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EXACT SOLUTION OF MODE III CRACK IN ELASTIC HALF-SPACE

The elastic half-space contains a straight – line crack which lies in some distance from the tangentially loaded boundary. Fourier transform technique is used to reduce the problem to the solution of the Fredholm integral equation of the second kind. This equation is solved exactly. Field intensity factors of stress, crack displacement and the energy release rate are determined explicitly. Accordingly to exact analytical solution, obtained here, which is new to the author'best knowledge, the behaviour of a crack which is located in the neighbourhood of the boundary of a half-space may be investigated exactly.

1. Introduction

Sih (1963, 1965) was apparently the first to publish the solution of an anti-plane shear crack for elastic medium. Sih and Chen (1981) did the same. They used integral transforms, and their solutions were very convoluted, difficult for numerical implementation and for estimation of the solution accuracy. The most recent publications can be quoted (Hu et al. (2005); Li and Kardomateas (2006); Zhou et al (2005)), where reader can find numerous other references. Additionally the numerical procedures are used to obtain the results. When the crack lies near the boundary of the medium the numerical procedures become illposed in the sense of Hadamard, i.e. small perturbation of the data can yield arbitrarily large changes in the result. This makes the numerical solution of governing integral equation of the problem quite difficult when the crack is in the neighbourhood of the boundary medium.

Motivated by this consideration the author reconsiders the problem in this elaboration to shown exact solution.

2. Basic equations

For a linearly elastic medium under anti-plane shear there are only the non-trivial antiplane displacement w :

$$u_x = 0 \quad , \quad u_y = 0 \quad , \quad u_z = w(x, y) \tag{1}$$

strain components γ_{xz} and γ_{yz}:

$$\gamma_{xz} = \frac{\partial w}{\partial x} \quad , \quad \gamma_{yz} = \frac{\partial w}{\partial y} \tag{2}$$

and stress components τ_{xz} and τ_{yz}

$$\tau_{\alpha z} = c_{44}\gamma_{\alpha z} \quad , \quad \alpha = x, y \tag{3}$$

where c_{44} is the shear modulus along the z-direction.
The equilibrium equation (Einstein's summation convention is used)

$$\tau_{z\alpha,\alpha} = 0 \tag{4}$$

yields the harmonic equation for displacement w

$$\nabla^2 w = 0 \tag{5}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplace operator.

3. Formulation of the crack problem

Consider an elastic half-space containing straight-line crack of length $2a$, parallel to the surface of a half-space which is subjected to mechanical loads τ_0. The crack is located along the x-axis from $-a$ to a at a depth h from the loaded surface with a rectangular coordinate system, as shown in Fig.1.

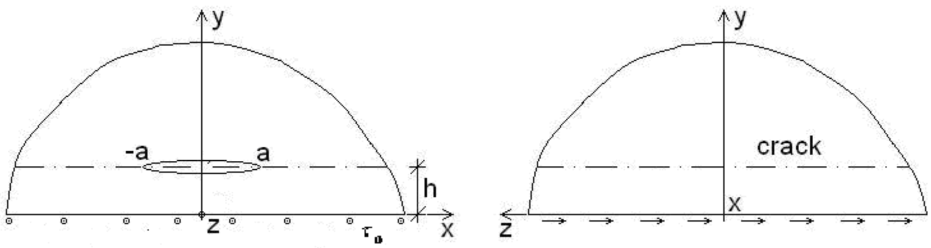

Fig. 1. The elastic half-space with a crack parallel to its surface under an anti-plane mechanical load

To solve the crack problem in linear elastic solids, the superposition technique is usually used. The elementary solution of the medium without the crack is $\tau_{yz} = \tau_0$. Therefore, we use equal and opposite value as the crack surface traction. Thus, in this study, $-\tau_0$, is mechanical loading applied on the crack surfaces (the so called perturbation problem).
The boundary conditions can be written as:

$$\tau_{zy}(x, h\pm) = -\tau_0 \quad , \quad |x| < a \tag{6}$$

$$\left[\left|\tau_{zy}\right|\right]=0 \quad , \quad |x|<\infty \qquad y=h \tag{7}$$

$$\left[\left|w\right|\right]=0 \quad , \quad |x|\geq a \qquad y=h \tag{8}$$

$$\tau_{zy}(x,0)=0 \quad , \quad |x|<\infty \tag{9}$$

where the notation $\left[\left|f\right|\right]=f^{+}-f^{-}$ and f^{+} denotes the values for $h+$ while f^{-} for $h-$.

Of course, in perturbation problem the surface of the half-space is free.

