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## **DYNAMIC MODELLING OF THIN PLATES MADE OF LONGITUDINALLY FUNCTIONALLY GRADED MATERIAL**

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*Subject of this paper are thin plates with characteristic geometry: periodic in one direction and smoothly varying along another. The aim of the contribution is to formulate and investigate an averaged model describing the vibrations of this plate. Modelling procedure is based on the tolerance averaging technique. We are to analyze the plate in rectangular and polar coordinate systems.*

### **1. Introduction**

#### **1.1. Subject of the consideration**

The subject of this paper is a thin plate made of the two-phase functionally graded composite. The plate has periodically inhomogeneous microstructure which is slowly varying in space: the structure is 1-periodic along  $x_1$  coordinate, but smoothly graded apparent (averaged) properties in the perpendicular direction of the  $x_1$ , along  $x_2$  axis. (Fig. 1 and Fig. 2.)

#### **1.2. The aim of contribution**

We would like to derive and apply a deterministic macroscopic model describing the dynamic behaviour of the microheterogeneous plate made of two components.

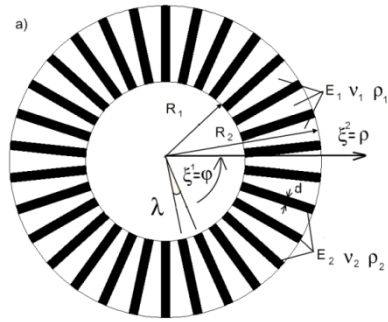


Fig. 1. Example of considered microheterogeneous plate in polar coordinates

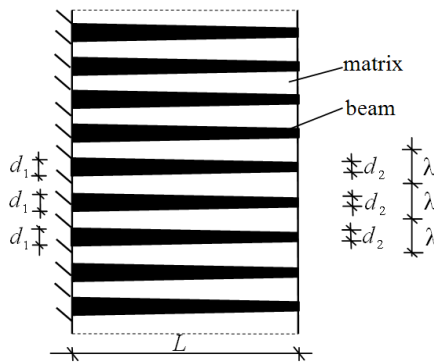


Fig. 2. Example of considered microheterogeneous plate in rectangular coordinates

### 1.3. The assumptions

We shall assume that the generalized period  $\lambda$  is sufficiently small when compared to measure of the domain of argument  $x^1$ .

## 2. Modelling

### 2.1. Basic concepts of tolerance averaging technique

Subsequently we use tolerance averaging technique (TAT) for the modeling of dynamic behavior of thin plates. It was presented by Woźniak and Wierzbicki [9]. You can find many examples from TAT and bibliography in monograph [8]. Some operators and lemmas are formulated in this theory [5]. The most important of them will be rewritten:

- the averaging operator:

Let  $f \in TP_\delta^\alpha(\Omega, \square)$ . By the averaging of tolerance periodic function  $f \equiv \partial^0 f$  or its derivatives  $\partial^k f$ ,  $k = 1, 2, \dots, \alpha$ , we shall mean function  $\langle \partial^k f \rangle(x), x \in \overline{\Omega}$ , defined by:

$$\langle \partial^k f \rangle(x) \equiv \frac{1}{|\square|} \int_{\square(x)} \tilde{f}^{(k)}(x, z) dz, \quad k = 0, 1, \dots, \alpha, \quad x \in \overline{\Omega}, \quad (1)$$

where for the sake of simplicity we have denoted:  $\langle \partial^k f \rangle = \langle \partial^k f(\cdot) \rangle(x)$

- the micro-macro decomposition:

Let us assume that  $w \in H^1(\Omega \times \Xi)$ ,  $w = w(z, \xi)$ ,  $z \in \Omega, \xi \in \Xi$ . We shall deal with the integral functional:

$$A(w(\cdot)) = \int_{\Omega \times \Xi} L(z, \nabla w(z, \xi), w(z, \xi)) d\xi dz, \quad (2)$$

where  $L(z, \nabla w(z, \xi), w(z, \xi)) \in HO_\delta^\alpha(\Omega, \square)$ . It implies that for every  $x \in \overline{\Omega}$  there exists periodic approximation  $\tilde{L} = \tilde{L}(z, \nabla w, w)$  of integrand  $L(z, \nabla w, w)$ ,  $z \in \square(x)$ .

