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# UNIDIRECTIONAL AND STATIONARY HEAT CONDUCTION PROBLEM IN TWO-PHASE HOLLOW CYLINDER WITH FUNCTIONALLY GRADED AND TEMPERATURE DEPENDENT EFFECTIVE MATERIAL PROPERTIES

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*The unidirectional stationary heat conduction in two-phase hollow cylinder is considered. The conductor is made of two-phase stratified composites and has a smooth gradation of effective properties in the radial direction. Therefore, we deal here with a special case of functionally graded materials, FGM (cf. [6]). Moreover, these properties are isotropic but temperature dependent. The formulation of mathematical model of the conductor is based on the tolerance averaging approach (TAA), cf. [9]. Applications to the stationary heat conduction with φ-constant temperature on the boundaries will be shown. The effect of the nonlinearity parameter, fibres shape and quantity on the temperature field will be examined. Additionally, the differences in temperature field for tolerance and asymptotic model will be considered.*

## 1. Basic concepts

### 1.1. Subject of contribution

Let us introduce orthogonal curvilinear coordinate system $O\rho\varphi z$ in the physical space $\Omega$ occupied by the two-phase hollow cylinder (Fig. 1a), which has invariable structure and constant material properties in the infinite $z$-direction. The main aim of this paper is to consider the stationary heat transfer problem in this composite, where material properties for both phases are temperature dependent. The deterministic microstructure of the two-component conductor under consideration is, for a fixed radius, periodic along the angular axis and has smooth and functional effective material properties in the radial direction (Fig. 1b). Therefore, we deal here with a special case of functionally graded materials, FGM, cf. [6].

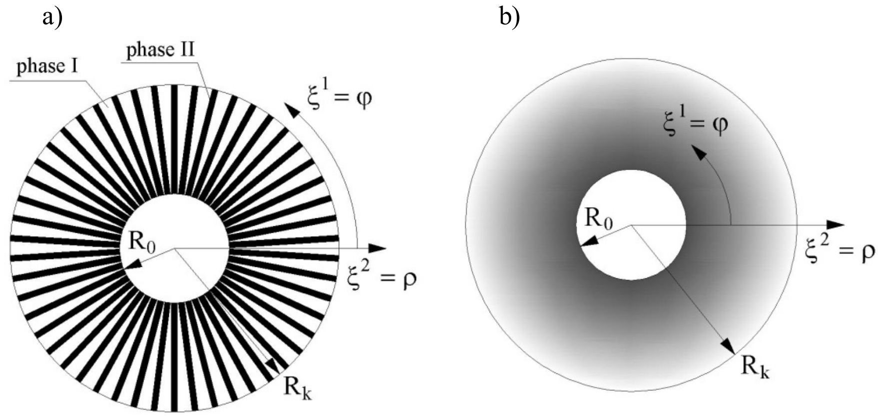


Fig. 1. Structure of the two-phase functionally graded composite in a) micro- and b) macro-scale

The analysis of the heat transfer in the hollow cylinder made from functionally graded materials can be found in [2], [5], where material properties are expressed as power or exponential functions of the radial coordinate. In paper [1] we can find application of higher-order theory for thermal analysis in functionally graded materials.

The formulation of the macroscopic mathematical model for the analysis of heat conduction in the conductor under consideration will be based on the tolerance averaging technique, cf. [8-9]. The general description of this technique and application to analysis of longitudinally graded stratified media can be found in [4], [7-8].

## 1.2. Subject of contribution

The physical phenomenon of the stationary heat transfer is described by the well known Fourier's law equation

$$\nabla \cdot (\boldsymbol{K} \cdot \nabla \Theta) = 0 , \tag{1.1}$$

which contains (in this case) highly oscillating and discontinuous coefficients of $K$ – heat conduction tensor. Additionally, these properties are temperature dependent and assumed in the polynomial form

$$\boldsymbol{K}(\Theta) = \boldsymbol{k}_0 + \boldsymbol{k}_1 \cdot \Theta , \tag{1.2}$$

where $\boldsymbol{k}_0, \boldsymbol{k}_1$ are in general arbitrary positive real number tensors of conductivity.