4. The solution for half-space with discontinuity at y = h

Define the Fourier transform pair by equations

$$\hat{f}(s)=\int_{0}^{\infty} f(x)\cos(sx)dx \quad , \quad f(x)=\frac{2}{\pi}\int_{0}^{\infty}\hat{f}(s)\cos(sx)ds \tag{10}$$

Considering the symmetry about y-axis the Fourier cosine transform is only applied in Eqs (5) resulting in ordinary differential equations and their solutions

$$\hat{w}(s,y)=A_1(s)e^{-sy} \quad , \qquad y\rangle h \tag{11a}$$

$$\hat{w}(s,y)=A_2(s)e^{-sy}+A_3(s)e^{sy} \quad , \qquad 0\leq y\langle h \tag{11b}$$

In the domain $y>h$ the solution has the form (11a) to ensure the regularity conditions at infinity.

The unknown functions $A_i(s)$, $i=1,2,3$, are obtained from the boundary conditions (7) and (9), which in transform domain are:

$$\left[\left|\hat{\tau}_{zy}\right|\right]=0 \quad , \quad y=h \tag{12}$$

$$\hat{\tau}_{zy}=0 \quad , \quad y=0$$

where $\left[\left|\hat{f}\right|\right]=\hat{f}(s,h+)-\hat{f}(s,h-)$.

The result is:

$$\begin{aligned} A_1(s)&=\hat{f}(s)\left(e^{-sh}-e^{sh}\right) \\ A_2(s)&=A_3(s)=\hat{f}(s)\,e^{-sh} \end{aligned} \tag{13}$$

Finally, the solution for the half-space with dislocation density functions $f(s)$ in the domain $y\geq 0$, $|x|\langle\infty$ is given by:

$$w(x,y) = -\frac{2}{\pi}\int_0^\infty \hat{f}(s)\left[\operatorname{sgn}(y-h)e^{-s|y-h|} - e^{-s(y+h)}\right]\cos(sx)ds \tag{14}$$

$$\tau_{zy}(x,y) = \frac{2}{\pi}c_{44}\int_0^\infty s\hat{f}(s)\left[e^{-s|y-h|} - e^{-s(y+h)}\right]\cos(sx)ds$$

where sgn($y-h$) equals 1 as $y-h > 0$ and -1 as $y-h < 0$.

5. Fredholm integral equation of the second kind

The unknown function $f(s)$ can be obtained from the mixed boundary conditions (6) and (8) which yield

$$\frac{2}{\pi}\int_0^\infty s\hat{f}(s)\left[1-e^{-2sh}\right]\cos(sx)ds = -\frac{\tau_0}{c_{44}} \quad , \quad |x|\langle a \tag{15a}$$

$$\int_0^\infty \hat{f}(s)\cos(sx)ds = 0 \quad , \quad |x| \geq a \tag{15b}$$

The integral equations (15a) may be rewritten as

$$\frac{2}{\pi}\int_0^\infty \hat{f}(s)\left[1-e^{-2sh}\right]\sin(sx)ds = -\frac{\tau_0 x}{c_{44}} \quad , \quad |x|\langle a \tag{16}$$

We introduce the integral representation of the unknown function $\hat{f}(s)$ as follows

$$\hat{f}(s) = -\frac{\tau_0}{c_{44}}\int_0^a f(u)uJ_0(su)du \tag{17}$$

where $J_0(su)$ is the Bessel function of the first kind and zero order and $f(u)$ is new auxiliary functions. This representation satisfies equation (15b) automatically and converts equation (16) to the Abel integral equation, which can be solved explicitly. The result is the Fredholm integral equation of the second kind

$$f(u) - \int_0^a f(v)K(u,v)\,dv = 1 \tag{18}$$

with the kernel

$$K(u,v) = v\int_0^\infty se^{-2sh}J_0(su)J_0(sv)ds \tag{19}$$

6. The solution of Fredholm integral equation of the second kind

The kernel function $K(u,v)$ may be presented in more useful form. Using the Neumann's theorem (Watson, 1966)

$$J_0(su)J_0(sv)=\frac{1}{\pi}\int_0^{\pi}J_0(sR)d\alpha \quad , \quad R^2=u^2+v^2-2uv\cos\alpha \tag{20}$$

and the integral

$$\int_0^{\infty}sJ_0(Rs)e^{-2sh}ds=\frac{2h}{\left[R^2+(2h)^2\right]^{3/2}} \tag{21}$$

the kernel function becomes

$$K(u,v)=\frac{4hv}{\pi l^{3/2}}\int_0^{\pi/2}\frac{d\alpha}{\left(1-k^2\cos^2\alpha\right)^{3/2}} \tag{22}$$

$$l^2=(u+v)^2+4h^2 \quad , \quad k^2=\frac{4uv}{l^2}$$

The kernel function is presented by means of elliptic integral. The integral equation (18) can be solved by consecutive iterations.
The recurrence formula is

$$f_i(u)=1+\int_0^{a}f_{i-1}(v)K(u,v)dv \quad , \quad f_0(v)=1 \quad , \quad i=1,2,\ldots n \tag{23}$$