Let  $h = \{h^A(\cdot) \in HO_\delta^1(\Omega, \square), A = 1, \dots, N\}$  be a system of  $N$  linear independent functions which is assumed to be postulated *a priori* in every modelling problem under consideration. We shall assume that for every  $x \in \overline{\Omega}$  condition

$$\langle \rho h^A \rangle(x) = 0, \quad (3)$$

is satisfied for a certain given *a priori* positive function  $\rho \in TP_\delta^0(\Omega, \square)$ .

The fundamental assumption imposed on integrand  $L$  in the framework of the tolerance averaging approach is that field  $w$  in  $L$  will be assumed in the form:

$$w(z, \xi) = w_h(z, \xi) = u(z, \xi) + h^A(z) v_A(z, \xi), \quad A = 1, \dots, N, \quad (4)$$

Where summation over  $A = 1, \dots, N$  holds and  $u(\cdot, \xi), v_A(\cdot, \xi) \in SV_\delta^1(\Omega, \square)$ ,  $\bar{\nabla} u(\cdot, \xi), \bar{\nabla} v_A(\cdot, \xi) \in SV_\delta^0(\Omega, \square)$  and  $w(\cdot, \xi) \in TP_\delta^1(\Omega, \square)$  for every  $\xi \in \Xi$ .

Formula (4) will be referred to as *the micrio-macro decomposition*. Functions  $u, v_A, A = 1, \dots, N$  will play the role of the arguments of the averaged functional and are called *averaged (macroscopic) variable* and *fluctuation (microscopic) amplitudes*, respectively. The postulated *a priori* functions  $h^A(\cdot), A = 1, \dots, N$  are referred to as *fluctuation shape functions*.

## 2.2. Modelling procedure

Modelling procedure is based on the action functional in the form [6]:

$$A(w(\cdot)) = \int_{\Omega} \int_{\Xi} (L(z, \nabla w(z, \xi), w(z, \xi)) + p(z, \xi)w(z, \xi)) d\xi, dz \quad (5)$$

where  $L(z, \nabla w(z, \xi), w(z, \xi)) \in HO^0_\delta(\Omega, \square)$ ,  $p(z, \xi) \in HO^0_\delta(\Omega, \square)$ .

We obtain Euler-Lagrange equations in the form:

$$\nabla \cdot \frac{\partial L}{\partial \nabla w} - \frac{\partial L}{\partial w} = p, \quad (6)$$

Tolerance modelling procedure for equation (6) is realized in two steps. The first step is the tolerance averaging of action functional (5) by means of (1) under the micro-macro decomposition (4). It has to be emphasized that in this step function  $p$  is given a priori. The second step is to apply the extended principle of stationary action. Using this procedure, from (1) and (5), we obtain averaged functional:

$$A(u(\cdot), v_A(\cdot), p(\cdot)) = \int_{\Omega} \int_{\Xi} (\langle L_h \rangle(x, \nabla u, u, \bar{\nabla} v_A, v_A) + \langle p \rangle(x)u + \langle \rho h^A \rangle(x)v_A) d\xi, dx, \quad (7)$$

where

$$\begin{aligned} \langle L_h \rangle(x, \nabla u, u, \bar{\nabla} v_A, v_A) &= \frac{1}{|\square|} \int_{\square(x)} \tilde{L} \left( \begin{array}{l} x, z, \nabla u(x, \xi) + \\ + \partial \tilde{h}^A(x, z) \bar{\nabla} v_A(x, \xi), u(x, \xi) + \\ + \tilde{h}^A(x, z) v_A(x, \xi) \end{array} \right) dz \\ \langle p \rangle(x) &= \frac{1}{|\square|} \int_{\square(x)} \tilde{p}(x, z, \xi) dz, \\ \langle \rho h^A \rangle(x) &= \frac{1}{|\square|} \int_{\square(x)} \tilde{h}^A(x, z) \tilde{p}(x, z, \xi) dz, \end{aligned}$$

for every  $\xi \in \Xi$ .