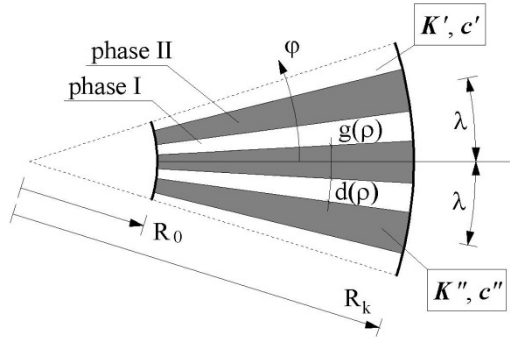


Fig. 2. Deterministic microstructure of composite in the $O\rho\varphi$ plane

The object of our considerations is a hollow conductor with a microstructure given on Fig.2. This microstructure is determined by the unit cell Δ with the diameter of $\lambda = 2\pi / N$, where $N$ is a number of cells in considered composite. Every two-component cell area consists of the so called 'fibre' and 'matrice', where the fibres width $g$ is in general given by

$$g(\rho) = g_0 + (g_k - g_0)\cdot\zeta(\rho),\ \zeta(\rho) = \frac{\rho - R_0}{R_k - R_0}, \tag{1.3}$$

where $g_0 = \lambda\omega_0 R_0$, $g_k = \lambda\omega_k R_k$ for $\omega_0, \omega_k \in (0,1)$, $\rho \in [R_0, R_k]$. Volume fractions of homogeneous layers are denoted by $\nu'(\rho) = d(\rho)/\lambda\rho$ and $\nu''(\rho) = g(\rho)/\lambda\rho$, where $g(\rho) + d(\rho) = \lambda\rho$. Dimensionless function $\nu = \sqrt{\nu'\nu''}$ is referred to as the distribution of heterogeneity.

The one of the fundamental assumptions in tolerance averaging approach concerns the temperature field decomposition

$$\Theta(\rho,\varphi) = \theta(\rho,\varphi) + h(\rho,\varphi)\cdot\psi(\rho,\varphi), \tag{1.4}$$

where $\varphi \in [0,2\pi)$ and $\rho \in [R_0, R_k]$. Functions of averaged temperature $\theta$ and oscillation amplitude temperature $\psi$ are assumed to be slowly varying, i.e. $\theta(\rho,\cdot), \psi(\rho,\cdot) \in SV_\delta^1(\Omega,\Delta)$. The exact definition of the *slowly varying* and *tolerance periodic* function can be found in [8-9]. The expected form of the temperature oscillations, caused by discontinuity of the coefficients in (1.1), is assured by the "saw-type" *locally periodic* function (Fig. 3), which would be called the *fluctuation shape function h*

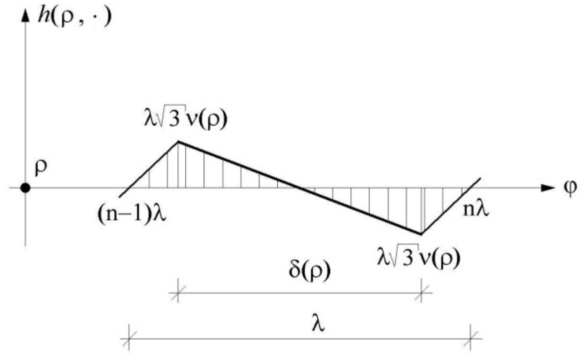


Fig. 3. Fluctuation shape function

where $\delta(\rho)=d(\rho)/\rho$.

The second concept of the modelling technique is the averaging operation

$$\langle f\rangle(\rho,\varphi)=\frac{1}{|\Delta|}\int_{\varphi-\lambda/2}^{\varphi+\lambda/2} f(\rho,z)dz\,, \tag{1.5}$$

where $|\Delta| = \lambda$. On the grounds of this definition we can formulate the second modelling assumption, the tolerance averaging approximation. In the course of modelling it is assumed that terms $O(\varepsilon)$ are negligibly small, where $\varepsilon$ is a certain tolerance parameter, cf. [8]. For the arbitrary tolerance periodic function $f\in TP_\varepsilon^1(\Omega,\Delta)$, slowly varying function $F\in SV_\varepsilon^1(\Omega,\Delta)$ and fluctuation shape function $h\in FS_\varepsilon^1(\Omega,\Delta)$ we have

$$\begin{bmatrix}\langle fF\rangle=\langle f\rangle F+O(\varepsilon)\\ \langle f\nabla(hF)\rangle=\langle f\partial h\rangle F+\langle fh\rangle\overline{\nabla}F+O(\varepsilon)\end{bmatrix}. \tag{1.6}$$