The n-th approximation gives

$$(u)=1+\frac{a}{a+u}\left[1-\frac{4h}{\pi}\frac{K(k_0)}{l_0}\right]+\left(\frac{a}{a+u}\right)^2\left[1-\frac{4h}{\pi}\frac{K(k_0)}{l_0}\right]^2+\ldots\ldots\ldots+\left(\frac{a}{a+u}\right)^n\left[1-\frac{4h}{\pi}\frac{K(k_0)}{l_0}\right]^n \tag{24}$$

where $K(k_0)$ is the elliptic integral of the first kind defined by

$$K(k_0)=\int_0^{\pi/2}\frac{d\alpha}{(1-k_0^2\cos^2\alpha)^{1/2}} \tag{25}$$

$$l_0^2=(a+u)^2+4h^2 \quad , \quad k_0^2=\frac{4au}{l_0^2}$$

The sum of infinite geometric series converges to the solution as $n \to \infty$, giving

$$f(u) = \left[1 - \frac{a}{a+u}\left(1 - \frac{2}{\pi}\frac{K(k_0)}{l_0/2h}\right)\right]^{-1} \qquad |u| \le a \tag{26}$$

The range of convergence is given by inequality

$$\frac{2}{\pi}K(k_0) < \left(2 + \frac{u}{a}\right)\frac{l_0}{2h} \qquad |u| \le a \tag{27}$$

and is satisfied for all of u and a/h.
For $h \to \infty$, $(2/\pi)K(k_0) \to 1$ and $l_0/2h \to 1$ while for $h \to 0$ we have the logarithmic singularity of $K(k_0)$ for $u = a$

$$K(k_0) \sim \ln\left(\frac{1}{1 - \frac{2\sqrt{au}}{a+u}}\right) \tag{28}$$

But $hK(k_0)/l_0$ tends to zero as $a/h \to \infty$.
Thus we have the values

$$f\left(\frac{a}{h}\right) = 2\left[1 + \frac{2}{\pi}\frac{1}{\sqrt{1+\delta^2}}K\left(\frac{\delta}{\sqrt{1+\delta^2}}\right)\right]^{-1} \quad , \quad f(0) = \sqrt{1 + \frac{\delta^2}{4}} \quad , \quad \delta = \frac{a}{h} \tag{29}$$

The values of $f(a/h)$ changes from 1 to 2 for all of a/h and $f(u)$ is given explicitly by Eq. (26).
This analytical solution is new to the author' best knowledge.

7. Field intensity factors

The shear stress outside of the crack surface can be expressed by

$$\tau_{zy}(x, h\pm) = -\frac{2}{\pi}\tau_0\int_0^a f(u)udu\int_0^\infty sJ_0(su)\left(1 - e^{-2sh}\right)\cos(sx)ds \tag{30}$$

Using the integral (Rogowski, 2006)

$$\int_0^\infty e^{-2sh}\sin(sx)J_0(su)ds = \frac{\eta}{x(\xi^2+\eta^2)}$$
$$u^2 = x^2(1+\xi^2)(1-\eta^2) \qquad 2h = x\xi\eta \qquad , \qquad x\xi > 0 \quad , \quad \eta > 0 \tag{31}$$

equations (30) may be written as

$$\tau_{zy}(x,h\pm) = -\frac{2}{\pi}\tau_0\frac{d}{dx}\int_0^a f(u)udu\left[\frac{|x|}{x\sqrt{x^2-u^2}} - \frac{\eta}{x(\xi^2+\eta^2)}\right] \tag{32}$$

The singular term is included in the first term as $|x| \to a^+$. Since the singular field near the crack tip exhibits the inverse square-root singularity we define the stress, intensity factor as follows

$$K_\tau = \lim_{|x|\to a^+}\sqrt{2(|x|-a)}\,\tau_{xy} \tag{33}$$

The intensity factor is obtained as

$$K_\tau = \frac{2}{\pi}\tau_0 f(a)\sqrt{a} \tag{34}$$

The jump of displacement on the crack surfaces can be expressed as

$$[\![w]\!] = \frac{4}{\pi}\frac{\tau_0}{c_{44}}\int_x^a \frac{f(u)udu}{\sqrt{u^2-x^2}} \tag{35}$$

If we define the jump of displacement intensity factor as

$$K_w = \lim_{|x|\to a^-}\frac{1}{2\sqrt{2(a-|x|)}}[\![w]\!] \tag{36}$$

then in view of the results in Eq. (35), we have

$$K_w = \frac{1}{c_{44}}K_\tau \tag{37}$$

The energy release rate of the crack-tip is obtained from the following integral:

$$G = \frac{1}{2}\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta \{\tau_{yz}(r+a,0)[\![w]\!](r+a-\delta)\}dr \tag{38}$$

The energy release rate is defined as

$$G = \frac{1}{2} K_\tau K_w \tag{39}$$

or

$$G = \frac{2}{\pi^2} f^2(a) a \frac{\tau_0^2}{c_{44}} \tag{40}$$

8. Result and discussion

The stress intensity factor K_τ is proportional to the applied mechanical load, as Eq. (34) implies. The K_τ therefore is just a function of the geometry of the cracked elastic half-space as shown in Fig. 2.

