^k \to \mathbb{R}^k$ are continuous and $g = (g_j)_{j=1}^k : [0,1] \to \mathbb{R}^k$ has a bounded variation. The symbol $\int_0^1 x\,dg$ stands for the vector with coordinates $\int_0^1 x_j\,dg_j$, $j = 1,\dots,k$. We assume that all coordinates have sufficiently large jump at 0 :

$$\lim_{\varepsilon\to 0^+} Var(g,[\varepsilon,1]) \le \min_{j\le k} |g_j(0^+) - g_j(0)|.$$

The remaining assumptions are: there exists $R > 0$ such that

$$\langle f(t,x), x\rangle \le 0, \qquad t \in [0,1], \quad |x| = R$$

– the euclidean norm in $\mathbb{R}^k$, and

$$\deg(h, B(0,r), 0) \neq 0$$

for some $r \in (r_-, r_+]$ where

$$r_- := R\left(\min_{j\le k} |g_j(0^+) - g_j(0)| - \lim_{\varepsilon\to 0^+} Var(g, [\varepsilon, 1])\right),$$

$$r_+ := R\left(|g(0^+) - g(0)| + \lim_{\varepsilon\to 0^+} Var(g, [\varepsilon, 1])\right).$$

The proof has two steps. First, we assume strong inequalities comparing jumps of g_j and the limits of its total variation. The Mawhin's scheme can be applied with $X = C([0,1], \mathbb{R}^k)$, $\mathrm{dom}(L) = C^1([0,1], \mathbb{R}^k)$, $Z = X \times \mathbb{R}^k$ – the last with the norm

$$\|(z, x)\| := \sup_t |z(t)| + |x|,$$

$Lx := (x', 0)$,

$$N(x) := \left(F(x), h(\int_0^1 x\, dg)\right)$$

where F denotes the Nemytski operator defined by f. If we take projectors $Px(t) := x(0)$ and $Q(z, \alpha) := (-\alpha, \alpha)$, (we indentify vectors in $\mathbb{R}^k$ with constant functions where it is needed), then $K_P(z, \alpha)(t) := \int_0^t z + t\alpha$ and

$$K_P N x(t) = \int_0^t f(s, x(s))\, ds + th\left(\int_0^1 x\, dg\right).$$

The family of BVPs $x' = \lambda f(t, x)$ with the nonlocal condition has no solution on the boundary of the ball centered at 0 with radius R. This is proved by standard arguments: if φ is a solution, then $\psi(t) := |\varphi(t)|^2$ takes the maximum R^2 at a point $t_0 \in [0,1]$. If $t_0 > 0$, then

$$0 \le \psi(t_0) - \psi(t) = \psi'(\xi)(t_0 - t) = 2\lambda\langle f(\xi, \varphi(\xi)), \varphi(\xi)\rangle(t_0 - t)$$

which contradicts the first assumption for $t \in [0, t_0)$. Thus $\psi(0) = R^2$. But one can estimate the Riemann-Stieltjes integral

$$\left|\int_0^1 \varphi(s)\, dg(s)\right| = \left|\varphi(0)(g(0^+) - g(0)) + \lim_{\varepsilon\to 0^+} \int_\varepsilon^1 \varphi\, dg\right| > r_-,$$

$$\left| \int_0^1 \varphi(s)\, dg(s) \right| \le r_+.$$

Hence the existence of the Brouwer degree contradicts $\psi(0) = R^2$. The second assumptions is exactly the Brouwer degree from Mawhin's Theorem. The proof under weak inequalities goes by perturbation of f by $-\frac{1}{n}x$.

The second application of the method to more general nonlinear nonlocal BVPs comes from [59], where an earlier idea of Jean Mawhin was used. Consider

$$x'' = f(t, x, x'), \qquad x(0) = a, \qquad x'(1) = N(x'), \tag{5.10}$$

where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $N : C \to \mathbb{R}^n$ are continuous functions, $C := C([0,1], \mathbb{R}^n)$ and $a \in \mathbb{R}^n$. Moreover N maps bounded sets in C onto bounded ones. The problem is equivalent to the first order BVP:

$$y'(t) = f\left(t, a + \int_0^t y, y(t)\right), \qquad y(1) = N(y),$$

which reduces to the fixed point problem for operator $T : C \to C$ with T defined by the formula:

$$Ty(t) := N(y) - \int_t^1 f\left(s, a + \int_0^s y, y(s)\right) ds.$$

This operator is compact by the Ascoli-Arzelá Theorem and the compactness of N.

Theorem 17. *([59]) If there exists an open bounded neighbourhood $\mathcal{C}$ of 0 in $\mathbb{R}^n$ containing the closed ball $B(0, |a|)$ such that*

(A) for any $y \in \partial\mathcal{C}$ there exists an outer normal vector *$\nu(y)$ i.e.*

$$\bar{\mathcal{C}} \subset \{z \in \mathbb{R}^n : \ \langle \nu(y), z - y \rangle \le 0\},$$

such that

$$\langle \nu(y), f(t, x, y) \rangle \ge 0$$

for $t \in [0,1]$, $x - a \in \bar{\mathcal{C}}$,

(B) for every $y \in C$ taking values in $\bar{\mathcal{C}}$ and $y(1) \in \partial\mathcal{C}$ we have $y(1) \neq N(y)$,

(C) deg$(I - N, \mathcal{C}, 0) \neq 0$,

then there exists a solution x such that $x(t) - a \in \bar{\mathcal{C}}$, $x'(t) \in \bar{\mathcal{C}}$ for $t \in [0,1]$.

Proof. We consider the homotopy $H(\lambda, y)(t) := y(t) - T_\lambda y(t)$ where

$$T_\lambda y(t) := N(y) - \lambda \int_t^1 \left[f\left(s, a + \int_0^s y, y(s) \right) + (1-\lambda) y(s) \right] ds,$$

$\lambda \in [0,1]$, $t \in [0,1]$, $y \in C$. Let Ω be an open bounded set (in Banach space C) of continuous functions on $[0,1]$ taking values in $\mathcal{C}$. The assumptions guarantees that the homotopy does not vanish for $y \in \partial\Omega$ and $\lambda \in [0,1)$ (all technical details in [59]). If $H(1,y) = 0$ for some $y \in \partial\Omega$, then it is a solution satisfying the assertion. If $H(1,y) \neq 0$ for such y, then the Leray-Schauder degrees

$$\deg_{LS}(I - T, \Omega, 0) = \deg_{LS}(I - N, \Omega, 0).$$

But N sends the whole space C into the space of constant functions identified with $\mathbb{R}^n$, thus this last degree equal the Brouwer one from assumption (C). The Kronecker property gives the result.

In the above mentioned paper you can find many examples this theorem works.

Recently, the most popular are equations containing the so called p-Laplacian or even more general differential operators. In one dimensional case, p-Laplacian of a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is $(\phi_p(u'))'$, where $\phi_p(s) = |s|^{p-2}s$, for $s \neq 0$ and $\phi_p(0) = 0$. This operator is nonlinear for $p \neq 2$ and has a sense if $p > 1$. More generally, p-Laplacian can be replaced by ϕ-Laplacian $(\phi(u'))'$, where

$$\phi(s) = \begin{cases} \frac{\beta(|s|)}{|s|} s & \text{for} \quad s \neq 0, \\ 0 & \text{for} \quad s = 0, \end{cases}$$

$\beta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous increasing homeomorphism of the half-line. Although *phi*-Laplacian operator is nonlinear, it inherits many properties of the usual Laplacian. For example, one can consider the periodic problem for

$$(\phi(u'))' + f(t, u, u') = 0, \tag{5.11}$$

$f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and T-periodic w.r.t. t, in the frame of Mawhin's continuation method; the following theorem is proved in [53] (the assumptions on ϕ there are even more general). By a solution of such equations we mean C^1-functions u such that $\phi(u')$ is also of the class C^1 and the equations is satisfied in all t.

Theorem 18. *(*Manásevich-Mawhin *[53]) Let $\Omega \subset C^1_T(\mathbb{R},\mathbb{R}^n)$ be an open bounded set in the space of C^1 and T-periodic functions. If*
(i) for $\lambda \in (0,1)$ the equation $(\phi(u')' = f(t,u,u')$ has no T-periodic solutions $u \in \partial\Omega$;
(ii) the algebraic equation

$$F(x) := \frac{1}{T}\int_0^T f(t,x,0)\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}^n$;
(iii) the Brouwer degree deg$(F, \partial\Omega \cap \mathbb{R}^n, 0) \neq 0$,
then (5.11) *has a T-periodic solution in $\bar{\Omega}$.*

More abstract continuation theorem is proved in [26] for equation $Lx = N(x)$ with nonlinear operator L having some properties of Fredholm operators with index 0 but we omit so general approach. Instead, we present an application of the above Manásevich-Mawhin Theorem from [52]

Theorem 19. *Assume there exist $D > 0$ and $E > 0$ such that*
(B1) $\langle f(t,x,y), x\rangle < \beta(|y|)|y|$ for all t, $|x| = D$ and $|y| \leq E$;
(B2) $\langle f(t,x,y), y\rangle \neq 0$ for all t, $|x| < D$ and $|y| = E$.
Then (5.11) *has a T-periodic solution.*

Proof. Set $\Omega := \{u \in C^1(\mathbb{R},\mathbb{R}^n) : \ \sup|u| < D, \quad \sup|u'| < E\}$. We use M.-M. Theorem on this set and the crucial point is to check assumption (i). If u is a solution for some $\lambda \in (0,1)$ which sits in the part of $\partial\Omega$ with $\sup|u| = D$, then there is t_0, where $|u(t_0)| = D$. Consider the function

$$\psi(t) := \left\langle \frac{\beta(|u'(t)|)}{|u'(t)|}u'(t), u(t)\right\rangle = \frac{\beta(|u'(t)|)}{2|u'(t)|}(|u(t)|^2)'.$$

Then

$$\psi'(t_0) = -\lambda\langle f(t_0,u(t_0),u'(t_0)), u(t_0)\rangle + \beta(|u'(t_0)|)|u'(t_0)| > 0$$

by condition (B1). On the other hand

$$\langle u'(t_0), u(t_0)\rangle = \frac{1}{2}\frac{d}{dt}|u'(t)|^2|_{t=t_0} = 0.$$

Hence $\psi(t_0) = 0$ and therefore $\psi(t) > D$ for some $t > t_0$ contrary to $u \in \bar{\Omega}$.

Let u be a solution from the second part of $\partial\Omega : \sup|u(t)| < D, \sup|u'(t)| = E$ and t_0 be such that $|u'(t_0)| = E$ is the maximum of the norm of the derivative. Consider function

$$\psi_1(t) := \beta^2(|u'(t)| = \left\langle \frac{\beta(|u'(t)|)}{|u'(t)|}u'(t), \frac{\beta(|u'(t)|)}{|u'(t)|}u'(t) \right\rangle.$$

Observe it takes the maximum at t_0 equal to $\beta^2(E)$. From condition (B2) we shall get a contradiction:

$$0 = \psi_1'(t_0) = -2\lambda \frac{\beta(|u'(t_0)|)}{|u'(t_0)|} \langle f(t_0, u(t_0), u'(t_0)), u'(t_0) \rangle \neq 0.$$

Assumption (ii) of Manásevich-Mawhin Theorem follows from (B1) since $\beta(0) = 0$ and the degree in (iii) is $(-1)^n$ due to

$$\langle F(x), x \rangle < 0$$

for $|x| = D$ (see the proof of Theorem 6 (Krasnosielski)). This ends the proof.

The above result can be applied to the relativistic pendulum

$$\left(\frac{u'}{\sqrt{1 - |u'|^2}} \right)' + f(t, u) = 0$$

(here f does not depend on u'). We refer to [52] for details.

There are many papers with resonant problems studied by different methods [16, 17, 24, 35, 38, 61, 83, 84].

Chapter 6

BVP on unbounded domains

We refer to [6] for a survey of results concerning BVPs on unbounded intervals but our approach will be in the spirit of this monograph.

6.1 Nonresonant example

Topological methods cannot be applied to BVPs on unbounded domains in most cases since they lead to integral equations with operators which are not compact or even the linear part is not a Fredholm operator. You can see all typical difficulties with this topic if you study BVP:

$$x'' = f(t), \qquad x(0) = 0 = \lim_{t\to\infty} x'(t). \tag{6.1}$$

Integrating the equation we get

$$x'(t) = c + \int_0^t f(s)\,ds$$

and the second BC gives $c = -\int_0^\infty f(s)\,ds$ that suggests we should assume $f \in L^1(0,\infty)$. Tha last equation implies

$$x'(t) = -\int_t^\infty f(s)\,ds.$$

The second integration gives

$$x(t) = -\int_0^t sf(s)\,ds - t\int_t^\infty f(s)\,ds.$$

This is an explicit formula for the unique solution of the linear BVP given for any $f \in L^1$, although x usually does not belong to this space: for $f(t) := \frac{2}{t^2+1}$, we have $x(t) = 2t \arctan t - \pi t - \ln(t^2+1)$ which is unbounded. If one passes to a corresponding nonlinear problem

$$x'' = f(t,x), \qquad x(0) = 0 = \lim_{t\to\infty} x'(t), \tag{6.2}$$

then one needs very strong assumptions on $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ to get a solution to

$$x(t) = \int_0^\infty G(t,s) f(s, x(s))\, ds,$$

where

$$G(t,s) := \begin{cases} -s & \text{for} \qquad s < t, \\ -t & \text{for} \qquad s > t, \end{cases}$$

i.e. $G(t,s) = -\min(t,s)$.