Bearing in mind the mean value definition (1.5) and all model assumptions, we conclude to the system of averaged equations (cf. [8]):

$$\begin{cases}\nabla\cdot\left(\langle \boldsymbol{k}_0\rangle\nabla\theta+\langle \boldsymbol{k}_0\partial h\rangle\psi+\langle \boldsymbol{k}_1\rangle\theta\nabla\theta+\langle \boldsymbol{k}_1\partial h\rangle\theta\psi+\left\langle \boldsymbol{k}_1 h^2\right\rangle\psi\,\overline{\nabla}\,\psi\right)=0\\ \overline{\nabla}\cdot\left(\left\langle \boldsymbol{k}_0 h^2\right\rangle\overline{\nabla}\,\psi+\left\langle \boldsymbol{k}_1 h^2\right\rangle\left(\theta\,\overline{\nabla}\,\psi+\psi\,\nabla\theta\right)+\left\langle \boldsymbol{k}_1 h^2\partial h\right\rangle\psi^2\right)-\langle \boldsymbol{k}_0\partial h\rangle\nabla\theta-\left\langle \boldsymbol{k}_0\partial h^2\right\rangle\psi+\\ \quad-\langle \boldsymbol{k}_1\partial h\rangle\theta\,\nabla\theta-\left\langle \boldsymbol{k}_1\partial h^2\right\rangle\theta\psi-\left\langle \boldsymbol{k}_1 h^2\partial h\right\rangle\psi\,\overline{\nabla}\,\psi=0\end{cases}, \tag{1.7}$$

describing two dimensional heat conduction in two-phase hollow cylinder, where the coefficients

$$\begin{aligned}&\text{a)}\quad \langle \boldsymbol{k}\rangle=\boldsymbol{k}'\nu'+\boldsymbol{k}''\nu''\,,\ \langle \boldsymbol{k}\partial h\rangle=2\nu\sqrt{3}\left(\boldsymbol{k}'-\boldsymbol{k}''\right),\ \left\langle \boldsymbol{k}\partial h^2\right\rangle=12\left(\boldsymbol{k}'\nu''+\boldsymbol{k}''\nu'\right),\\ &\text{b)}\quad \left\langle \boldsymbol{k}h^2\right\rangle=\lambda^2\nu^2\langle \boldsymbol{k}\rangle\,,\ \left\langle \boldsymbol{k}h^2\partial h\right\rangle=\lambda^2\nu^2\langle \boldsymbol{k}\partial h\rangle\,,\end{aligned} \tag{1.8}$$

are continues and functional but temperature dependent also. The gradient operators in the above equations have the form

$$\nabla=\left(\frac{\partial}{\partial\xi^{1}},\frac{\partial}{\partial\xi^{2}}\right),\ \partial=\left(\frac{\partial}{\partial\xi^{1}},0\right),\ \overline{\nabla}=\left(0,\frac{\partial}{\partial\xi^{2}}\right). \tag{1.9}$$

The obtained averaged differential equations (1.7), called *tolerance model* in further part of this paper, have smooth functional coefficients in contrast to the coefficients in equation (1.1). To obtain the results, numerical methods in most cases had to be used. This model takes into account an effect of the microstructure size on the overall heat transfer behaviour.

It is easy to notice, that coefficients (1.8b) depend explicitly on diameter $\lambda$. Hence, through the smooth passage with $\lambda$ to 0 we obtain the *asymptotic model* equations:

$$\begin{cases}\nabla\cdot\left(\langle\boldsymbol{k}_0\rangle\nabla\theta+\langle\boldsymbol{k}_0\partial h\rangle\psi+\langle\boldsymbol{k}_1\rangle\theta\nabla\theta+\langle\boldsymbol{k}_1\partial h\rangle\theta\psi\right)=0\\ -\langle\boldsymbol{k}_0\partial h\rangle\nabla\theta-\left\langle\boldsymbol{k}_0\partial h^2\right\rangle\psi-\langle\boldsymbol{k}_1\partial h\rangle\theta\nabla\theta-\left\langle\boldsymbol{k}_1\partial h^2\right\rangle\theta\psi=0\end{cases}, \tag{1.10}$$