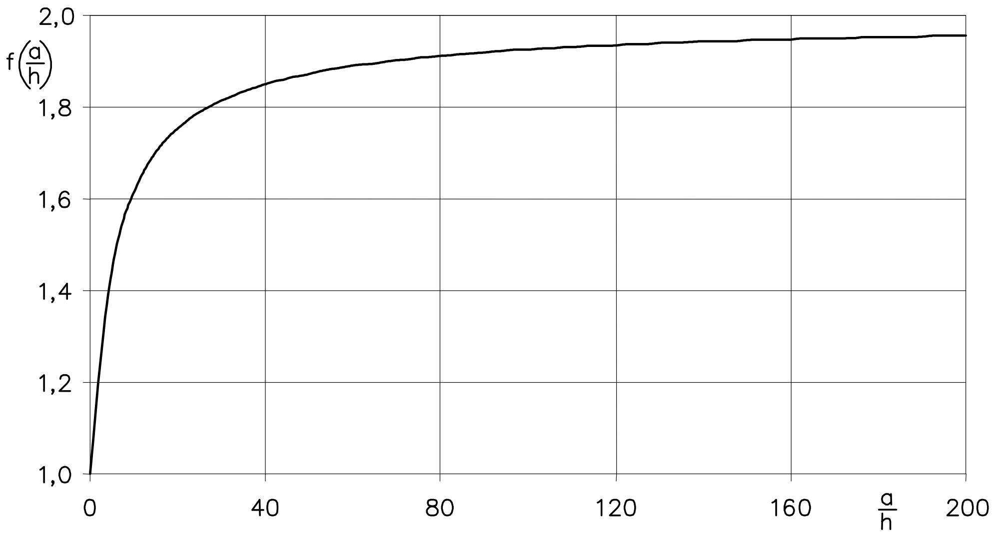

Fig. 2. Variation of $f(a/h)$ versus ratio of a/h; stress intensity factor is proportional to $f(a/h)$ since: $f(a/h) = K_\tau / \tau_0 \sqrt{a} (2/\pi)$

From the figure 2 we can see that the stress intensity factor increase with a/h. For small values of a/h these quantities grow at an approximately constant rate with increasing a/h. For very large a/h (the crack near the boundary of a half – space) $f(a/h)$ increases slowly tending to 2.

9. Conclusions

The following conclusions can be reached from the results of this study:

- The stress intensity factor of mode III changes as follows: $K_\tau(0) \le K_\tau\left(\frac{a}{h}\right) \le 2K_\tau(0)$, where $K_\tau(0) = (2/\pi)\tau_0\sqrt{a}$.
- The energy release rate of a crack mode III changes as follows: $G(0) \le G\left(\frac{a}{h}\right) \le 4G(0)$, where $G(0) = (2/\pi^2)\,\tau_0^2 a/c_{44}$, where a is half –length of a crack and h is the distance of one from the boundary of a half-space.
- The analytical solution (26) is new to the author' best knowledge. Accordingly, the behaviour of a crack which lies in the neighbourhood of the boundary of the medium may be investigated exactly.
- Note that the solution presented here is also the solution of quarter plane with an edge crack of length a since the plane $x = 0$ is the plane of symmetry and τ_{xz} vanishes on this plane.

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DOKŁADNE ROZWIĄZANIE DLA SZCZELINY TRZECIEGO RODZAJU W SPRĘŻYSTEJ PÓŁPRZESTRZENI

Streszczenie

Sprężysta półprzestrzeń zawiera prostoliniową szczelinę usytuowaną w pewnej odległości od stycznie obciążonego brzegu. Zastosowana technika transformacji całkowej Fouriera sprowadza zagadnienie do rozwiązania równania całkowego Fredholma drugiego rodzaju. Równanie to jest rozwiązane dokładnie. Współczynniki intensywności pola naprężenia i przemieszczenia oraz energia odkształcenia szczeliny są wyznaczone w sposób jawny. Dzięki dokładnemu analitycznemu rozwiązaniu, otrzymanemu tutaj, które jest nowym o ile autor dobrze wie, możemy badać dokładnie zachowanie się szczeliny usytuowanej w bliskim sąsiedztwie brzegu.