The most natural method for such problems is the *truncation*. Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be continuous,

(i) for any $M > 0$, there exists bounded and integrable on $\mathbb{R}_+$ function h_M such that

$$\sup_{|x|\le M} |f(t,x)| \le h_M(t),$$

(ii) there exists $M_0 > 0$ such that, for each t and $|x| \ge M_0$, $x \cdot f(t,x) > 0$.

Then (6.2) has a solution.

Proof. Fix $k \in \mathbb{N}$. We shall show that

$$x'' = f(t,x), \qquad x(0) = 0 = x'(k)$$

has a solution. It is equivalent to integral equation

$$x(t) = T_k x(t) := -\int_0^k \min(t,s) f(s, x(s))\, ds.$$

The integral operator T_k is compact as a map of $C[0,k]$ and the following homotopy $I - \lambda T_k$ is admissible on the ball $B(0, M_0+1)$. In fact, a fixed point of λT_k should satisfy $x'' = \lambda f(t,x)$, with boundary conditions and if sits on $\partial B(0, M_0+1)$, then it has the maximum $x(t_0) = M_0 + 1$ or minimum $= -M_0 - 1$. If $t_0 < k$, then it is a local maximum (resp. minimum) and $x''(t_0) \le 0$. (resp. $x''(t_0) \ge 0$) that contradicts (ii). If $t_0 = k$, then again by (ii), x' is negative (resp. positive) in a left neighborhood of k what contradicts

$x \in \bar{B}$. Hence the Leray-Schauder degree of $I - T_k$ and I in this ball at 0 are the same, the second degree is 1 by the normalisation property and the Kronecker property gives the existence of a solution to the BVP on $[0, k]$.

Denote by x_k the function which is the above solution on $[0, k]$ and is constant $x_k(k)$ for $t \geq k$. We know that supremum norms of x_k satisfies $\|x_k\| \leq M_0 + 1 =: M_1$. Moreover

$$|x_k''(t)| = |f(t, x_k(t))| \leq h_{M_1}(t) \leq \sup h_{M_1}(t)$$

by (i). Since $x_k'(t) = -\int_t^k f(s, x_k(s))\, ds$, also $|x_k'(t)|$ are equibounded. Thus the sequence (x_k) has a subsequence which is uniformly convergent on any compact interval and the sequence of derivatives (x_k') has the same property. It follows that $\varphi := \lim_{k\to\infty} x_k$ satisfies the differential equation on $\mathbb{R}_+$, $\varphi(0) = 0$. Choose any $\varepsilon > 0$ and k_0 such that

$$\int_{k_0}^{\infty} h_{M_1}(t)\, dt < \varepsilon.$$

Then, for $t > k_0$ and $k \geq k_0$, we have $|x_k'(t)| \leq \varepsilon$ hence $|\varphi'(t)| \leq \varepsilon$ that gives the second BC. Thus φ is a solution to (6.2).

6.2 Resonant example

The above problem has been nonresonant, since the only solution to $x'' = 0$ satsfying $x(0) = 0 = \lim_{t\to\infty} x'(t)$ is the null function. If we slightly change the first BC $x'(0) = 0$, the problem will have constant solutions – the resonance. Then one can apply more sophisticated methods. We shall present a result of K. Szymańska-Dębowska [91], where a main tool was *Miranda's Theorem:*

Theorem 20. *(Miranda [62]) Let $F = (F_1, \ldots, F_n) : [-M, M]^n \to \mathbb{R}^n$ be continuous and its coordinates has the property:*

$$F_j(x_1, \ldots, x_{j-1}, -M, x_{j+1}, \ldots, x_n) \leq 0 \leq F_j(x_1, \ldots, x_{j-1}, +M, x_{j+1}, \ldots, x_n)$$

for each $j = 1, \ldots, n$. Then there exists x such that $F(x) = 0$.

It can be proved with sharp inequalities in the assumption by using homotopy $\lambda F + (1 - \lambda)I$, $\lambda \in [0.1]$. Next, it suffices to study the sequence of problems with $F(x) + x/k$, $k \in \mathbb{N}$, that have zeros and take a convergent subsequence.

Theorem 21. *The following BVP:*

$$x'' = f(t, x, x'), \qquad x'(0) = 0 = \lim_{t\to+\infty} x'(t), \tag{6.3}$$

has a solution provided that $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous and Lipschitz continuous w.r.t.* x*, and* x'*,*
(i) there exist $b, c \in L^1(\mathbb{R}_+)$ *such that*

$$|f(t, x, y)| \leq b(t)|y| + c(t),$$

(ii) there exists $M > 0$ *such that, for each* $j \in \{1, 2, \ldots, n\}$*,*

$$x_j \cdot f_j(t, x, y) \geq 0, \qquad \text{for } (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}^{2n},\ \ |x_j| \geq M.$$

Sketch of the proof. The initial value problem

$$x'' = f(t, x, x'), \qquad x(0) = c, \quad x'(0) = 0,$$

has a local (unique) solution for any $c \in \mathbb{R}^n$ – it is the only point, where we use the assumption on Lipschitz continuity of the nonlinear term. This solution is global in time; an a priori bound is obtained for the equivalent problem:

$$y' = f\left(t, c + \int_0^t y, y(t)\right), \qquad y(0) = 0.$$

In fact, this function y_c satisfies

$$y_c(t) = \int_0^t f\left(s, c + \int_0^s y_c, y_c(s)\right) ds \tag{6.4}$$

and we have

$$|y_c(t)| \leq \int_0^t (b(s)|y_c(s)| + c(s))\, ds \leq C + \int_0^t b(s)|y_c(s)|\, ds,$$

where $C = \int_{\mathbb{R}_+} c(s)\, ds$ and from Gronwall's Lemma

$$|y_c(t)| \leq C \exp \int_{\mathbb{R}_+} b(s)\, ds.$$

We have proved that the solution of IVPs are global and their derivatives are bounded on $\mathbb{R}_+$. Moreover, functions y_c have finite limits as $t \to \infty$ by the integrability of the right-hand side of (6.4).

Define $F : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$F(c) = \lim_{t\to\infty} y_c(t).$$

It is a continuous mapping (verify it). We shall show the assumptions of the Miranda Theorem hold on the cube $[-M-1, M+1]^n$. Let $c_j = M+1$ and φ be j-th coordinate of y_c. We have $\varphi(0) = 0$. If $\varphi(t) < 0$ for some $t > 0$, then take t_* being the infimum of such t's. Then $\varphi(t_*) = 0$ and, by continuity of φ there is $t_1 > t_*$ such that

$$\int_{t_*}^{t_1} |\varphi| \le 1.$$

It follows that

$$x_j(t) = c_j + \int_{t_*}^{t} \varphi \ge M$$

for $t \in [t_*, t_1]$. From (ii), $\varphi'(t) \ge 0$ for such t that contradicts the definition of t_*. Thus $\varphi(t) \ge 0$ for any $t > 0$ and $F_j(c) = \lim_{t\to\infty} \varphi(t) \ge 0$. Similar arguments applied to $c_j = -M-1$ give $F_j(c) \le 0$. Due to the Miranda Theorem, there exists $c \in \mathbb{R}^n$ such that $F(c) = 0$ and this ends the proof.

Remarks. Miranda's theorem holds if we reverse inequalities in its assumption for some $j \in J_0$ – the homotopy to

$$(x_j)_{j=1}^n \mapsto (\epsilon_j x_j)_{j=1}^n,$$

where $\epsilon_j = -1$ for $j \in J_0$ and $= +1$ for other indices. It enables us to get the existence of a solution (6.3) under condition

(ii') there exists $M > 0$ such that, for each $j \in J_0$,

$$x_j \cdot f_j(t,x,y) \le 0, \qquad \text{for } (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^{2n}, \ |x_j| \ge M$$

and the reversed inequalities for $j \notin J_0$.

One can also remove the Lipschitz continuity of f but then the mapping F will be a multivalued function. There is a theory of topological degree for such multis (with values of a special kind) developed by L. Górniewicz and one can get the Miranda Theorem for such multivalued functions (see [92]).

6.3 Perturbation method revisited

We have seen that resonant problems $Lx = N(x)$ with $\ker L \neq \{0\}$, when L is not a Fredholm operator (equations with Fredholm linear part of nonzero index were studied by Nirenberg [60] but they are not so important from applicational point of view) can occur for BVPs on unbounded domains. This means that either $\ker L$ is infinite dimensional, or $\operatorname{im} L$ has infinite codimension or $\operatorname{im} L$ is not even closed subspace of Z. The third situation seems to be the most difficult but this is the case for many boundary value problems on unbounded domains. It is surprising that the perturbation method developped in the above mentioned paper [64] for Fredholm linear part works here, as well.

The method is applicable if our abstract equation can be embedded into a continuos family $L(\lambda)$ and, for $\lambda > 0$ and small (or $\lambda < 0$), the linear operator $L(\lambda)$ is invertible. Usually, its inverse is not compact because L is non-Fredholm, hence, in order to obtain the solution x_n of $L(\lambda_n)x = N(x)$ for $\lambda_n \to 0$, we need assumptions guaranteeing $L(\lambda_n)^{-1}N$ to be compact and has a fixed point x_n. This is the first group of assumptions. The second one is necessary to get that the sequence (x_n) cannot be unbounded. They rely on conditions of the asymptotic behaviour of $N|\ker L$ and are of Landesman-Lazer type. The third question is if the sequence (x_n) which is bounded, is relatively compact. Usually, it does not need additional assumptions and any cluster point x of this sequence is a solution of $L(0)x = N(x)$. We shall show three examples of the above procedure.

In the paper [39], W. Karpińska studied the existence of solutions to an ordinary differential equation of the first order which are bounded on the whole line. This question can be considered as the boundary value problem:

$$x' = Ax + f(t,x), \quad x \text{ bounded on } \mathbb{R}, \tag{6.5}$$

where A is a linear selfadjoint operator on $\mathbb{R}^k$ with eigenvalue 0, $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous. The space $\mathbb{R}^k$ can be represented as a direct sum of linear invariant subspaces: X_+ where A has positive eigenvalues, X_- where it has negative eigenvalues, and X_0 – its kernel. Let f_+, f_- and f_0 stand for respective superpositions of f with projectors onto these subspaces.

Theorem 22. *If f is bounded,*

(a) $\lim_{t\to\pm\infty} |f(t,x)| = 0$ *uniformly on any ball,*

(b) *the scalar product* $\langle x, f_0(t,x)\rangle \le 0$ *for vectors* x *with large projections on* X_0,
then the problem (6.5) has a solution.

Theorem 23. *If* f *satisfies (a), (b) and (instead of the boundedness)*
(c) $\langle x, f_+(t,x)\rangle \ge 0$ *for* x *with large projections on* X_+,
(d) $\langle x, f_-(t,x)\rangle \le 0$ *for* x *with large projections on* X_-,
then the problem (6.5) has a solution.

The problem is examined in the space of bounded and continuous functions $x : \mathbb{R} \to \mathbb{R}^k$ denoted by $BC(\mathbb{R}, \mathbb{R}^k)$ with the supremum norm; the linear part $L : x \mapsto x' - Ax$ with the domain $\mathrm{dom}L = \{x \in BC(\mathbb{R}, \mathbb{R}^k) : x \in C^1\}$. The role of the Landesman-Lazer type condition plays assumption (b). The existence of solutions to perturbed equations is obtain by using the Schauder Fixed Point Theorem in the case of Theorem 22 and the Leray-Schauder degree in the case of Theorem 23. The results of [39] are formulated for general Hilbert space instead of $\mathbb{R}^k$ but we restrict ourselves for simplicity here.

Karpińska studied separately [40] the case of second order systems and its bounded solutions. This problem is not a special case of (6.5) – it is the existence of bounded with the first derivative solutions.

R. Stańczy [88] considers the question of the existence of bounded solutions for semilinear elliptic problem:

$$\begin{aligned} &\Delta u = f(x,u) \ \text{ for } \ |x| > 1, \ x \in \mathbb{R}^n, \ n \ge 3 \\ &u(x) = 0 \ \text{ for } \ |x| = 1. \end{aligned} \tag{6.6}$$

The problem is resonant, since the homogeneous BVP:

$$\Delta u = 0 \ \text{ for } \ |x| > 1,$$

$$u(x) = 0 \ \text{ for } \ |x| = 1,$$

has a nontrivial bounded solution $u(x) = 1 - |x|^{2-n}$. The Laplace operator is a natural candidate for linear part L, but there is no natural choice of Banach spaces X and Z – the space of bounded and continuous functions is too large and Hölder spaces on unbounded domains are not uniquely defined. If nonlinear part f has the radial symmetry, i.e. $f(x,u) = g(|x|,u)$ where $g : [1,\infty) \times \mathbb{R} \to \mathbb{R}$, then the problem (6.6) leads to

$$v'' + \frac{n-1}{r} v' = g(r,v), \quad v(1) = 0.$$

When v is a solution of the last problem, then $u(x) = v(|x|)$ is a solution of (6.6) called a radial solution. The complete answer to the question of the existence of radial solutions gives the following

Theorem 24. *([88]). Suppose that the function g is continuous,*

(i) for each $R > 0$,

$$\lim_{r\to\infty} \sup_{|v|\le R} |g(r,v)| = 0,$$

(ii) there exists $M > 0$ such that, for all $|v| \ge M$ and all r,

$$vg(r,v) \ge 0.$$

Then the boundary value problem (6.6) *has a bounded radial solution.*

The proof is based on the perturbation scheme; assumption (ii) plays the role of Landesman-Lazer type condition (notice that it is not asymptotic). When nonlinear term f is not radially symmetric, the question is much more complicated. The perturbed linear operator $\Delta - \lambda I$ is invertible in appropriate Hölder spaces [19] but the boundedness of a sequence of solutions (u_n) for $\lambda_n \to 0$ is not obvious. However, it seems almost sure that the existence of a bounded solution to (6.6) can be obtained under very similar conditions as in Theorem 24. The question of the existence of decaying at infinity solutions is simpler (comp. [89] and its references).