Applying the principle of stationary action to  $A_h$  we obtain the following system of equations:

$$\nabla \cdot \frac{\partial \langle L_h \rangle}{\partial \nabla u} - \frac{\partial \langle L_h \rangle}{\partial u} = \langle p \rangle, \quad (8)$$

$$\bar{\nabla} \cdot \frac{\partial \langle L_h \rangle}{\partial \bar{\nabla} v_A} - \frac{\partial \langle L_h \rangle}{\partial v_A} = \langle \rho h^A \rangle, \quad A=1, \dots, N, \quad (9)$$

We obtain the final system of equations for  $u = u(x, \xi), v_A = v_A(x, \xi), A=1, \dots, N$  as the new basic unknowns. The above equations represent *the tolerance model equations* under the micro-macro decomposition.

### 3. Dynamic analysis of fgm plate

#### 3.1. Direct description

As a basis of modeling procedure of FGM plate the following formulas are taken into account [4]:

- strain-displacements relations:

$$\kappa_{\alpha\beta} = -w_{|\alpha\beta}, \tag{10}$$

where:  $\kappa_{ab}$  is curvature,  $w$  is displacement field.

- constitutive equations:

$$m^{\alpha\beta} = BH^{\alpha\beta\gamma\delta} \kappa_{\gamma\delta}, \tag{11}$$

where:

$$H^{\alpha\beta\gamma\delta} = \frac{1}{2} \left\{ g^{\alpha\mu} g^{\beta\gamma} + g^{\alpha\gamma} g^{\beta\mu} - \right. \\ \left. + \nu \left( \epsilon^{\alpha\gamma} \epsilon^{\beta\mu} + \epsilon^{\alpha\mu} \epsilon^{\beta\gamma} \right) \right\}, \tag{12}$$

$$B = \frac{E\delta^3}{12(1-\nu^2)}, \tag{13}$$

where:

$E$  = Young module,  $\delta$  = thickness of plate,  $\nu$  = Poisson number,  $\epsilon^{ij}$  = component of Ricci tensor,  $g$  = component of contravariant metric tensor.

These equations have highly oscillating coefficients, and hence they are difficult to solve.

#### 3.2. Averaging description

The density of elastic and kinetic energy we write as a functional in the form:

$$L_{\Pi} = -\frac{1}{2} BH^{\alpha\beta\gamma\mu} w_{|\alpha\beta} w_{|\gamma\mu} + \frac{1}{2} \mu \dot{w} \ddot{w} \tag{14}$$

Taking into account micro-macro decomposition (4) and using averaging operator (1) we obtain averaging density of elastic and kinetic energy as follows:

$$\begin{aligned}
\langle L_{\Pi} \rangle = & -\frac{1}{2} \langle BH^{\alpha\beta\gamma\mu} \rangle u_{|\alpha\beta} u_{|\gamma\mu} - \langle BH^{11\gamma\mu} h_{|11}^A \rangle v_A u_{|\gamma\mu} - 2 \langle BH^{12\gamma\mu} h_{|11}^A \rangle v_{A|2} u_{|\gamma\mu} + \\
& - \langle BH^{22\gamma\mu} h^A \rangle v_{A|22} u_{|\gamma\mu} - \frac{1}{2} \langle BH^{1111} h_{|11}^A h_{|11}^A \rangle v_A v_A - \langle BH^{1122} h_{|11}^A h^A \rangle v_A v_{A|22} + \\
& - 2 \langle BH^{1212} h_{|11}^A h_{|11}^A \rangle v_{A|2} v_{A|2} - \frac{1}{2} \langle BH^{2222} h^A h^A \rangle v_{A|22} v_{A|22} + \\
& + \frac{1}{2} \langle \mu \rangle \ddot{u}\ddot{u} + \langle \mu h^A \rangle \ddot{v}_A \ddot{u} + \frac{1}{2} \langle \mu h^A h^A \rangle \ddot{v}_A \ddot{v}_A
\end{aligned} \tag{15}$$