Moreover, in this model the amplitude oscillation temperature $\psi$ strictly depends on the averaged temperature $\theta$. Therefore, we obtain

$$\nabla\cdot\left(\left(\langle\boldsymbol{k}_0\rangle+\langle\boldsymbol{k}_1\rangle\theta-\frac{\left(\langle\boldsymbol{k}_0\partial h\rangle+\langle\boldsymbol{k}_1\partial h\rangle\theta\right)^2}{\left\langle\boldsymbol{k}_0\partial h^2\right\rangle+\left\langle\boldsymbol{k}_1\partial h^2\right\rangle\theta}\right)\nabla\theta\right)=0 \text{ and}$$

$$\psi=-\frac{\langle\boldsymbol{k}_0\partial h\rangle+\langle\boldsymbol{k}_1\partial h\rangle\theta}{\left\langle\boldsymbol{k}_0\partial h^2\right\rangle+\left\langle\boldsymbol{k}_1\partial h^2\right\rangle\theta}\nabla\theta\ . \tag{1.11}$$

## 2. Special cases

Through the various assumptions and modifications of geometry or material properties equations (1.7) and (1.11) can take a particular form. Sometimes, there would analytical solution to these equations possible to find. In general, each of component of considered composite has a priori anisotropic properties, i.e.

$$\boldsymbol{k}_i=k_i\cdot\begin{bmatrix}1 & \alpha_i\\ \alpha_i & 1\end{bmatrix},\ \alpha_i\in[0,1), \tag{2.1}$$

for $i$ = 1, 2. Let us consider the following special cases.

## 2.1. Isotropic (orthotropic) material properties

If we assume $\alpha_i = 0$ for $i$ = 1, 2 in (2.1) for each of components, some of the coefficients (1.8) vanish instantly. Hence, the tolerance model

$$\begin{cases} \nabla \cdot \left( \langle \boldsymbol{k}_0 \rangle \nabla \theta + \langle \boldsymbol{k}_1 \rangle \theta \nabla \theta + \left\langle \boldsymbol{k}_1 h^2 \right\rangle \psi \, \overline{\nabla} \psi \right) = 0 \\ \overline{\nabla} \cdot \left( \left\langle \boldsymbol{k}_0 h^2 \right\rangle \overline{\nabla} \psi + \left\langle \boldsymbol{k}_1 h^2 \right\rangle \left( \theta \, \overline{\nabla} \psi + \psi \nabla \theta \right) \right) - \left\langle \boldsymbol{k}_0 \partial h^2 \right\rangle \psi - \left\langle \boldsymbol{k}_1 \partial h^2 \right\rangle \theta \psi = 0 \end{cases}, \tag{2.2}$$

and asymptotic model

$$\nabla \cdot \left( \left( \langle \boldsymbol{k}_0 \rangle + \langle \boldsymbol{k}_1 \rangle \theta \right) \nabla \theta \right) = 0 \;\text{ and }\; \psi = 0 . \tag{2.3}$$

The obtained equations are far more easier to solve then the original. Moreover, in polar coordinate system $O\rho\varphi$ there exist analytical solution to the asymptotic model of the unidirectional heat transfer problem

$$\theta = \frac{\langle \boldsymbol{k}_0 \rangle}{\langle \boldsymbol{k}_1 \rangle} \left( \sqrt{1 + \frac{2\langle \boldsymbol{k}_1 \rangle}{\langle \boldsymbol{k}_0 \rangle} \left( A \cdot ln\left( \rho \langle \boldsymbol{k}_0 \rangle \right) + B \right)} - 1 \right) \text{ and } \psi = 0 , \tag{2.4}$$

where $A$, $B$ are constant values to evaluate from the boundary conditions. When the number of cells in considered conductor is sufficiently large and in some special cases, asymptotic model should completely suffice in describing the heat transfer. Therefore, it would be easier to use solution (2.4) rather then compute it from (2.2).

## 2.2. Fully periodic microstructure

As mentioned in the introduction part, the microstructure of considered composite is periodic but only for a fixed radius. Assuming $\omega_k = \omega_0$ in (1.3) we come to the case where the microstructure is periodic for every radius, i.e. $\delta(\rho) = const$ for every $\rho \in [R_0, R_k]$. The coefficients (1.8) are no longer functional but became constant along radial direction. The equations of tolerance model and asymptotic model looks the same as (1.7) and (1.11), respectively.