In [75] we look for a solution of the nonlinear parabolic system

$$v_t = \Delta v - f(v, a\cdot x - ct) \tag{6.7}$$

where $x, a \in \mathbb{R}^l$, $|a| = 1$, $c > 0$, $v = (v^1,\dots,v^k)$, $\Delta v = (\sum_{j=1}^{l} v^i_{x_j x_j})_{i=1}^{k}$, $f : \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}^k$. This solution is supposed to be of a special form

$$v(x,t) = w(a\cdot x - ct)$$

with $w : \mathbb{R} \to \mathbb{R}^k$ having finite limits

$$\lim_{s\to\pm\infty} w(s) = \pm w_\pm$$

and is called a travelling wave (of the front wave type). Usually, f depends on v only, and the speed c of the wave and its direction a is not determined by the system (comp. [93]).

If one substitutes $w = u + \psi$ where $\psi(s) = \omega(s)w_- + (1-\omega(s))w_+$ with ω – a smooth real function that equals 1 for $s \le -1$ and 0 for $s \ge 1$, then the function u should satisfy the second order ordinary differential system in $\mathbb{R}^k$

$$u'' + cu' = f(u + \psi(s), s) - \psi''(s) - c\psi'(s)$$

and vanish at $\pm\infty$. This is equivalent to an integral Hammerstein equation on the real line

$$u(t) = -\frac{1}{c}\int_{-\infty}^{t} \mathrm{e}^{-c(t-s)} f(u(s)+\psi(s), s)\, ds - \frac{1}{c}\int_{t}^{\infty} f(u(s)+\psi(s), s)\, ds + w_+ - w_-$$

with the additional condition

$$\int_{-\infty}^{\infty} f(u(s) + \psi(s), s)\, ds = c(w_+ - w_-).$$

This system of equations can be considered as a kind of equations (5.7). The perturbation of the above ODE by λu causes that the linear operator begins invertible and the condition on integral over the whole line is omitted. This is similar as in our abstract scheme.

Theorem 25. *([75]). Under the following assumptions on f :*
1) continuity;
2) $|f(x,s)| \le \alpha(s)|x|^\rho + \beta(s)$ with $\rho < 1$, α and β vanishing at $\pm\infty$,
$\sup_s |\alpha(s)| \le \alpha_0$ with the constant α_0 sufficiently small;
3) there exists a function γ_0 vanishing at infinity such that for every coordinate f_i of f, $i = 1, \dots, k$, every s, $|u_i| \ge \gamma_0(s)$, and every $|u_j| \le |u_i|$ $(j \ne i)$

$$u_i f_i(u + \psi(s), s) \ge 0,$$

the parabolic system (6.7) *has a solution being of the above form.*

In the proof, we perturb the above ODE by $\lambda_n u$ with positive $\lambda_n \to 0$. The related question is reduced to the fixed point problem for some compact operator in the space $C_0(\mathbb{R}, \mathbb{R}^k)$ of functions $u : \mathbb{R} \to \mathbb{R}^k$ vanishing at both infinities. The sequence u_{λ_n} of fixed points is then relatively compact in the above space and any cluster point is a solution of the main problem. Assumption 3) is the Landesman-Lazer type condition and excludes the unboundedness of the sequence.

Chapter 7

Carathéodory solutions

Integrals operators, which we encountered earlier has the form

$$Tx(t) := \int_{\alpha}^{\beta} G(t,s) f(s, x(s))\, ds,$$

where G has a jumping discontinuity at $s = t$ and is continuous at remaining points and f is continuous. Such operators act in the space of continuous functions $C = C([\alpha, \beta], \mathbb{R}^n)$, but it is rather obvious that the continuity of f is not necessary for $T(C) \subset C$ and has next needed properties. We should have only integrability of $s \mapsto f(s, x(s))$ for every $x \in C$. A weaker assumption is: f is a *Carathéodory function,* i.e.:

(i) $f(\cdot, x)$ is measurable for any $x \in \mathbb{R}^n$,

(ii) $f(t, \cdot)$ is continuous for a.e. $t \in [\alpha, \beta]$,

(iii) for each $M > 0$, there exists an integrable function $h_M \in L^1(\alpha, \beta)$ such that $|f(t,x)| \le h_M(t)$ for a.e.t and $|x| \le M$.

Thus we can extend most of our results replacing continuity of f by Carathéodory condition. However fixed points of integral operators are not functions differentiable everywhere. They are the so called absolutely continuous functions (see [33]) and such functions are only differentiable almost everywhere with the appropriate value.

Function $\varphi : [\alpha, \beta] \to \mathbb{R}^n$ is called *absolutely continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sum_j |\varphi(b_j) - \varphi(a_j)| < \varepsilon$$

for any finite family of mutually disjoint intervals $[a_j, b_j]$, $j = 1, \ldots, p$, such that $\sum_j (b_j - a_j) < \delta$. If $\psi : [\alpha, \beta] \to \mathbb{R}^n$ is integrable, then

$$\varphi(t) := \int_\alpha^t \psi(s)\, ds$$

is absolutely continuous and $\varphi'(t) = \psi(t)$ almost everywhere (a.e.). Therefore differential equations with Carathéodory right-hand side have solutions satisfying the equation only on a set of full measure.

As an example we shall show

Theorem 26. *Let $f : [\alpha, \beta] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and $x_0 \in \mathbb{R}^n$. If $h(t) := \sup_{M>0} h_M(t)$ is integrable, then the initial value problem*

$$x' = f(t, x), \qquad x(\alpha) = x_0,$$

has a solution $\varphi : [\alpha, \beta] \to \mathbb{R}^n$. If, moreover, there exists locally integrable function $L : (\alpha, \beta) \to (0, \infty)$ such that

$$|f(t, x) - f(t, y)| \le L(t)|x - y|, \qquad a.e.t \quad x, y \in \mathbb{R}^n,$$

then the solution is unique.

Proof. For any continuous function $x : [\alpha, \beta] \to \mathbb{R}^n$, the composition $t \mapsto f(t, x(t))$ is integrable (prove it!). Consider the operator

$$T(x)(t) := x_0 + \int_\alpha^t f(s, x(s))\, ds$$

on the space $C := C([\alpha, \beta], \mathbb{R}^n)$. Since $T(x)$ is absolutely continuous, we have: $T : C \to C$, Tx is a.e. differentiable with $(Tx)'(t) = f(t, x(t))$ a.e.. Moreover, T is continuous by the Lebesgue Domination Convergence Theorem and even compact by the integrability of h_M for any $M > 0$. Since

$$\sup_t |T(x)(t)| \le \int_\alpha^\beta h(s)\, ds =: R,$$

the whole space is mapped into the ball $\bar{B}(0, R)$ and T has a fixed point due to the Schauder Fixed Point Theorem.
(Uniqueness) If we have two solutions x, $y : [\alpha, \beta] \to \mathbb{R}^n$, denote $\vartheta(t) :=$

$|x(t) - y(t)|$ and $\mu(t) := \sup_{s\in[\alpha,t]} \vartheta(s)$. Function μ is continuous and nondecreasing, hence there is t_0 such that $\mu(t) = 0$ for $t \in [\alpha, t_0]$ and $\mu(t) > 0$ for $t > t_0$. We have

$$\mu(t) = \sup_{t_1\le t} \vartheta(t_1) = \sup_{t_1} \left| \int_{t_0}^{t_1} (f(s, x(s)) - f(s, y(s)))\, ds \right| \le \int_{t_0}^{t} L(s)\vartheta(s)\, ds$$

$$\le \mu(t) \int_{t_0}^{t} L(s)\, ds$$

for $t > t_0$. It follows that $\mu(t) = 0$ or $\int_{[t_0,t]} L \ge 1$. The last option contradicts the local integrability of L, the first contradicts the definition of t_0.

If we drop the assumption about h, the existence will hold on some small interval $[\alpha, \gamma)$; the local existence.

The theorem also holds if, instead of $|f(t, x)| \le h(t) \in L^1$ we assume a linear bound:

$$|f(t, x)| \le M(t)|x| + N(t), \qquad x, y \in \mathbb{R}^n, \quad a.e.\ t$$

for some integrable functions M, N. The proof relies on the Gronwall Lemma as in the continuous case.

Exercise 24. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and bounded and $h : (0, 1) \to \mathbb{R}^n$ be integrable. Show that BVP:*

$$x'' = g(x) + h(t), \qquad x(0) = 0 = x(1),$$

has a solution (in the above Carathéodory sense).

The nonlinear term f can depend also on the first derivative but then the integral operator has to be defined on the space of C^1 functions. Consider

$$x'' = g_1(x) + g_2(x') + h(t), \qquad x(0) = 0 = x(1)$$

with $g_{1,2} : \mathbb{R}^n \to \mathbb{R}^n$ bounded continuous and $h : [0, 1] \to \mathbb{R}^n$ integrable. The problem is equivalent to seeking a fixed point to the integral operator

$$Tx(t) = \int_0^1 G(t, s) \left(g_1(x(s)) + g_2(x'(s)) + h(s)\right) ds$$

acting in $C^1([0, 1], \mathbb{R}^n)$.

Exercise 25. *Show that T is compact and its range is contained in a ball. Thus a solution exists due to the Schauder Theorem.*

One can consider this operator on a larger space $W^{1,p}([0,1],\mathbb{R}^n)$ consisting of L^p functions having the first derivative defined everywhere which belongs to L^p. It is an example of Sobolev spaces. It is equipped with the norm

$$\|x\|_{1,p}^p := \int_0^1 \left(|x(t)|^p + |x'(t)|^p\right)\, dt.$$

Sometimes it is easier to get an apriori bound in this norm than in the supremum one.

The above BVP is an example of singular boundary value problems (singularity in t at end points of the interval). One can also study singularities in the dependent argument x, for example

$$x'' = \frac{1}{|x|^p} + h(t), \qquad x(0) = 0 = x(1).$$

Notice that, here, the equation cannot be satisfied at the end points by BCs. The integral operator which fixed points are solutions

$$T(x)(t) := \int_0^1 G(t,s)(|x(s)|^{-p} + h(s))\, ds$$

does not act on the whole space $C[0,1]$. We can avoid this trouble by using another Banach spaces – see [87].

Chapter 8

Positive solutions

Sometimes, equations derived in applications have right-hand sides defined on some special subsets of function spaces. Usually, the unknowns have natural meaning only if their values are nonnegative (density of species in population ecology, mass in physics, number of sicks in epidemiology). From the mathematical point of view $f : [\alpha, \beta] \times \mathbb{R}^n_+ \to \mathbb{R}^n$ in differential equations x' or $x'' = f(t, x)$. Then a solution belongs to the cone P of the space $C([\alpha, \beta], \mathbb{R}^n)$ including functions with all coordinates nonnegative and it is a fixed point of an integral operator defined on P.

Let E be a real Banach space. A closed subset $P \subset E$ is called a *cone* if
(i) $x, y \in P \;\Rightarrow\; x + y \in P$;
(ii) $x \in P,\ \lambda \geq 0 \;\Rightarrow\; \lambda x \in P$;
(iii) $P \cap (-P) = \{0\}$.

Any cone defines a partial order in E :

$$x \leq y \;\Leftrightarrow\; y - x \in P.$$

Examples. 1) $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) :\ x_j \geq 0,\ j = 1, \ldots, n\}$ is a cone in $\mathbb{R}^n$.

2) The set of all continuous functions $\varphi : X \to \mathbb{R}^n$ with values in $\mathbb{R}^n_+$ is a cone in $C(X, \mathbb{R}^n)$. Similar set is a cone in $L^p(X, \mu)$.

3) H – a Hilbert space, $E := L(H)$ – the space of all linear bounded operators on H. The set of all $A \in L(H)$ such that $\langle Ax, x\rangle \geq 0$ for any $x \in H$ is a cone (of nonnegative operators).

There are many fixed point theorems for nonlinear operators $T : P \to E$. We present the most known *Krasnosielski's theorem* (cone compression or expansion). It is motivated by a simple observation: for continuous T :

$\mathbb{R}_+ \to \mathbb{R}_+$ such that there are $0 < r < R$, with properties $T(r) < r$ and $T(R) > R$ (or with both reversed inequalities), there exists a fixed point.

Theorem 27. *([30]) Let $T : P \to P$ be a compact operator on a cone P in a Banach space E. Suppose that there are $0 < r < R$ such that one of condition holds:*

(cone-expansion)

$$\|Tx\| \le \|x\| \quad \textit{for} \quad \|x\| = r \quad \textit{and} \quad \|Tx\| \ge \|x\| \quad \textit{for} \quad \|x\| = R$$

or

(cone-compression)

$$\|Tx\| \ge \|x\| \quad \textit{for} \quad \|x\| = r \quad \textit{and} \quad \|Tx\| \le \|x\| \quad \textit{for} \quad \|x\| = R.$$

Then there exists at least one fixed point $x \in \bar{B}(0, R) \setminus B(0, r)$.