Subsequently, we shall use the tolerance averaging procedure and hence we obtain the following system of equations:

$$\begin{aligned}
& \langle BH^{\alpha\beta\gamma\delta} w_{|\gamma\delta}^0 \rangle_{|\alpha\beta} + \langle BH^{\alpha\beta 11} q_{|11}^A \rangle V_A_{|\alpha\beta} + \langle BH^{\alpha\beta 22} q^A \rangle V_{A|22}_{|\alpha\beta} + \langle \rho \dot{w} \rangle = \langle p \rangle \\
& \langle q^A_{|11} BH^{11\gamma\delta} \rangle w_{|\gamma\delta}^0 + \langle q^A_{|11} BH^{1111} q_{|11}^B \rangle V_B + \langle q^A_{|11} BH^{1122} q^B \rangle V_{B|22} + \\
& + \langle q^A_{|11} BH^{22\gamma\delta} \rangle w_{|\gamma\delta}^0_{|22} + \langle q^A_{|11} BH^{2211} q_{|11}^B \rangle V_B_{|22} + \langle q^A_{|11} BH^{2222} q^B \rangle V_{B|22}_{|22} + \\
& + \langle q^A \rho q^B \rangle \ddot{V}_B = \langle q^A p \rangle
\end{aligned} \tag{16ab}$$

The above system consists of  $N+I$  differential equations. Coefficients in the above system are continuous and slowly-varying functions.

## 4. Equations in rectangular and polar coordinates

### 4.1. Equations for band plate

For example we show equations for band plate in rectangular coordinates and for annular plate in polar coordinates. After simple manipulation we obtain from equations (16ab) the following system of two differential equations describing dynamic behavior of the plate band:

$$\begin{aligned}
& \partial_{22} \langle B^{2222} \rangle \partial_{22} w^0 + \langle B^{2211} q_{|11} \rangle V + \langle B^{2222} q \rangle \partial_{22} V + \langle \mu \rangle \dot{w}^0 = 0, \\
& \partial_{22} \langle B^{2222} q \rangle \partial_{22} w^0 + \langle B^{1122} q \rangle \partial_{11} q + V + \langle B^{2222} qq \rangle \partial_{22} V + \\
& - 4 \partial_2 \langle B^{1212} \partial_1 q \partial_1 q \rangle \partial_2 V + \langle B^{1122} \partial_{11} q \rangle \partial_{22} w^0 + \langle B^{1122} \partial_{11} qq \rangle \partial_{22} V + \\
& + \langle B^{1111} \partial_{11} q \partial_{11} q \rangle V + \langle \mu qq \rangle \dot{V} = 0
\end{aligned} \tag{17}$$

Equations (17) represent a system of two partial differential equations for the averaged deflection  $w^0(\cdot, t)$  and fluctuation amplitude  $V(\cdot, t)$ .

Substituting into (17) we obtain equations for  $\tilde{w}^0(x_2)$  and  $\tilde{V}(x_2)$ .

$$\begin{aligned}
 w^0(x_2, t) &= \tilde{w}^0(x_2) e^{i\omega t} & V(x_2, t) &= \tilde{V}(x_2) e^{i\omega t}. & (18) \\
 \partial_{22}(\langle B^{2222} \rangle \partial_{22} w^0 + \langle B^{2211} g_{|11} \rangle V + \langle B^{2222} q \rangle \partial_{22} V) + \langle \mu \rangle \omega^2 \tilde{w}^0 &= 0, \\
 \partial_{22}(\langle B^{2222} q \rangle \partial_{22} w^0 + \langle B^{1122} q \rangle \partial_{11} q + V + \langle B^{2222} g g \rangle \partial_{22} V) + \\
 -4 \partial_2(\langle B^{1212} \partial_1 q \partial_1 q + \partial_2 V) + \langle B^{1122} \partial_{11} q + \partial_{22} w^0 + \langle B^{1122} \partial_{11} q q + \partial_{22} V + \\
 + \langle B^{1111} \partial_{11} q \partial_{11} q + V + \langle \mu q q \rangle \omega^2 \tilde{V} &= 0 & (19)
 \end{aligned}$$