## 2.3. Linear heat transfer problem

Since the material properties are expressed as in (1.2), the non-linear heat conduction problem is considered. However, if we come with $k_1$ in (2.1) to 0, for each of components we conclude to the linear problem. The obtained this way equations and analytical solutions for tolerance and asymptotic model are the same as those derived in [7].

# 3. Examples of application

The main aim of this chapter is to display mostly an effect of the nonlinearity on the temperature field. These considerations concern with the unidirectional heat transfer for a two-phase conductor with deterministic microstructure (Fig. 2) for geometric values of $R_0 = 1\,[m]$ and $R_k = 2\,[m]$. In following examples we shall assume isotropic properties ($\alpha_i = 0$, $i$ = 1,2) and

$$\boldsymbol{k}_1 = \eta \cdot \boldsymbol{k}_0\,,\ \eta \geq 0\,, \tag{3.1}$$

for each of components, where fixed values of conductivity are listed below.

Table 1. Material properties

| | phase I | phase II |
|---|---|---|
| $k_0\,\left[Wm^{-1}K^{-1}\right]$ | 58 | 0.045 |

Bearing in mind (3.1) we conclude to the system equations for tolerance model

$$\begin{cases} \nabla \cdot \left( \langle \boldsymbol{k}_0 \rangle \nabla \theta (1+\eta\theta) + \eta \left\langle \boldsymbol{k}_0 h^2 \right\rangle \psi\, \overline{\nabla}\, \psi \right) = 0 \\ \overline{\nabla} \cdot \left( \left\langle \boldsymbol{k}_0 h^2 \right\rangle \overline{\nabla}\, \psi (1+\eta\theta) + \eta \left\langle \boldsymbol{k}_0 h^2 \right\rangle \psi\, \nabla \theta \right) - \left\langle \boldsymbol{k}_0 \partial h^2 \right\rangle \psi (1+\eta\theta) = 0 \end{cases}, \tag{3.2}$$

and asymptotic model

$$\nabla \cdot \left( \langle \boldsymbol{k}_0 \rangle (1+\eta\theta) \nabla \theta \right) = 0 \ \text{ and } \ \psi = 0\,. \tag{3.3}$$

Because of the decomposition (1.4), twice more boundary conditions are needed as distinct from deterministic problem approach. Hence, for the first kind of boundary conditions (temperature Θ given on the boundary) we denote:

$$\begin{cases} \Theta\big|_{\rho=R_0} = \Theta_0 \\ \Theta\big|_{\rho=R_k} = \Theta_k \end{cases} \Rightarrow \begin{cases} \theta\big|_{\rho=R_0} = \theta_0 = \langle \Theta_0 \rangle\big|_{\rho=R_0} \\ \theta\big|_{\rho=R_k} = \theta_k = \langle \Theta_k \rangle\big|_{\rho=R_k} \end{cases} \text{ and } \begin{cases} \psi\big|_{\rho=R_0} = \psi_0 = \langle \Theta_0 h \rangle\big|_{\rho=R_0} \\ \psi\big|_{\rho=R_k} = \psi_k = \langle \Theta_k h \rangle\big|_{\rho=R_k} \end{cases}. \tag{3.4}$$

It is important to remember, that if $F$ is a $\varphi$-constant function, then $\langle Fh \rangle = 0$. All above conditions and formulations will be used for all following examples in subsequent part of this paper.

### 3.1. Nonlinearity effect

Let us consider two-phase hollow cylinder (Fig. 1) under constant temperature, $\Theta_0 = 100\left[{}^0C\right]$ and $\Theta_k = 0\left[{}^0C\right]$ on the boundaries, which are equivalent to $\theta_0 = 100\left[{}^0C\right]$ and $\theta_k = \psi_0 = \psi_k = 0\left[{}^0C\right]$. The fibres width is expressed by (1.3) with $\omega_0 = 1/2$ and $\omega_k = 1/6$ (constant fibers width along $\rho$-axis). The obtained results