Notice that a fixed point in the cone compression case can be obtained via Schauder's Theorem even without condition on the sphere with radius r but it can be $x = 0$. The theorem is obtained by using the fixed point index on a cone. Let $T : \bar{\Omega} \cap P \to P$ be compact, Ω is open bounded set in E and T has no fixed points on $\partial\Omega \cap P$. Then there exists a retraction $\mathfrak{r} : E \to P$. Define

$$i(T, \Omega, P) := \deg_{LS}(I - T\mathfrak{r}, \mathfrak{r}^{-1}(\Omega) \cap B(0, M), 0)$$

with M so large that $\Omega \subset B(0, M)$. One can easily show that the definition does not depend on $\mathfrak{r}$ and all properties of the degree can be repeated. There are two facts if there are no fixed points for $\|x\| = M$:

– if for all $\|x\| = M$, $\|Tx\| \le M$, then $i(T, B(0, M), P) = 1$;

– if for all $\|x\| = M$, $\|Tx\| \ge M$, then $i(T, B(0, M), P) = 0$.

Thus $i(T, B(0, R)\setminus, \bar{B}(0, r), P) \ne 0$ in both cases and we get a fixed point in this set. See [30] for details and another applications of order defined by cones in Banach spaces to get fixed points of mappings and solutions to BVPs at last. More sophisticated theory can be found in original Krasnosielski's boog [50].

Example. Consider BVP:

$$-x'' = f(t, x), \qquad x(0) = 0 = x(1).$$

where $f : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Suppose that there are two positive constants a, b with $b > 4a$ such that:

$$f(t, x) \le 8a \quad \text{for} \quad t \in [0, 1], \quad x \in [0, a], \tag{8.1}$$

$$f(t,x) \geq \frac{128}{3} b \quad \text{for} \quad t \in [\frac{1}{4}, \frac{3}{4}], \quad x \in [\frac{1}{4}b, b]. \tag{8.2}$$

Then there exists a solution such that $\|x\| \in [a, b]$.

We shall apply the cone expansion case of the Krasnosielski Theorem with: the Banach space $C[0,1]$, the cone

$$P := \{x \in C[0,1] : \ x(t) \geq 0 \text{ for } t \in [0,1], \quad \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} x(t) \geq \frac{1}{4}\|x\|\},$$

$r = a$, $R = b$. Remind of the Green function $G(t,s) = s(1-t)$ for $s < t$ and $= t(1-s)$ for $s > t$. It is easy to see that

$$G(t,s) \leq s(1-s) \quad \text{for} \quad s, t \in [0,1],$$

$$\frac{1}{4} s(1-s) \leq G(t,s) \quad \text{for} \quad s \in [0,1], t \in J := [\frac{1}{4}, \frac{3}{4}].$$

The BVP is equivalent to the fixed point problem for the following integral operator

$$T(x)(t) = \int_0^1 G(t,s) f(s, x(s))\, ds$$

acting the set of nonnegative functions $\subset C[0,1]$. We need $T(P) \subset P$. In fact, for $x \in P$ and $t \in J$, we have

$$T(x)(t) \geq \frac{1}{4} \int_0^1 s(1-s) f(s, x(s))\, ds \geq \frac{1}{4} \|T(x)\|.$$

Let $x \in P$ and $\|x\| = a$. Then for all $s \in [0,1]$, $f(s, x(s)) \leq 8a$ and

$$\|T(x)\| \leq \sup_t \int_0^1 G(t,s) 8a\, ds = 4a \sup_t t(1-t) = a.$$

On the other hand, for $x \in P$, $\|x\| = b$, we get if $t \in J$,

$$T(x)(t) \geq \frac{1}{4} \int_J G(t,s) f(s, x(s))\, ds \geq \frac{1}{4} \cdot \frac{128}{3} b \int_J G(t,s)\, ds = \frac{128}{24} b(-t^2 + t - \frac{1}{16}).$$

The supremum of the right-hand side is gained at $t = 1/2$ and we obtain $\|T(x)\| \geq b$. This ends the proof.

Similar assumptions enables to use cone compression case of Krasnosielski's theorem.

Our assumptions are of such a kind that we can find several solutions. In order to get k solutions, one needs $0 < R_1 < R_2 < \ldots < R_{k+1}$ and

$$\|Tx\| \leq \|x\| \quad \text{for} \quad \|x\| = R_{\text{odd}},$$
$$\|Tx\| > \|x\| \quad \text{for} \quad \|x\| = R_{\text{even}}$$

(or reversely). We need sharp inequalities to avoid the case that two solutions existing in sequel annuli coincide. It is the first result, where we obtain multiple solutions but not only at least one.

Exercise 26. *Try to find similar results under assumptions on* $\lim_{x\to 0+} \frac{f(x)}{x}$ *and* $\lim_{x\to\infty} \frac{f(x)}{x}$ *for BVP*

$$-x'' = f(x), \qquad x(0) = 0 = x(1).$$

If the limit of $f(x)/x$ *equals* ∞ *at* 0 *and equals* 0 *at* ∞*, then* f *can be singular at* 0.

Exercise 27. *Change the interval* $J = [1/4, 3/4]$ *for arbitrary* $[\alpha, \beta]$*,* $0 < \alpha < \beta < 1$ *in the above arguments to see what you have to change in the assumptions.*

The Krasnosielski Theorem can be applied to nonlocal BVPs. In [21], we have studied the existence of nonnegative solutions to

$$-\Delta u = f\left(u, \int_\Omega u\right), \qquad u|_{\partial\Omega} = 0, \tag{8.3}$$

where $\Omega := B(0,R) \setminus \bar{B}(0,\rho)$ is an annulus in $\mathbb{R}^n$, $n \geq 2$, and $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is continuous and satisfies

$$f(v, \lambda) \leq A(\lambda)v + B(\lambda), \tag{8.4}$$

A, B continuous functions. We have looked for radial solutions to this problem $u(x) = v(|x|)$, thus v should satisfy

$$-v''(r) - \frac{n-1}{r} v'(r) = f\left(v, \omega_n \int_\rho^R t^{n-1} v(t)\, dt\right), \qquad v(\rho) = 0 = v(R), \;\; v \geq 0.$$

Here, ω_n is the measure of the unit sphere in $\mathbb{R}^n$. This problem is nonresonant and his Green function has the form

$$G(r,t) = \frac{t(R^{n-2} - \max(r,t)^{n-2})(\min(r,t)^{n-2} - \rho^{n-2})}{(n-2)(R^{n-2} - \rho^{n-2})r^{n-2}}$$

for $n > 2$ and

$$G(r,t) = \frac{t(\ln R - \ln \max(r,t))(\ln \min(r,t) - \ln \rho)}{\ln R - \ln \rho}$$

for $n = 2$. Thus a solution is any fixed point of the integral operator

$$Tv(r) := \int_\rho^R G(r,t) f\left(v(t), \omega_n \int_\rho^R s^{n-1} v(s)\, ds\right) dt.$$

The Banach space E, where this operator acts, is $L^1(\rho, R)$ equipped with the norm

$$\|v\| := \omega_n \int_\rho^R s^{n-1} |v(s)|\, ds$$

which is equivalent to the standard one. The cone P in this space is the set of nonnegative (a.e.) functions and the growth condition (8.4) implies the existence of the integral, hence $T : P \to P$. This operator is compact (we were used the Ascoli-Arzelá Theorem since the image of a bounded set sits in the space of continuous functions).

Theorem 28. *([21]) If additionally there exist two positive numbers c_1, c_2 such that*

$$f(v, c_1) \le \frac{c_1}{\omega_n \gamma}, \qquad f(v, c_2) \ge \frac{c_2}{\omega_n \gamma}$$

for any $v \ge 0$, where

$$\gamma := \int_{[\rho,R]^2} r^{n-1} G(r,t)\, dt\, dr,$$

then (8.3) *has a radial nonnegative solution v with the norm between these constants.*

The proof relies on the cone compression version of the Krasnosielski Theorem if $c_1 > c_2$ and on the second kind of assumptions of this theorem if $c_1 < c_2$. If we have an increasing sequence of $2N$ positive constants c_j such that the inequalities of the above theorem are satisfied alternately then the existence of at least $N - 1$ solutions is guaranteed.

The same method can be applied to the essentially partial differential equation

$$-\sum_{i,j=1}^n D_i(a_{ij}(x) D_j u) = f\left(u, \int_\Omega g(u^p)\right), \qquad u|_{\partial\Omega} = 0, \;\; u \ge 0,$$

where $D_i u$ stands for the partial derivative of u w.r.t. x_i, $p > 1$, Ω is an open bounded set with sufficiently regular boundary in $\mathbb{R}^n$, $n \geq 3$, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous w.r.t. the first variable u and it satisfies (8.4) and $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$g(u) \in [b_1 u, b_2 u], \qquad u \geq 0$$

for some positive constants b_1, b_2. The differential operator on the left-hand side of the equation is uniformly elliptic with sufficiently regular coefficients $a_{i,j}$ so as the linear Dirichlet problem

$$-\sum_{i,j+1}^{n} D_i(a_{ij}(x) D_j u = h, \qquad u|_{\partial\Omega} = 0$$

has the unique solution for any h given by the Green function

$$u(x) = \int_\Omega G(x, y) h(y)\, dy.$$

Theorem 29. *([22]) If there exist positive c_1, c_2 such that*

$$f(u, c_1)^p \leq \frac{c_1}{b_2 \gamma^p}, \qquad f(u, c_2)^p \geq \frac{c_2}{b_1 \gamma^p}$$

for any $u \geq 0$, where

$$\gamma := \left(\int_\Omega \left(\int_\Omega G(x, y)\, dy \right)^p \right)^{1/p},$$

then the classical solution $u \in C^2(\Omega)$ exist with $\int_\Omega g(u^p)$ between c_1 and c_2.

The proof is very similar as in the case of radial solutions but the Banach space $E = L^p(\Omega)$, now, and the fixed points of the integral operator sits in the cone of nonnegative functions. However, we need the theory of linear elliptic PDEs to get the existence of the Green function and the estimates:

$$|G(x, y)| \leq C|x - y|^{2-n}, \qquad |\nabla_x G(x, y)| \leq C|x - y|^{1-n}$$

(see [19] or any other monograph on this theory).

Chapter 9

Variational methods

9.1 Introduction

There are some advantages of variational methods in BVPs: they are subtle, often enable us to find a solution with weaker assumptions; there are variational theorems that give multiple solutions (even infinite sequence). However, they work only in the case when the nonlinear term has a potential and not work for several boundary conditions.

Generally, these methods rely on observation that critical points of a functional are weak solutions of given BVP. The introduction of the notion of weak solution is necessary and important for PDEs, where even some simple linear problems have no solutions in the usual sense (classical or Carathéodory), but one can consider them also for ODEs.

Consider the Dirichlet problem:

$$x'' = f(t,x), \qquad x(0) = 0 = x(\pi), \qquad \text{where} \quad f(t,x) = \nabla_x F(t,x), \tag{9.1}$$

$F : [0,\pi] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function with continuous partial derivatives w.r.t. x_j, $j = 1, \ldots, n$, being coordinates of gradient $\nabla_x F$. The corresponding functional has then the form:

$$\Phi(x) := \int_0^\pi \left(\frac{1}{2}|x'(t)|^2 + F(t,x(t)) \right) dt \tag{9.2}$$

and a natural space for its domain is the space of C^1-functions vanishing at the ends of $[0,\pi]$. You should realize that the critical points of this functional need not be twice differentiable even a.e., hence they need not be classical or Carathéodory solutions of (9.1). By a weak solution of (9.1) we mean a critical point of Φ (i.e. a point where its Gâteaux derivative vanishes) but defined on a larger space $H_0^1(0,\pi)$ – the so called *Sobolev space.* It is the space consisting of all absolutely continuous on $[0,\pi]$ functions with $x' \in L^2$ and vanishing at the ends. The first summand under the integral suggests this choice of the domain and this space can be equipped with the norm given by a scalar product

$$\langle x, y\rangle := \int_0^\pi \sum_{j=1}^n (x_j'(t)y_j'(t) + x_j(t)y_j(t))\, dt.$$

Then this Sobolev space begins a Hilbert space. This is very important for our purpose that the domain is complete and reflexive and both these properties has H_0^1 but not C_0^1.

Exercise 28. *Prove the* Poincaré inequality

$$\int_0^\pi |x(t)|^2\, dt \le \int_0^\pi |x'(t)|^2\, dt$$

for $x \in H_0^1$ – use Fourier series, and show the L^2 norm of x' is an equivalent norm in H_0^1.

The Poincaré inequality implies that we can consider another norm in H_0^1 given by a scalar product:

$$\langle x, y\rangle := \int_0^\pi \langle x'(t), y'(t)\rangle dt, \qquad \|x\|^2 := \int_0^\pi |x'(t)|^2\, dt$$

and this norm is equivalent to the previous one. In the sequel, we shall use this norm not only for the case of $H_0^1(0,\pi)$ but for any open bounded set $\Omega \subset \mathbb{R}^n$ instead of the interval $(0,\pi)$.

The well known *theorem* due to *du Bois-Reymond* states that if $x \in H_0^1$ is a critical point of Φ, then there exists a constant c such that

$$x'(t) = \int_0^t \nabla_x F(s, x(s))\, ds + c \qquad a.e..$$

Hence critical points of Φ are classical solutions of (9.1).