Since  $g(\cdot) \in O(\lambda^2)$ , the inertial module  $\langle \mu q q \rangle$  and the underlined terms depend on the microstructure length parameter  $\lambda$ , hence aforementioned equations describe the microstructure length-scale effect on the natural frequencies of the plate under consideration.

### 4.2. Equations for annular plate

Let us consider the following polar coordinates: one circular coordinate  $\xi_1$  in angle measure and another radial coordinate  $\xi_2$  in linear measure. Model equations in these coordinates are more complicated than those written in Cartesian coordinates. Mathematical derivation following equations can be found in [5]:

$$\begin{aligned}
 w^0_{,112} \left( 2 \langle q^A B H^{2211} \rangle_{,2} \right) + w^0_{,1122} \left( \langle q^A B H^{2211} \rangle \right) + w^0_{,11} \left( \langle q^A_{|11} B H^{1111} \rangle \right) + \\
 \left( \langle q^A_{|11} B H^{2211} \rangle_{,22} \right) + \\
 + w^0_{,22} \left( \langle q^A_{|11} B H^{1122} \rangle + 2 \rho \langle q^A B H^{2211} \rangle_{,2} \right) + w^0_{,2222} \left( \langle q^A B H^{2222} \rangle_{,2222} \right) + \\
 \left( 2 \langle q^A B H^{2211} \rangle + \langle q^A B H^{2222} \rangle_{,22} \right) + \\
 w^0_{,222} \left( \rho \langle q^A B H^{2211} \rangle \right) + w^0_{,2} \left( \rho \langle q^A_{|11} B H^{1111} \rangle + \rho \langle q^A B H^{2211} \rangle_{,22} + 2 \langle q^A B H^{2211} \rangle_{,2} \right) + \\
 + V_B \left( \langle q^A_{|11} B H^{1111} q^B_{|11} \rangle + \langle q^A B H^{2211} q^B_{|11} \rangle_{,22} \right) + V_{B,2} \left( 2 \langle q^A B H^{2211} q^B_{|11} \rangle_{,2} \right) + \\
 + V_{B,22} \left( \langle q^A_{|11} B H^{1122} q^B \rangle + \langle q^A B H^{2211} q^B_{|11} \rangle + \langle q^A B H^{2222} q^B \rangle_{,22} \right) + \\
 + V_{B,222} \left( 2 \langle q^A B H^{2222} q^B \rangle_{,2} \right) + V_{B,2222} \left( \langle q^A B H^{2222} q^B \rangle \right) + \langle q^A \rho q^B \rangle V_B = \langle q^A \rho \rangle
 \end{aligned}$$