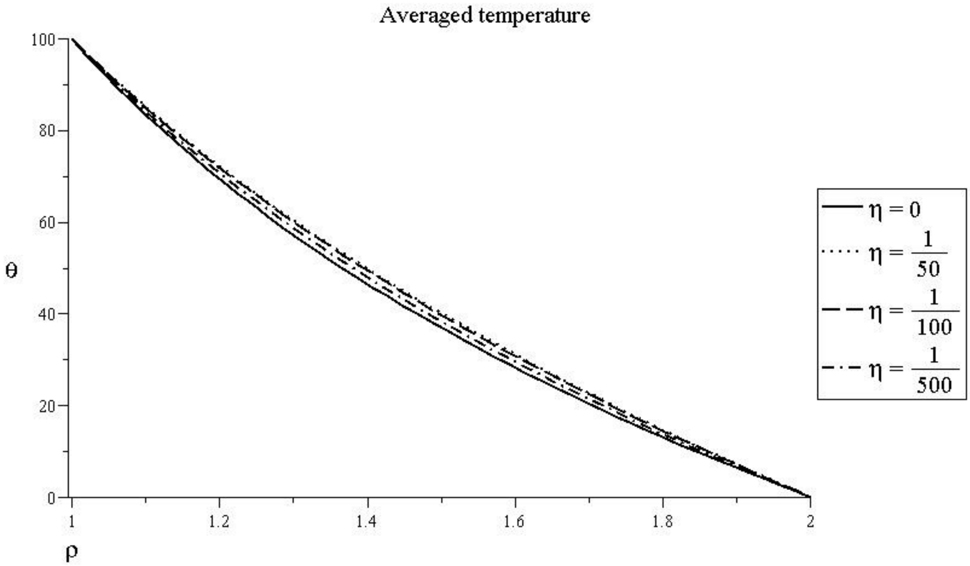


Fig. 4. Nonlinearity effect on the averaged temperature

was made for $N$ = 60 cells and material properties as in Table 1. There is no temperature oscillations in this case.

### 3.2. Fibres shape effect

Various shapes of fibres width, for $\omega_0 = 1/2$, are considered and their effect on the temperature field by $\eta = 0.01$ in (3.1). Let us consider three cases:

- $\omega_k = 1/6$ – constant fibres width along $\rho$-axis,
- $\omega_k = 1/2$ – periodic microstructure,
- $\omega_k = 5/6$ – constant matrices width along $\rho$-axis.

Boundary conditions and all other geometric characteristics are the same as in 3.1. The obtained results for averaged temperature:

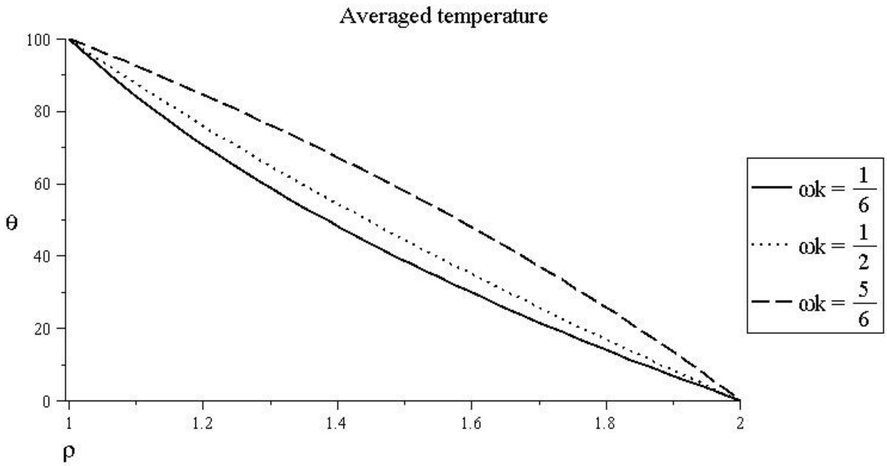


Fig. 5. Fibres width effect on the averaged temperature

and amplitude oscillation temperature is constant at zero.

## 3.3. Number of cells effect

The fibres width is assumed as $\rho$-constant ($\omega_0 = 1/2$, $\omega_k = 1/6$) and nonlinearity ratio $\eta = 0.01$ in (3.1). The rest of geometric characteristics and boundary conditions are the same as in example above. Under consideration in this example is only the microstructure size $\lambda$, which involves cells quantity $N$ ($\lambda = 2\pi / N$).Case of $''N = \infty''$ is described by equations (2.4) – asymptotic model.