9.2 Minima of functionals

Critical points can be obtained as global minima of Φ. For this purpose, Φ should be convex. Functional $\Phi : E \to \mathbb{R}$ is said to be *convex*, if $\Phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\Phi(x) + \lambda\Phi(y)$ for any $x, y \in E$ and $\lambda \in [0,1]$. We shall say that Φ is *coercive,* if $\Phi(x) \to +\infty$ as $\|x\| \to \infty$. We shall say that Φ is *lower semicontinuous* (resp. *weakly sequentially lower semicontinuous*), if

$$\liminf_{k\to\infty} \Phi(x_k) \geq \Phi(x) \qquad \text{for any} \quad x_k \to x \quad (resp.\ x_k \rightharpoonup x).$$

We have the following crucial (though simple) theorem

Theorem 30. *If Φ is a lower semicontinuous or weakly sequentially lower semicontinuous, coercive and convex functional on a reflexive Banach space E, then it has a global minimum. Moreover, if it is Gâteaux differentiable, then the set of its critical points is nonempty (closed and convex, as well).*

Proof. Take a sequence (x_k) such that $\Phi(x_k) \searrow \inf \Phi$ (we do not exclude this infimum is $-\infty$). Since Φ is coercive, this sequence is bounded. By E is reflexive, it contains a weakly convergent subsequence $x_k \rightharpoonup x$ (Eberlain-Schmulian). From *Mazur's theorem* (see [11], Cor. 3.8, p. 61), there exists a sequence of convex linear combinations

$$y_k = \sum_{j=1}^{p_k} a_{jk} x_k, \qquad a_{jk} \geq 0, \quad \sum_{j=1}^{p_k} a_{jk} = 1$$

which is strongly convergent to x. Thus, from the lower semicontunuity and the convexity,

$$\Phi(x) \leq \liminf \Phi(y_k) \leq \liminf \Phi(x_k) = \inf \Phi.$$

For weakly sequentially semicontinuous functionals the above arguments are simpler (without Mazur's theorem).

Functional defined by (9.2) is convex if $F(t, \cdot)$ is convex for all t. If it is bounded from below by a quadratic function

$$F(t,x) \geq -\frac{a}{2}|x|^2 - b|x| - c, \tag{9.3}$$

where $a < 1$, then Φ is coercive (on H_0^1). In fact,

$$\Phi(x) \geq \int_0^\pi \left(\frac{1}{2}(|x'(t)|^2 - a|x(t)|^2) - b|x(t)| - c \right) dt$$

$$\geq \frac{1-a}{2}\|x\|^2 - b\sqrt{\pi}\|x\| - c\pi.$$

We shall show that Φ is weakly sequentially lower semicontinuous. Let $x_k \rightharpoonup x$. It follows that (x_k') is bounded in L^2.

$$|x_k(t)| = \left|\int_0^t x_k'\right| \leq \sqrt{\pi}\|x_k\| \leq \sqrt{\pi}M,$$

i.e. (x_k) is equibounded in $C[0,\pi]$. Moreover,

$$|x_k(t) - x_k(s)| = \left|\int_s^t x_k'\right| \leq \sqrt{|t-s|}\|x_k\| \leq M\sqrt{|t-s|},$$

that implies equicontinuity of this sequence. By the Ascoli-Arzelá Theorem it has a uniformly convergent subsequence. But (x_k) is also weakly convergent, hence all subsequencences tends to the same limit. Thus $x_k \rightrightarrows x$. We have

$$0 \leq \int_0^\pi |x_k' - x'|^2 = \int_0^\pi (|x_k'|^2 - 2\langle x_k', x'\rangle + |x'|^2).$$

The sum of the second and the third summands tends to $-\|x\|^2$, hence

$$\liminf \int_0^\pi |x_k'|^2 \geq \int_0^\pi |x'|^2.$$

Since F is continuous (uniformly continuous on compacta) and $x_k \rightrightarrows x$, $F(t, x_k(t)) \rightrightarrows F(t, x(t))$ and

$$\lim \int_0^\pi F(t, x_k(t))\,dt = \int_0^\pi F(t, x(t))\,dt,$$

and it ends the proof.

We summarize all of above:

Theorem 31. *Let $F : [0,\pi] \times \mathbb{R}^n \to \mathbb{R}$ be continuous and has continuous gradient $\nabla_x F$, $F(t,\cdot)$ is convex for any t and F satisfies* (9.3)*. Then problem* (9.1) *has a solution. If F is strictly convex i.e.*

$$F(\cdot, (1-\lambda)x + \lambda y) < (1-\lambda)F(\cdot, x) + \lambda F(\cdot, y), \qquad \lambda \in (0,1),$$

then this solution is unique.

This result shows precisely all differences between topological and variational methods. If one takes $f(t,x) := \nabla_x F(t,x)$ and it has a linear growth $|f(t,x)| \leq a|x| + b(t)$, where $a < 1$, then one can find a sufficiently large ball $B(0,R)$ in C such that the integral operator with the Green function maps it into itself and by Schauder's theorem we have a solution. If we integrate the right-hand side of this estimate, we shall get a bound on $|F|$ by a quadratic function with coefficient $a < 1$. But it is an estimate from below and above ... . An upper bound for this constant a is natural, since $x'' + ax = 0$, $x(0) = 0 = x(\pi)$, is resonant for $a = 1$ and it is not resonant for $a \in (-\infty, 1)$. The variational method can ,,see" the difference between $a < 0$ and $a > 0$.

9.3 Saddle points

Although there are a lot of examples, where Φ is convex or, at least, it has a minimum, you can easily find functionals, where it fails. For instance, consider $\Phi : H_0^1(0,\pi) \to \mathbb{R}$

$$\Phi(x) := \int_0^\pi \left(\frac{1}{2}|x'(t)|^2 - \frac{1}{4}|x(t)|^4 \right) dt,$$

– a functional corresponding the Dirichlet BVP $x'' = -x^3$, $x(0) = 0 = x(\pi)$. If we choose $x \neq 0$ and $\lambda \in \mathbb{R}$, we get

$$\Phi(\lambda x) = \int_0^\pi \left(\frac{\lambda^2}{2}|x'(t)|^2 - \frac{\lambda^4}{4}|x(t)|^4 \right) dt \to -\infty$$

when $|\lambda| \to \infty$. On the other hand,

$$\Phi(\sin kt) = \frac{\pi}{4}(k^2 - 1) \to +\infty \qquad \text{when } \ k \to \infty.$$

Thus this functional is not bounded from below and from above. However, one can prove that it has a local minimum at $x = 0$.

For such functionals, you need different variational results and the simplest of them is the following one due to Paul Rabinowitz:

Theorem 32. (Saddle Point, *[79]) Let $E = V \oplus X$ be a Banach space such that V is nontrivial and finite dimensional and $\Phi : E \to \mathbb{R}$ is Fréchet differentiable with continuous derivative ($\Phi \in C^1$). Assume that* *(i) Φ satisfies* Palais-Smale condition, *i.e. any sequence $(x_k) \subset E$ such that $\Phi(x_k)$ is bounded and $\Phi'(x_k) \to 0$ contains a convergent subsequence;*

(ii) there exists a bounded neighborhood D of 0 in V such that $\Phi(x) \leq \alpha$ for $x \in \partial D$;

(iii) there exists a constant $\beta > \alpha$ such that $\Phi(x) \geq \beta$ for $x \in X$.

Then Φ possesses a critical value $c \geq \beta$ given by

$$c = \inf_{h \in \Gamma} \sup_{x \in \bar{D}} \Phi(h(x)),$$

where Γ is the set of all continuous extensions of the identity on ∂D : $h : \bar{D} \to E$ such that $h(x) = x$ for $x \in \partial D$.

The simplest proof invokes the Deformation Lemma [57] – very technical result describing possible deformations of sets $\{x : \Phi(x) \leq b\}$ near regular value. It is beyond our lectures.

It is easy to see that $\Phi \in C^1$ for Φ given by (9.2). Its derivative has the form

$$\Phi'(x) \cdot v = \int_0^\pi \left(\langle x'(t), v'(t)\rangle + \langle \nabla_x F(t, x(t)), v(t)\rangle\right) dt.$$

Exercise 29. *Prove it.*

We shall show that our functional satisfies the Palais-Smale condition if there exist $p > 2$, a and $R > 0$ such that

$$\langle \nabla_x F(t, x), x\rangle \leq pF(t, x)$$

for $t \in I := [0, \pi]$, $|x| \geq R$. If $(x_k) \subset H_0^1$ is such that $|\Phi(x_k)| \leq c_1$ and $\Phi'(x_k) \to 0$. It follows $\|\Phi'(x_k)\| \leq c_2$. Notice that integrals

$$\int_{J_k} F(t, x_k(t))\, dt, \qquad \int_{J_k} \langle \nabla_x F(t, x_k(t)), x_k(t)\rangle\, dt,$$

where $J_k := \{t \in I : \ |x_k(t)| \leq R\}$, are bounded. Thus we have

$$pc_1 + c_2\|x_k\| \geq p\Phi(x_k) - \Phi_k'(x_k) \cdot x_k$$

$$= \frac{p-2}{2}\|x_k\|^2 + \int_I \left(pF(t, x_k(t)) - \langle \nabla_x F(t, x_k(t)), x_k(t)\rangle\right) dt$$

$$\geq \frac{p-2}{2}\|x_k\|^2 + c_3$$

– the last inequality obtained through omitting positive integral $pF - \langle \nabla F, x_k \rangle$ over $I \setminus J_k$. Hence (x_k) is bounded in H_0^1. Since this space is reflexive, it contains a weakly convergent subsequence. But we have already known that it implies uniform convergence $x_k \rightrightarrows x$ in the space of continuous functions. Now,

$$(\Phi'(x_k) - \Phi'(x)) \cdot (x_k - x) = \int_I |x_k' - x'|^2 \, dt$$

$$+ \int_I \langle \nabla_x (F(t, x_k(t)) - F(t, x(t))), x_k(t) - x(t) \rangle \, dt$$

and the second summand tends to 0. By $\Phi'(x_k) - \Phi'(x) \to -\Phi'(x)$ and $x_k \to x$ weakly, the left-hand side converges to 0 and, therefore, the first summand tends to 0 that means $\|x_k - x\|^2 \to 0$.

We shall see how the Saddle Point Theorem works in the following resonant example:

Theorem 33. *Consider the one dimensional BVP*

$$x'' + m^2 x = f(t, x), \qquad x(0) = 0 = x(\pi),$$

where $m \in \mathbb{N}$, $f : I \times \mathbb{R} \to \mathbb{R}$ *is continuous and bounded function satisfying*

$$\lim_{|x| \to \infty} \inf_t F(t, x) = -\infty, \qquad \text{where} \quad F(t, x) := \int_0^x f(t, \xi) \, d\xi.$$

Then the problem possesses at least one solution.

Proof. We shall show that the corresponding functional

$$\Phi(x) = \int_I \left(\frac{1}{2}(|x'(t)|^2 - m^2 |x(t)|^2) + F(t, x(t)) \right) dt$$

defined on H_0^1 satisfies the assumption of the Saddle Point Theorem. The proof of the Palais-Smale condition will be given later. Take $V := Lin\{e_k : k = 1, \ldots, m\}$ (e_k is a normalized eigenfunction for the eigenvalue k) and $X := V^\perp$. Denote any $x \in V$ as the sum $\sum_k \lambda_k e_k$. For such x we have: since

$$\int_I |e_k'(t)|^2 \, dt = 1 = k^2 \int_I |e_k(t)|^2 \, dt,$$

we obtain by using the Mean Value Theorem

$$\Phi(x) = \frac{1}{2}\sum_{k=1}^{m-1}\lambda_k^2\left(1-\frac{m^2}{k^2}\right) + \int_I F(t,x_0(t))\,dt + \int_I f(t,u(t))x_-(t)\,dt,$$

where we write $x_0(t) = \lambda_m e_m(t)$, $x_-(t) = x(t) - x_0(t)$. The first summand is bounded by above

$$\sum_{k=1}^{m-1}\lambda_k^2\left(1-\frac{m^2}{k^2}\right) \le \left(1-\frac{m^2}{(m-1)^2}\right)\sum_{k<m}\lambda_k^2 = -c_1\|x_-\|^2,$$

and the third one

$$\int_I f(t,u(t))x_-(t)\,dt \le \sup|f|\int_I |x_-(t)|\,dt \le c_2\|x_-\|,$$

where we have used the Hölder and Poincaré inequalities. Therefore,

$$\Phi(x) \le -c_1\|x_-\|^2 + \int_I F(t,x_0(t))\,dt + c_2\|x_-\|$$

and if $\|x\|^2 = \|x_-\|^2 + \|x_0\|^2 \to \infty$, then either the first and the third terms $\to -\infty$ or the second one tends to $-\infty$ when $\|x_0\| \to \infty$. Taking $D = B_V(0,R)$ and changing the radius R, we can get α arbitrarily close to $-\infty$.

We shall verify (iii). Let $x \in X$ be write-down as $x = \sum_{k>m}\lambda_k e_k$. Then

$$\int_I \left(|x'(t)|^2 - m^2|x(t)|^2\right)\,dt = \sum_{k>m}\lambda_k^2\left(1-\frac{m^2}{k^2}\right) \ge \left(1-\frac{m^2}{(m+1)^2}\right)\|x\|^2$$

and

$$\left|\int_I F(t,x(t))\,dt\right| \le \sup|f|\int_I |x| \le \sup|f|\cdot\sqrt{\pi}\|x\|,$$

where we have applied the Hölder and Poincaré inequalities. It means that $\Phi(x) \to +\infty$ as $\|x\| \to \infty$, hence Φ is bounded from below by some constant β. Choose R in in the proof of (ii) so large that $\alpha < \beta$.