$$\begin{aligned}
& w^0_{,1111} \left( \langle BH^{1111} \rangle \right) + w^0_{,112} \left( \frac{4}{\rho} \langle BH^{1212} \rangle + \frac{2}{\rho} \langle BH^{2211} \rangle + 4 \langle BH^{1212} \rangle_{,2} + 2 \langle BH^{2211} \rangle_{,2} \right) + \\
& + w^0_{,1122} \left( 2 \langle BH^{2211} \rangle + 4 \langle BH^{1212} \rangle \right) + w^0_{,11} \left( \begin{aligned} & \left( -\frac{4}{\rho^2} \langle BH^{1212} \rangle - 2 \langle BH^{1111} \rangle - \frac{4}{\rho} \langle BH^{1212} \rangle_{,2} + \right. \\ & \left. + \langle BH^{2211} \rangle_{,22} + \frac{2}{\rho} \langle BH^{1122} \rangle_{,2} - \rho \langle BH^{1111} \rangle_{,2} \right) \end{aligned} \right) + \\
& + w^0_{,22} \left( 2 \langle BH^{2211} \rangle - \rho^2 \langle BH^{1111} \rangle + \rho \langle BH^{2211} \rangle_{,2} + \langle BH^{2222} \rangle_{,22} + \frac{1}{\rho} \langle BH^{2222} \rangle_{,2} \right) + \\
& + w^0_{,2222} \left( \langle BH^{2222} \rangle \right) + w^0_{,222} \left( \frac{2}{\rho} \langle BH^{2222} \rangle + 2 \langle BH^{2222} \rangle_{,2} \right) + \\
& + w^0_{,2} \left( \begin{aligned} & \left( -3\rho \langle BH^{1111} \rangle + \frac{2}{\rho} \langle BH^{2211} \rangle + \rho \langle BH^{2211} \rangle_{,22} + \right. \\ & \left. + 4 \langle BH^{2211} \rangle_{,2} - \rho^2 \langle BH^{1111} \rangle_{,2} + \frac{1}{\rho} \langle BH^{2222} \rangle_{,2} \right) \end{aligned} \right) + \\
& + V_{A,11} \left( \langle BH^{1111} q^A \rangle \right) + \\
& + V_A \left( \langle BH^{2211} q^A \rangle_{,22} + \frac{2}{\rho} \langle BH^{2211} q^A \rangle_{,2} - \rho \langle BH^{1111} q^A \rangle_{,2} - 2 \langle BH^{1111} q^A \rangle \right) + \\
& + V_{A,2} \left( 2 \langle BH^{2211} q^A \rangle_{,2} + \frac{2}{\rho} \langle BH^{2211} q^A \rangle - \rho \langle BH^{1111} q^A \rangle \right) + \\
& + V_{A,22} \left( \begin{aligned} & \left( \langle BH^{2211} q^A \rangle + \langle BH^{2222} q^A \rangle_{,22} - \frac{1}{\rho^2} \langle BH^{2222} q^A \rangle + \right. \\ & \left. \frac{2}{\rho} \langle BH^{2222} q^A \rangle_{,2} - \rho \langle BH^{1122} q^A \rangle_{,2} - 2 \langle BH^{1122} q^A \rangle \right) \end{aligned} \right) + \\
& + V_{A,1122} \left( \langle BH^{2211} q^A \rangle \right) + V_{A,222} \left( 2 \langle BH^{2222} q^A \rangle_{,2} + \frac{2}{\rho} \langle BH^{2222} q^A \rangle - \rho \langle BH^{1122} q^A \rangle \right) + \\
& + V_{A,2222} \left( \langle BH^{2222} q^A \rangle \right) + \langle \rho \rangle \ddot{w} = \langle p \rangle
\end{aligned}$$

(20)

where:

$$H^{1111} = \frac{1}{(\xi_2)^4}, H^{2222} = 1, H^{1122} = H^{2211} = \frac{\nu}{(\xi_2)^2}, H^{1212} = H^{2121} = H^{2112} = H^{1221} = \frac{1-\nu}{2(\xi_2)^2}$$



## 5. Conclusions

After modeling and analysis received results we can formulate some conclusions:

- the tolerance averaging technique can be successfully applied to formulate averaging model of dynamic behavior of composite plates with functionally graded material,
- the obtained model is described by equations with functional but smooth coefficients in contrast to direct description by equations with non-continuous and highly oscillating coefficients.

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# **MODELOWANIE DYNAMIKI PŁYT CIENKICH WYKONANYCH Z MATERIAŁÓW GRADIENTOWYCH O PODŁUŻNIE ZMIENNYCH WŁAŚCIWOŚCIACH**

## **Streszczenie**

Przedmiotem pracy są płyty cienkie o określonej geometrii: periodyczne w jednym kierunku i wolnozmiennie w drugim. Celem opracowania jest sformułowanie uśrednionego modelu opisującego drgania takiej płyty. Procedura modelowania jest oparta na technice tolerancyjnego uśrednienia. Analizowane płyty są w biegunowym i kartezjańskim układzie współrzędnych.