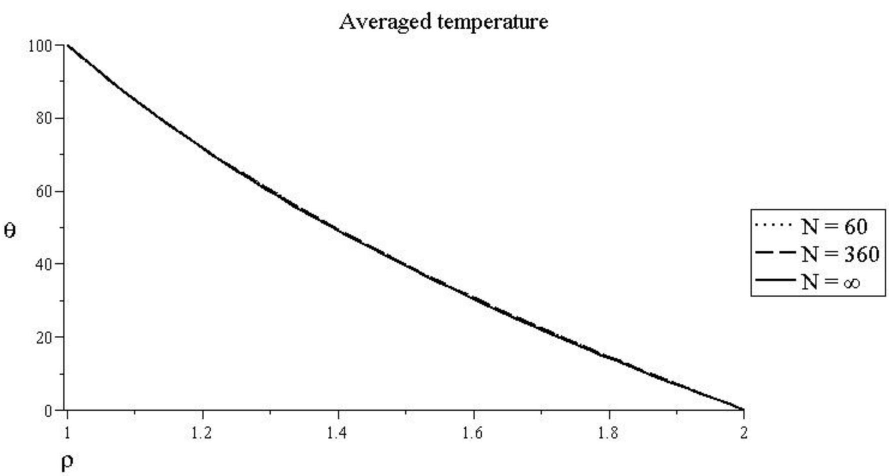


Fig. 6. Averaged temperature by various number of cells

The obtained results shows that there is no significant error in assuming asymptotic model instead of tolerance model for $N \geq 60$. On Fig. 6 all diagrams are overlayed.

## 4. Summary

The tolerance averaging approximation leads to the mathematical model of composites conductor with functionally graded material properties. The obtained model equations have continues coefficients in opposition to a discrete model, where they are strongly oscillating. Since the proposed model equations have smooth functional coefficients then in most cases solutions to specific problem, for heat conductor under consideration, have to be obtained using well known numerical methods. The tolerance model takes into account an effect of the microstructure size on the temperature field. Moreover, by changing fibres shape, we can obtain desirable temperature field inside composite. The comparison of tolerance and asymptotic model in case of isotropic properties reveals, that there is no particular loss of accuracy on temperature field values, if asymptotic model is used. Hence, more simple equation need to be solved.

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# JEDNOKIERUNKOWY STACJONARNYPRZEPŁYW CIEPŁA W DWUFAZOWYM PRZEWODNIKU CYLINDRYCZNYM O FUNKCYJNIE ZMIENNYCH I ZALEŻNYCH OD TEMPERATURY EFEKTYWNYCH WŁASNOŚCIACH MATERIAŁOWYCH


## Streszczenie

Głównym celem niniejszej pracy jest rozważenie nieliniowego zagadnienia przewodnictwa ciepła w nieskończonym przewodniku cylindrycznym, w którym efektywne własności materiałowe zmieniają się w sposób wolnozmienny w kierunku promieniowym. W tym dwuskładnikowym kompozycie o deterministycznej mikrostrukturze, będzie rozpatrywane zagadnienie stacjonarne. Z uwagi na mikrostrukturę kompozytu, która jest periodyczna dla ustalonej wartości promienia, mamy tu do czynienia z materiałem o funkcyjnej gradacji własności, FGM (por. Suresh, Mortensen, 1988). Zagadnienie przepływu ciepła jest opisane prawem Fouriera, w którym współczynniki są nieciągłe oraz silnie oscylujące. Własności materiałowe są zależne od temperatury, określone w postaci wielomianu, co powoduje nieliniowość zagadnienia. Model matematyczny opisujący rozważany kompozyt jest oparty na technice tolerancyjnego uśrednienia (por. Woźniak, Wierzbicki, 2000). Ogólny opis i zastosowania dla materiałów o podłużnej gradacji własności można znaleźć w [Woźniak, Michalak, Jędrysiak, 2008] oraz [Michalak i inni, 2007]. Model ten uwzględnia wpływ wielkości mikrostruktury na całkowite pole temperatury, a uśredniony układ równań posiada ciągłe i wolnozmienne współczynniki. Rozważany tu będzie jednokierunkowy przepływ ciepła w mikroheterogenicznym przewodniku pierścieniowym, w którym będą rozpatrywane m.in. różne funkcje szerokości inkluzji $g = g(\rho)$ i ich wpływ na rozkład pola temperatury.