Let us come to the Palais-Smale condition. Let (x_k) be a sequence in H_0^1 such that $|\Phi(x_k)| \le c_1$ and $\Phi'(x_k) \to 0$. Write $x_k = x_k^0 + x_k^+ + x_k^-$, where $x_k^0 = \lambda_k e_m$, $x_k^- = \sum_{j<m}\lambda_{jk}e_j \in V$, $x_k^+ = \sum_{j>m}\lambda_{jk}e_j \in X$. Then

$$|\Phi'(x_k)x_k^\pm| = \left|\int_I \left(x_k'\cdot x_k^{\pm\prime} - m^2 x_k\cdot x_k^\pm + f(t,x_k(t))\cdot x_k^\pm(t)\right)dt\right| \le \|x_k^\pm\|$$

for large k. Then

$$\|x_k^+\| \geq \left(1 - \frac{m^2}{(m+1)^2}\right)\|x_k^+\|^2 - \sup|f|\|x_k^+\|,$$

and similarly for x_k^-. It means that both these sequences $(x_k^\pm)$ are bounded. Then

$$c_1 \geq |\Phi(x_k)| = \left|\int_I \left(\frac{1}{2}(|{x_k^+}'|^2 + |{x_k^-}'|^2 - m^2|x_k^+|^2 - m^2|x_k^-|^2\right.\right.$$

$$\left.\left. + F(t, x_k(t)) - F(t, x_k^0(t))\right) dt + \int_I F(t, x_k^0(t))\, dt\right|.$$

But the first summand is bounded by a constant independent of k (for $F(t, x_k) - F(t, x_k^0)$ we apply the Mean Value Theorem), hence

$$c_1 \geq \left|\int_I F(t, x_k^0(t))\, dt\right| - c_2.$$

Exercise 30. *Prove that*

$$\lim_{|\lambda|\to\infty} \int_I F(t, \lambda e_0(t))\, dt = -\infty.$$

It follows that (x_k^0) is bounded. The rest of the proof of Palais-Smale's condition is similar as above. Thus all assumptions of the Saddle Point Theorems hold and we have a solution.

You can find many other variational results consulting [57] or [79]. Especially, you should study the Mountain Pass Lemma and its applications.

Chapter 10

Multiple solutions

Up to now, the main question considered in this book has been the existence of at least one solution. The uniqueness of this solution is also important from the mathematical point of view, however if our BVPs come from applications of the real world (in my opinion, cannot be a kind of art only), we usually believe the solution is unique since the model should predict the behaviour of the real system. We shall not study the question of the uniqueness in this book, but in this section we shall show some results giving the existence of at least m solutions, since they can appear in the real world problems. For example, if we search for stationary solutions of some evolutionary problems, then the BVP describing them can have many solutions and the evolutionary equation have time dependent solutions tending to different stationary ones – the limit stationary solution depends on the initial state of the system. The methods necessary to find many solutions are the same as in the previous sections, however they are used more gently.

10.1 Standard methods

First we can see how to use the Leray-Schauder degree. Consider the simplest periodic problem

$$x'' = f(t,x), \qquad x(0) = x(1), \quad x'(0) = x'(1),$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and choose

$$a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < \ldots < a_m < c_m < b_m,$$

all $c_j \neq 0$. The second derivative is not invertible on the space of 1-periodic functions, but $x'' - \varepsilon x$ has this property for most $\varepsilon \in \mathbb{R}$. You can calculate eigenvalues ε, when the invertibility fails $\varepsilon_k = -(2\pi k)^2$, $k = 0, 1, 2, \dots$. Hence ε can be arbitrary positive.

Theorem 34. *Suppose that, for $c_j < 0$,*

$$f(t, a_j) > 0 \ \ \textit{for} \ \ t \in [0,1],$$

$$f(t, b_j) < 0 \ \ \textit{for} \ \ t \in [0,1]$$

and for $c_j > 0$, we have the opposite inequalities for $j = 1, \dots, m$. Then the problem has at least m solutions φ_j, $j = 1, \dots, m$, such that $\varphi_j(t) \in (a_j, b_j)$ for all $t \in (0,1)$, $j = 1, \dots, m$.

Proof. Take $\varepsilon > 0$ sufficiently large such that

$$f(t, a_j) - \varepsilon x < -\frac{c_j}{\varepsilon} \ \ \text{for} \ \ t \in [0,1], \ x \in (a_j, b_j),$$

$$f(t, b_j) - \varepsilon x > +\frac{c_j}{\varepsilon} \ \ \text{for} \ \ t \in [0,1], \ x \in (a_j, b_j)$$

for $j = 1, \dots, m$. Let

$$\Omega_j := \{x \in C[0,1] : \ a_j < x(t) < b_j \text{ for } t \in [0,1]\}, \qquad j = 1, \dots, m.$$

The operator $Lx := x'' - \varepsilon x$ is invertible on the space of T-periodic functions. The studied problem is equivalent to the fixed point problem for the operator

$$Tx(t) := \int_0^1 G(t,s)[f(s, x(s)) - \varepsilon x] \, ds$$

where G is the Green function for T-periodic problem and differential operator L. We should show that the Leray-Schauder degrees of $I - T$ on Ω_j at the point 0 do not vanish for all of j. To do this we use homotopies

$$H(\lambda, \cdot) := I - \lambda T + (1 - \lambda)\frac{c_j}{\varepsilon},$$

$\lambda \in [0,1]$. It is easy to see that it does not touch 0 at $x \in \partial\Omega_j$, since such functions take the minimum equal to a_j or the maximum b_j at some point

t_0 and for the minimum $x''(t_0) \geq 0$ and for the maximum $x''(t_0) \leq 0$ – a contradiction. Hence

$$\deg_{LS}(I - T, \Omega_j, 0) = \deg_{LS}(I, \Omega_j, c_j) = 1,$$

since T maps the constant function $\frac{c_j}{\varepsilon}$ to c_j, and this ends the proof.

Similar result can be obtained if we take $\varepsilon < 0$ but then it cannot be arbitrary, $\varepsilon > -4\pi^2$, for example. You can notice that this arguments are similar to proofs of results giving many solutions by using the Krasnosielski Cone-Compression/Expansion Theorem.

The second completely different approach to the quaestion of multiple solutions appeared in the article by A. Ambrosetti and G. Prodi [5]. The authors wondered how generalize

Theorem 35. (Global Inversion Theorem) *If $N : X \to Y$ is a C^1 mapping between Banach spaces which satisfies $N'(x) \in L(X, Y)$ is an isomorphism for each $x \in X$ (this implies a local invertibility at every point) and N is proper, i.e. $F^{-1}(K)$ is compact for K a compact set. then N maps diffeomorfically X onto Y.*

The earliest version of this theorem was found by J. Hadamard. The properness of a mapping is usually difficult to prove and some sufficient conditions are needed. For instance, we can assume $\sup_x \|N'(x)^{-1}\| < \infty$. Later many mathematicians tried to generalize it (see discussion in [4]) but theorem of this kind give exactly one solution to any equation $N(x) = y$. In [5], mapping N has singular points but of a special kind:

$$\ker N'(x_0) = \mathrm{Lin}(v_0), \qquad \mathrm{im} N'(x_0) = \ker \gamma_0,$$

where $\gamma_0 \in Y^*$. It means that the derivative $N'(x_0)$ is a Fredholm operator of index 0 with one dimensional kernel. Denote the set of singular points of N by Σ, suppose that N is of C^2-class and

$$\gamma_0(N''(x_0) \cdot (v_0, v_0)) \neq 0$$

for any $x_0 \in \Sigma$.

Theorem 36. *([5]) Suppose N is a C^2 proper mapping whose all singular points x_0 are such that $N'(x_0)$ is not isomorphism, they satisfy the above assumptions and Σ is connected. Moreover, equation $N(x) = y$ has exactly one solution for each $y \in \mathbb{N}(\Sigma)$. Then $N(\Sigma)$ is a C^2 manifold of codimension 1 in Y which cuts Y into two open connected sets Y_1, Y_2 and equation $N(x) = y$ has no solution for $y \in Y_1$ and it has exactly two solutions for $y \in Y_2$.*

The geometry of N described by this theorem can be visualized as a fold of a piece of paper and set it onto the second plane Y : for points $y \in Y$ touched by the fold is one solution, for other points under folded paper are two solutions, points not covered correspond to equations without any solution.

This theorem is applied to the semilinear equation with the Laplacian and the null Dirichlet condition but we would like to avoid Hölder spaces hence we consider:

$$x'' + p(x) = f(t), \qquad x(0) = 0 = x(\pi), \tag{10.1}$$

where $p : \mathbb{R} \to \mathbb{R}$ is a C^2-function with $p(0) = 0$, $p''(x) >$,

$$-0 < \lim_{x \to -\infty} p'(x) < 1 < \lim_{x \to \infty} p'(x) < 4,$$

$f \in C[0, \pi]$. The operator $N(x)(t) := x''(t) + p(x(t))$ map the Banach space $X := \{x \in C^2[0, \pi] : \ x(0) = 0 = x(\pi)\}$ into $Y := C[0, \pi]$ has continuous Fréchet derivatives N' and N''. We prove that N is proper by using the Ascoli-Arzelá Theorem. Take $(f_n)_n \subset Y$ such that $N(x_n) = f_n$ for some $x_n \in X$. First, we shall show that functions x_n are equibounded. If it is not, after passing to subsequence, $\|x_n\|_Y \to \infty$. Then $u_n = x_n/\|x_n\|_Y$ satisfies

$$u_n'' + \phi(x_n)u_n = \frac{f_n(t)}{\|x_n\|_Y},$$

where

$$\phi(x) := \begin{cases} \frac{p(x)}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Since ϕ is bounded, thus u_n'' are equibounded and it is easy to see that u_n' are equibounded (comp. earlier proof in section 5.2). Hence for a subsequence, (u_n) is uniformly convergent with derivatives to U. Multiplying both sides of equation on u_n by any test function $w \in C_0^\infty(0, \pi)$ and integrating by part, we get

$$-\int_0^\pi w'u_n' + \int_0^\pi w\phi(x_n)u_n = \int_0^\pi w\frac{f_n}{\|x_n\|_Y}.$$

We can pass to limits under integrals due to the Lebesgue Dominated Convergence Theorem and obtain

$$-\int_0^\pi w'U' + \int_0^\pi wr(t)U = 0,$$

where $r(t) = \lim_{x\to-\infty} p'(x)$ for such t that $U(t) < 0$, $= \lim_{x\to\infty} p'(x) =: \lambda_+$ if $U(t) > 0$ and $= p'(0)$ if $U(t) = 0$. Since U is twice differentiable, it is not only a weak solution but a strong one to

$$U'' + r(t)U = 0, \qquad U(0) = 0 = U(\pi).$$

But r is positive, so U is concave and, therefore positive on $(0, \pi)$. It means U is a solution $U'' + \lambda_+ U = 0$ but it is impossible $\lambda_+ \neq 1$. Hence the sequence $\|x_n\|_X$, $n \in \mathbb{N}$, is bounded.

If $N(x_n) = f_n \in K$ and K is compact in Y, then it has a convergent subsequence $f_n \to f$ and the above arguments show that $x_n \to x$ in X with $N(x) = f$. Thus $N^{-1}(K)$ is compact and N is proper. For the proof of the remaining assumptions of Theorem 36 we refer to [4], chap. 4.2.

The arguments in the proof of the above theorems are difficult since the assertions give the exact number of solutions. If someone needs only lower estimates of this number and not for all elements on the right-hand side, then the perturbation method or the coincidence degree theory suffices. For example, in [73], we have proved some abstract result and its application to:

Theorem 37. *Consider*

$$x'' + x = f(t, x) + h_1(t) + \lambda \sin t, \qquad x(0) = 0 = x(\pi),$$

where $f : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ *is continuous bounded from above,*

$$\limsup_{|x|\to\infty} \sup_t \frac{f(t, x)}{x} = 0,$$

both sets $\{t : \lim_{x\to\pm\infty} f(t, x) = -\infty\}$ *have positive measure,*

$$\int_0^\pi h_1(t) \sin t \, dt = 0$$

and $\lambda \in \mathbb{R}$, *then there exists* $\lambda_0 > 0$ *such that, the BVP has no solution for* $\lambda \leq -\lambda_0$ *and it has at least two solutions for* $\lambda \geq \lambda_0$.

The asymptotic behaviour of f implies the range of the operator $x \mapsto x'' + x - f(\cdot, x(\cdot))$ cannot reach functions sitting on one half-space

$$C[0, \pi] \setminus \{h : \int h(t) \sin t = 0\}$$

in a long distance to this linear subspace of codimension 1 and it reach it twice for the second half-space. The geometry is globally similar as above mentioned folding but only globally. The proof (of the existence of two solutions) relies on an appropriate choice of open bounded sets

$$\{h_1 + d\sin(\cdot) : \ \|h_1\| < R, \ d \in (d_0, d_1)\},$$

$$\{h_1 + d\sin(\cdot) : \ \|h_1\| < R, \ d \in (-d_1, -d_0)\}$$

with $0 < d_0 < d_1$. Similar problems have been studied in [71, 72], periodic problems in [20] and elliptic problems in [2].

10.2 Applications of the Morse Theory

There is a topological tool which is very subtle: the Morse Theory. It was developped to study isolated nondegenerate critical point of C^2-functionals defined on finite dimensional compact manifolds. Such functionals in a neighbourhood of a critical point x_0 have the following form in a certain map:

$$\varphi(x) = \varphi(x_0) - \sum_{j=1}^{p} x_j^2 + \sum_{j=p+1}^{n} x_j^2$$

and the number p is called the *Morse index* of φ at x_0 and denoted by $\mathrm{ind}_{x_0}(\varphi)$. The sum

$$\sum_{p=0}^{n} (-1)^p \sharp\{x_0 : \ \mathrm{ind}_{x_0}(\varphi) = p\},$$

where $\sharp Z$ is the cardinality of the set Z, equals the main topological invariant of the manifold – its Euler characterisic. The infinite dimensional version of this theory started from seminal papers due to R. Palais and it is beyond this short survey. We shall show only one special result which is applicable to study BVPs.

Theorem 38. *Let $\varphi : X \to \mathbb{R}$ be a C^2-functional defined on a real Banach space X satisfying Palais-Smale condition and bounded from below. Let x_0 be an isolated critical point with finite Morse index which is not a point of the global minimum and such that $\varphi'(x_0) = I - T$, where T is compact and $deg_{LS}(I - T, B(0, R), 0) = \pm 1$. Then φ has at least three critical points (x_0, an argument of the global minimum and one another).*

We applied this theorem to the so-called Lidstone problem for an even order ODE [37]. Here, we shall show the simplest case

$$x'' + f(t,x) = 0, \qquad x(0) = 0 = x(\pi),$$

where $f : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is continuous. If we use the spectral theory in $L^2(0,\pi)$ to the operator $Lx = -x''$ with boundary conditions $x(0) = 0 = x(\pi)$, then we find its spectrum: $\mu_n := n^2$, $n \in \mathbb{N}$, with the corresponding eigenfunctions $e_n(t) := \sqrt{\frac{2}{\pi}} \sin nt$ that gives an orthonormal basis in $L^2(0,\pi)$. The inverse operator obtained by using this theory is as follows:

$$L^{-1}h := \sum_n \frac{1}{n} \langle h, e_n \rangle e_n$$

hence it is the integral operator given by the formula:

$$L^{-1}h(t) = \frac{2}{\pi} \int_0^\pi \left(\sum_n \frac{1}{n^2} \sin ns \sin nt \right) h(s)\, ds$$

(the series in the parenthesis is convergent in L^2 to the Green function $G(t,s)$ of our problem). Since L^{-1} is a bounded positive operator on $L^2 : \langle L^{-1}h, h\rangle \geq \|h\|^2$, it has the square root – the unique positive operator S on L^2 such that $S^2 = L^{-1}$ and it is given by the formula

$$Sh := \sum_n \frac{1}{n} \langle h, e_n \rangle e_n,$$

which is also a compact integral operator. If we denote by N the Nemytski operator defined by f, then our BVP is equivalent to the equation $x = S^2 N(x)$ (solutions in L^2 are C^2-functions being classical solutions of the BVP). Let $F : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ be a potential of f given by

$$F(t,x) := \int_0^\xi f(t,\xi)\, d\xi.$$

Notice that the functional $\varphi : L^2 \to \mathbb{R}$

$$\varphi(y) := \frac{1}{2}\|y\|^2 - \int_0^\pi F(t, Sy(t))\, dt$$

is well-defined and its Fréchet derivative is the following:

$$\varphi'(y) \cdot h := \langle y, h\rangle - \langle N(Sy), Sh\rangle = \langle y - SNS(y), h\rangle,$$

therefore its critical points are solutions to the equation $y = SNS(y)$. If y is such a function, then $x = Sy$ is a solution of the BVP.

Theorem 39. *Assume a continuous function f satisfies for some $m \in N$:*
(C1) there exist limits

$$\lim_{x\to 0} \frac{f(t,x)}{x} =: \frac{\partial f}{\partial x}(t,0) \in (m^2, (m+1)^2)$$

for any t;
(C2) the following infinite system of equations:

$$c_j = \frac{2}{\pi j}\sum_n \frac{1}{n}\int_0^\pi \frac{\partial f}{\partial x}(t,0)\sin nt \sin jt\, dt \cdot c_n, \qquad j \in \mathbb{N},$$

has only the null solution;
(C3) there exist $\alpha < \frac{1}{2}$ and $\beta \in \mathbb{R}$ such that

$$F(t,x) \leq \alpha x^2 + \beta$$

for all arguments. Then the BVP has at least three solutions.

Proof. We shall apply the Theorem 38. First, for any $y \in L^2$,

$$\varphi(y) \geq \frac{1}{2}\|y\|^2 - \int_0^\pi (\alpha(Sy(t))^2 + \beta)\, dt$$

$$\geq \left(\frac{1}{2} - \alpha\|S\|^2\right)\|y\|^2 - \beta\pi = \left(\frac{1}{2} - \alpha\right)\|y\|^2 - \text{const}$$

which means that φ is bounded from below and coercive. If (y_k) is a sequence such that $\varphi(y_k)$ is bounded and $\varphi'(x_k) \to 0$, then $y_k - SNS(y_k) \to 0$. But SNS is compact operator, hence $SNS(y_k)$ has a convergent subsequence and, therefore, the Palais-Smale condition holds.

We shall show, 0 is a nondegenerate critical point of φ. $\varphi'(0) = 0$ by $f(t,0) \equiv 0$ hidden in condition (C1). Moreover,

$$\varphi''(0)(u,v) = \langle u, v\rangle - \int_0^\pi \frac{\partial f}{\partial x}(t,0)Su(t)Sv(t)\, dt = \langle u, v\rangle - \left\langle \frac{\partial f}{\partial x}(\cdot,0)SU, Sv\right\rangle.$$

This bilinear form is symmetric and continuous, thus it defines a self-adjoint operator T acting on L^2 :

$$Tu := u - S\left(\frac{\partial f}{\partial x}(\cdot, 0)Su\right)$$

and 0 is nondegenerate if T is one-to-one. The equation $Tu = 0$ means $u = S\left(\frac{\partial f}{\partial x}(\cdot, 0)Su\right)$ and if we expand u in the Fourier series $u = \sum_n c_n e_n$, then the system of equations from condition (C2) is satisfied. Hence, $u = 0$.

Now, we shall find the finite Morse index of 0. Let

$$H_- := \mathrm{Lin}\{e_n : n \leq m\},$$

$$H_+ := \mathrm{Lin}\{e_n : n \geq m+1\},$$

obviously $L^2(0, \pi) = H_- \oplus H_+$. We shall show that

$$\varphi''(0)(x, x) \leq -C\|x\|^2, \qquad x \in H_-,$$

for some positive constant C. Take $\varepsilon > 0$ such that

$$m^2 + \varepsilon \leq \frac{\partial f}{\partial x}(t, 0) \leq (m+1)^2 - \varepsilon, \qquad t \in [0, 1].$$

Then, for $x \in H_-$,

$$\varphi''(0)(x, x) = \|x\|^2 - \int_0^\pi \frac{\partial f}{\partial x}(t, 0)|Sx(t)|^2\, dt \leq -\varepsilon m^2\|x\|^2$$

and similarly for $x \in H_+$,

$$\varphi''(0)(x, x) \geq \varepsilon(m+1)^2\|x\|^2.$$

Since $T = I - \hat{S}$, where $\hat{S}u := S(\frac{\partial f}{\partial x}(\cdot, 0)SU)$ is a compact self-adjoint operator, it can be diagonalized. It means there exists a unitary operator U on L^2 such that $\hat{e}_n := U(e_n)$, $n \in \mathbb{N}$, and

$$U^{-1}\hat{S}U\left(\sum_n c_n\hat{e}_n\right) = \sum_n \hat{\mu}_n c_n \hat{e}_n.$$

Hence the above calculations show $\varphi''(0)$ is negative on the invariant subspace $U(H_-)$ and positive on $U(H_+)$. Thus the Morse index of 0 equals m.

It remains to prove that there is some x such that $\varphi(x) < \varphi(0) = 0$. Take $\varepsilon > 0$ as above and choose $\delta > 0$ such that $f(t,x) \geq (m^2+\varepsilon)x$ for all t and $|x|\delta$ – it exists due to (C1). Hence

$$F(t,x) \geq \frac{1}{2}(m^2+\varepsilon)x^2, \qquad |x| \leq \delta, \ \ t \in [0,\pi].$$

Then, for $y \in H_-$ such that $\sup_t |Sy(t)| \leq \delta$ we have

$$\varphi(y) = \frac{1}{2}\|y\|^2 - \int_0^\pi F(t, Sy(t))\, dt \leq \frac{\varepsilon}{2}\|y\|^2 < 0.$$

All assumptions of Theorem 38 are satisfied, thus at least three critical points exist.

Remark. In particular, condition (C2) holds if f does not depend on t or more generally, if $\frac{\partial f}{x}(t,0)$ is a constant, since The equations of the system are not splitted.

The *Lidstone problem* studied in [37] is more complicated:

$$x^{(2k)} - \sum_{j=1}^{k} \lambda_j x^{(2k-2j)} = f(t, x^{(2i-2)}),$$

$$x^{(2j)}(0) = 0 = x^{(2j)}(1), \qquad j = 0, 1, \ldots, k-1,$$

however all important ideas are included in our case $k = i = 1$ and differences are technical in nature. Earlier, this Lidstone problem were studied by M. Jurkiewicz in [36], where the existence of one solution were showed in the case the linear part is resonant.

You can find other results about the existence of many solutions in [2, 20, 82].

Chapter 11

BVPs for ODEs in Banach spaces

One can study ODEs in an arbitrary Banach space E, especially infinite dimensional. Let $f : [\alpha, \beta] \times E \to E$ be continuous function and $x_0 \in$ be fixed. Consider the initial value problem

$$x' = f(t, x), \qquad x(\alpha) = x_0. \tag{11.1}$$

If f satisfies the Lipschitz condition w.r.t. x., then the unique solution exists by the Contraction Principle as for $E = \mathbb{R}^n$. If f is only continuous, we cannot apply the Schauder Fixed Point Theorem even for bounded f, since the integral operator:

$$T(x)(t) := x_0 + \int_\alpha f(s, x(s))\, ds$$

acting on $C([\alpha, \beta], E)$ is not completely continuous (the Ascoli-Arzelá Theorem for $C(X, E)$ needs pointwise compactness of a family).

Exercise 31. *Prove that, if $f : [\alpha, \beta] \times E \to E$ is bounded, continuous and $f(t, \cdot) : E \to E$ maps bounded sets into relatively compact ones, then* (11.1) *has a global solution.*

Example. Let $f : c_0 \to c_0$ be defined by

$$f(x = (x_n)_{n=1}^\infty) = \left(\sqrt{|x_n|} + \frac{1}{n}\right)_{n=1}^\infty$$

on the space of real sequences converging to 0 with the supremum norm. f is uniformly continuous but Lipschitz's condition does non hold (check it). Suppose that the initial value problem $x' = f(x)$, $x(0) = 0$, has a solution $\varphi = (\varphi_n) : (-\delta, \delta) \to c_0$ where φ_n take real values. Since $\varphi'(t) = \sqrt{|\varphi_n(t)|} + n^{-1} > 0$ for sufficiently small $t > 0$, we can omit the absolute value

$$\varphi_n'(t) = \sqrt{\varphi_n(t)} + n^{-1}, \qquad \varphi_n(0) = 0.$$

One can solve it explicitly:

$$n\sqrt{\varphi_n(t)} - \ln\left(n\sqrt{\varphi_n(t)} + 1\right) = \frac{nt}{2}$$

for $t \in [0, \delta)$. If $h(y) = y - \ln(y+1)$, $y \geq 0$, then $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a homeomorphism (exercise). Hence

$$\varphi_n(t) = \left(\frac{h^{-1}(nt/2)}{n}\right)^2.$$

Since $h^{-1}(y) > y$ for each $y > 0$, we get

$$\left(\frac{h^{-1}(nt/2)}{n}\right)^2 > \frac{n^2t^2}{4n^2} = \frac{t^2}{4} > 0$$

and the sequence $(\varphi_n(t))$ cannot tends to 0 for any $t > 0$. Therefore the initial value problem has no local solution.

If you want to catch more sophisticated examples, you need richer tools from Nonlinear Analysis. The most natural is the theory of measure of noncompactness and topological methods related to it (see [7]).

Function γ defined on the family of all bounded subsets of a Banach space E taking values in $[0, \infty]$ is called a *measure of noncompactness,* if it has the following properties:

1. $\gamma(\overline{conv} A) = \gamma(A)$ for any $A \subset E$,
2. $\gamma(A) = 0$ iff A is relatively compact,
3. if $A \subset B$, then $\gamma(A) \leq \gamma(B)$,
4. $\gamma(A \cup B) = \max(\gamma(A), \gamma(B))$,

5. $\gamma(A+B) \le \gamma(A) + \gamma(B)$ where $A+B := \{a+b : \ a \in A, \ b \in B\}$,
6. $\gamma(\lambda \cdot A) = |\lambda|\gamma(A)$ where $\lambda A := \{\lambda a : \ a \in A\}$,
7. if $T : E \to E$ is linear bounded operator, then $\gamma(T(A)) \le \|T\|\gamma(A)$ for each A.

There are typical measures of noncompactness:

– Kuratowski's measure

$$\gamma_K(A) := \inf\{d > 0 : \text{there exists finite covering of } A \subset \bigcup_{k=1}^{p} \Omega_k, \ \operatorname{diam}(\Omega_k) \le d\}$$

diam means the diameter of the set $\operatorname{diam}(A) := \sup\{d(x,y) : \ x, y \in A\}$.

– Hausdorff's measure

$$\gamma_H(A) := \inf\{\varepsilon > 0 : \ \text{there exists finite } \varepsilon - \text{net of } A\}.$$

– Istrătescu's measure

$$\gamma_I(A) := \sup\{\varepsilon > 0 : \ \text{there exists sequence } (x_n) \subset A, \ \|x_n - x_m\| \ge \varepsilon\}.$$

– in any separable Hilbert space with a complete orthonormal set $\mathcal{E} = e_n$, $n \in \mathbb{N}$,

$$\gamma_{\mathcal{E}}(A) := \limsup_{n\to\infty} \sup_{x \in A} \left(\sum_{k=n+1}^{\infty} |\langle x, e_k \rangle|^2 \right)^{1/2}.$$

– in the space of continuous real function $C[a,b]$

$$\omega_0(A) := \lim_{\varepsilon \to 0+} \sup_{x \in A} \sup\{|x(t) - x(s)| : \ |t - s| \le \varepsilon\}$$

is a part of a measure of noncompactness; the second part is $\omega_1(A) := \sup_{x \in A} \sup_{t \in [0,1]} |x(t)|$; $\gamma_C(A) := \omega_0(A) + \omega_1(A)$.

We do not prove that all functions satisfy needed conditions (it is not true for all of them). Especially difficult are proofs of the first property which seems to be essential. We shall show that this notion enables us to join two result on intersections:

If (A_n) is a decreasing sequence of closed subsets of
– a complete metric space with $\operatorname{diam}(A_n) \to 0$,
– or a Hausdorff topological space with at least one being compact, then

$$\bigcap_{n=1}^{\infty} A_n \ne \emptyset$$

(in the first case it is a single point).

Theorem 40. *If (A_n) is a decreasing sequence of closed subsets of a Banach space such that $\gamma(A_n) \to 0$ for some measure of noncompactness, then the set*

$$A = \bigcap_{n=1}^{\infty} A_n \neq \emptyset$$

and is compact.

Proof. Choose $x_n \in A_n$ for each n. Since

$$\gamma(\{x_n : \ n \geq 1\} = \gamma(\{x_n : \ n \geq k\} \leq \gamma(A_k),$$

the sequence is relatively compact, hence contains convergent subsequence: $x_n \to x$ (we do not change notation for simplicity). Since A_k are closed, the limit x belongs to every A_k, thus to their intersection A. Moreover, $\gamma(A) \leq \gamma(A_n)$ for any n what implies $\gamma(A) = 0$. Therefore, A is compact as a closed relatively compact set.

Let $T : E \supset X \to E$ is a continuous map in a Banach space with a measure of noncompactness γ. We shall say that T is a γ-*contraction,* if there exists a constant $q < 1$ such that $\gamma(T(A)) \leq q \cdot \gamma(A)$ for any bounded A;
and T is a γ-*condensing,* if $\gamma(T(A)) < \gamma(A)$ for any A with $\gamma(A) > 0$.

It is easy to see that any compact map is a γ-contraction with arbitrarily small q, and that all γ-contractions are γ-condensing.

Exercise 32. *Every metric contraction is a γ-contraction for γ_K or γ_H with the same constant $q < 1$.*

Theorem 41. *(*Darbo*) Let C be a closed convex and bounded subset of a Banach space E and $T : C \to C$ is a γ_K- or γ_H-contraction. Then T has a fixed point.*

Proof. Let $C_1 := C$, $C_{n+1} := \overline{conv} T(C_n)$ for $n > 1$, where $\overline{conv} K$ stands for the closed convex hull of set K (i.e. the smallest closed and convex set containing K). Then (C_n) is a decreasing sequence of closed sets with

$$\gamma(C_{n+1}) \leq q\gamma(C_n) \leq \dots q^{n+1}\gamma(C)$$

what gives $\gamma(C_n) \to 0$. Thus the intersection C_∞ of this sets is nonempty, compact and convex and $T(C_\infty) \subset C_\infty$. Due to the Schauder Fixed Point Theorem it has a fixed point.

There is a more general theorem (due to Sadovskii) that gives the same for condensing maps. There is also a degree theory for maps of the form $I-$ condensing (see [15]).

Exercise 33. *(Krasnosielskii) If $T : C \to E$ is (metric) contraction and $S : C \to E$ is a compact mapping, where C is closed, convex and bounded set in E and $(T + S)(C) \subset C$, then T possesses a fixed point.*

We can apply this theory to initial or boundary value problems by using the following

Theorem 42. *([65]) Let $G : I \times I \to L(E)$ be a mapping bounded and continuous for $t \neq s$, where I is a compact interval, E is a Banach space with a measure of noncompactness γ, ($L(E)$ stands for a space of linear bounded operators $E \to E$), let $f : I \times E \to E$ be continuous and*

$$\gamma(f(t, A)) \leq q(t)\gamma(A)$$

for some integrable function q and any $t \in I$ and bounded $A \subset E$. If

$$\sup_t \int_I \|G(t, s)\| q(s)\, ds < 1,$$

and there exists $R > 0$ such that the integral operator

$$T(x)(t) := \int_I G(t, s) f(s, x(s))\, ds, \qquad x \in C(I, E)$$

maps $\bar{B}(0, R)$ into itself, then T has a fixed point.

The proof is based on the Darbo Theorem applied to $T : \bar{B} \to \bar{B}$ and the measure of noncompactness in $C(I, E)$ given by the formula

$$\gamma_C(\mathcal{A}) := \sup_{t \in I} \gamma(\mathcal{A}_t) + \omega_0(\mathcal{A}),$$

where $\mathcal{A}_t := \{x(t) :\ x \in \mathcal{A}\}$. The crucial point is the estimate:

$$\gamma \left\{ \int_I G(t, s) f(s, x(s))\, ds :\ x \in \mathcal{A} \right\} \leq \int_I \|G(t, s)\| q(s) \gamma(\mathcal{A}_s)\, ds.$$

Chapter 12

Nonlinear BVPs for PDEs

The same methods of Nonlinear Analysis work for boundary value problems for partial differential equations. However, there are many difficulties connected with linear equations and linear conditions. The problems and, implicitly, methods depend on the kind of PDE. Dirichlet, Neumann and Robin boundary conditions are typical in the case of elliptic equations. Then the theory is similar as for ODEs: there exists a Green function that enables to replace the BVP

$$-\Delta u = f(x,u), \qquad u|\partial\Omega = 0, \tag{12.1}$$

for example by the integral equation

$$u(x) = \int_\Omega G(x,y) f(y,u(y))$$

and use the Leray-Schauder degree theory. The Green function is usually explicitly unknown, but its qualitative properties are an important part of the classical theory which you can find in numerous textbooks (Courant-Hilbert or Gilbarg-Trudinger, for instance). Since Green functions have singularities as $x \to y$, even the continuity of corresponding integral operators is a problem. Moreover, one cannot work in spaces of continuous (or C^1, C^2) functions with supremum norms as we used to in the case of ODE, because even simple linear BVPs do not possess solutions in these spaces. A full family of Sobolev spaces is the natural framework for PDEs. The second possibility is one of Hölder spaces (see the fundamental monography of Evans). We do not develop this rich theory and bound ourselves to a special case of (12.1), where

$f(x, u) = \lambda e^u$. As usually, Ω is an open bounded set in $\mathbb{R}^n$ with sufficiently regular boundary. This regularity comes from the linear theory; the Green function exists only for such boundaries. It is well known that $-\Delta u = \lambda u$ with Dirichlet's condition $u|\partial\Omega = 0$, has a nontrivial solutions for a sequence of positive λ and the first eigenvalue λ_1 is simple with the eigenfunctions being multiplicities of a positive eigenfunction u_1. Suppose that u is a solution of (12.1) with our f. Then (integration by parts formula)

$$\lambda \int_\Omega e^u u_1 = -\int_\Omega \Delta u \cdot u_1 = \int_\Omega \nabla u \cdot \nabla u_1 = -\int_\Omega u \Delta u_1 = \lambda_1 \int_\Omega u u_1.$$

But $e^u > u$, hence

$$\lambda_1 \int_\Omega u u_1 > \lambda \int_\Omega u u_1$$

and $\lambda < \lambda_1$. We have proved that the problem has no solution for $\lambda \geq \lambda_1$ and it is true for solutions from Sobolev spaces, as well.

Theorem 43. *If $f : \mathbb{R} \to \mathbb{R}_+$ is continuous and decreasing function, then*

$$-\Delta u = f(u), \qquad u|\partial\Omega = 0,$$

has the unique solution.

Proof. For this problem, we can work in the space of continuous function vanishing on the boundary $C_0(\bar{\Omega})$ with the supremum norm. We look for a fixed point of the integral operator

$$T(u)(x) := \int_\Omega G(x, y) f(u(y))\, dy$$

which acts on this space. We need the following classical estimates for the Green function (for $n = 2$, it is slightly different):

$$0 < G(x, y) < \frac{c_0}{|x - y|^{n-2}}, \qquad |\nabla_x G(x, y)| \leq \frac{c_1}{|x - y|^{n-1}}.$$

The first bound gives $T : C_0(\bar{\Omega}) \to C_0(\bar{\Omega})$ and is continuous, both guarantee that for bounded $A \subset C_0(\bar{\Omega})$, $T(A)$ is equibounded and the set of their derivatives is bounded too. Hence, the Ascoli-Arzelá Theorem implies T is compact.

We shall apply the Leray-Schauder degree of $I - \lambda T$ with $\lambda \in [0, 1]$ on a ball $B(0, R)$ with sufficiently large radius, if we shall show a priori estimate on solution to $u = \lambda T(u)$. Since G and f takes nonnegative values, $u(x) \geq 0$. From the monotonicity of f,

$$u(x) \leq \lambda \int_\Omega G(x, y) f(0)\, dy \leq f(0) \sup_x \int_\Omega G(x, y)\, dy =: R.$$

Thus $\deg_{LS}(I - T, B(0, R), 0) = \deg_{LS}(I, B(0, R), 0) = 1$ and the problem has a solution.

If u, v are two solutions of the BVP, then $-\Delta(u - v) = f(u) - f(v)$ and after multiplying by $u - v$ and integrating, we get

$$\int_\Omega |\nabla(u - v)|^2 = \int_\Omega (f(u) - f(v)) \cdot (u - v) \leq 0.$$

Thus $u - v$ is constant and by the boundary condition $u - v \equiv 0$.

The are another boundary conditions for parabolic and hyperbolic equations, usually initial w.r.t. variable t and Dirichlet or Neumann w.r.t. spatial variable x. Most questions consider asymptotic behaviour of solutions when $t \to +\infty$. There is numerous applications of variational methods to BVPs defined for PDEs (compare [79]).

Chapter 13

Miscellaneous exercises

1. Prove *Massera's theorem:* Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous, T-periodic w.r.t. t and all initial value problems have unique solutions; then equation $x' = f(t, x)$ has a T-periodic solution iff it has a bounded on $\mathbb{R}$ solution. Hint. For bounded solution $\varphi : \mathbb{R} \to \mathbb{R}$ define the sequence $\varphi_n(t) := \varphi(t + nT)$. Then you have $\varphi_1(t) = \varphi(t)$ for some t or this sequence is monotonic thus convergent.
Apply this theorem to the case: there are $a < b$ such that $f(t, a) \geq 0 \geq f(t, b)$.

2. For the initial value problem

$$x' = \frac{1}{t^2 + x^2}, \qquad x(0) = x_0 \neq 0,$$

show that the solution exists on the whole line $\mathbb{R}$, it have finite limits as $t \to \pm\infty$. Find an estimate on the difference of the above limits by x_0.

3. Prove that BVP

$$x'' = -x^2, \qquad x(-c) = 0 = x(c),$$

admits exactly two solutions for any $c > 0$.

4. Let $f : [t_0, t_0 + \delta] \times \mathbb{R}^k$ be a continuous function. Divide the interval $[t_0, t_0 + \delta]$ by points

$$t_{m,j} := t_0 + \frac{j\delta}{m}, \qquad j = 0, 1, \ldots, m,$$

and define $\varphi_m, \psi_m : [t_0, t_0 + \delta] \to \mathbb{R}^k$ by formulas $\varphi_m(t_0) := x_0$,

$$\varphi_m(t) := \varphi_m(t_{m,j}) + (t - t_{m.j}) f(t_{m,j}, \varphi_m(t_{m,j})), \qquad t \in (t_{m,j}, t_{m,j+1}]$$

and next
$$\psi_m(t) := f(t_{j,m}, \varphi(t_{j,m})), \qquad t \in (t_{m,j}, t_{m,j+1}]$$
for $j = 0, 1, \dots m-1$. Prove that
$$\varphi_m(t) = x_0 + \int_{t_0}^{t} \psi_m(s)\, ds$$
and functions φ_m are equibounded and equicontinuous. It leads to another proof of the existence of solutions to the initial value problem
$$x' = f(t, x), \qquad x(t_0) = x_0$$
without using the Schauder Fixed Point Theorem. The sequence given by the above procedure is known in literature as the *Tonelli approximation*.

5. Let $f : [0, \infty) \times X \to X$ be a continuous mapping, X – a Banach space, and there exists an integrable function $h : [0, \infty) \to \mathbb{R}$ such that
$$|f(t, x) - f(t, y)| \leq h(t)|x - y|, \qquad t \in [0, \infty), \quad x, y \in X.$$
Prove the existence of the unique solution to the initial value problem
$$x' = f(t, x), \qquad x(0) = x_0,$$
provided that function $[0, \infty) \ni t \mapsto f(t, 0) \in X$ is integrable.

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