DYNAMICS AND STABILITY OF NONPERIODIC MULTILAYERED PLATES

S. KONIECZNY, B. TOMCZYK
Department of Structural Mechanics, Technical Uniwersity of Łódź
Al. Politechniki 6, 93-590 Łódź

The method of macro-modelling of nonperiodic multilayered elastic plates has been proposed in [1]. The proposed method is based on certain concepts of the nonstandard analysis [2,3] combined with some a priori postulated physical assumptions. In this paper, using this method the homogenized model of nonperiodic plate will be derived and applied to the evaluation of inhomogeneity effects on a critical force and a free vibration frequency for a simply supported laminated plate.

1. Introduction

An underformed plate which occupies a region Ω in physical space (parametrized by cartesian orthogonal coordinates x_1,x_2,x_3) bounded by the coordinate planes $x_3=h^+$, $x_3=h^-$ where $h^+>0$, $h^-<0$ and by cylindrical surface $\Gamma\equiv\partial\Pi\times(h^-,h^+)$, where Π is a regular region on $0x_1x_2$ plane is considered in the analysis. We define $\mathbf{x}\equiv(x_1,x_2,x_3)\in\Omega$, $x_\alpha\equiv(x_1,x_2)\in\Pi$, $x^3\in[h^-,h^+]$, $\tau\in[\tau_0,\tau_f]$ stands for a time coordinate.

The plate is made of N basic layers bounded by the coordinate planes $x_3=h^-+\zeta_K$, $K=0,1,2,\dots,N$, with $\zeta_0=0$, $\zeta_N=h$,where $h=h^+-h^-$ denotes the thickness of the plate; ζ_{K-1} describes the distance of K-th basic lamina from the boundry plane $x_3=h^-$. The thickness $\varepsilon_K\equiv\zeta_K-\zeta_{K-1}$,$K=1,2,\dots,N$, $(\zeta_K>\zeta_{K-1})$, of every basic layer is assumed to be sufficiently small when compared to the thickness h of the plate; this means that we shall deal with the nonperiodic plates made of a large number of laminae. Moreover, let every basic layer (ζ_{K-1},ζ_K) consists of three sublayers, made of three different homogeneous anisotropic linear-elastic materials; by δ_K, $\tilde{\delta}_K$ we denote the thicknesses of upper and middle sublayer of K-th basic unit, respectively.

Throughout the paper subscripts i,j run over 1,2,3, subscripts $\alpha,\beta,\gamma,\delta$ and indices a,d run over the sequence 1,2. The summation convetion holds with respect to all aforementioned indices.

The composite is loaded on the boundary planes $x_3=h^+$, $x_3=h^-$ by the known normal surface tractions $p^3_+(x_\alpha,\tau)$, $p^3_-(x_\alpha,\tau)$, respectively, and on the part Γ of the boundary the displacements $u_\Gamma(\mathbf{x},\tau)$, $\mathbf{x}\in\Gamma$, are known. By $u_i(\mathbf{x},\tau),e_{ij}(\mathbf{x},\tau),t_{ij}(\mathbf{x},\tau),b^3(\mathbf{x},\tau)$ we denote displacements, strains, stresses and body forces, respectively, as functions defined (almost everywhere) on Ω.

We shall define the subsets of $[h^-, h^+]$ by means of:

$$L \equiv \bigcup_{K=1}^{N}\left(h^- + \zeta_{K-1}, h^- + \zeta_{K-1} + \eta_K \varepsilon_K\right); \qquad \eta_K = \delta_K / \varepsilon_K; \quad \tilde{\eta}_K = \tilde{\delta}_K / \varepsilon_K;$$

$$S \equiv \bigcup_{K=1}^{N}\left(h^- + \zeta_{K-1} + \eta_K \varepsilon_K, h^- + \zeta_{K-1} + \left(\eta_K + \tilde{\eta}_K\right)\varepsilon_K\right);$$

$$U \equiv \bigcup_{K=1}^{N}\left(h^- + \zeta_{K-1} + \left(\eta_K + \tilde{\eta}_K\right)\varepsilon_K, h^- + \zeta_K\right) \quad ; \quad K=1, 2, \ldots, N \quad ; \qquad (1.1)$$

Hence the mass denisity $\rho(x_3)$ and the tensor of elastic constants $c^{ijkl}(x_3)$ of the nonperiodic plate under consideration will be given by:

$$\left(\rho(x_3), c^{ijkl}(x_3)\right) = \begin{cases} \left({}^L\rho, {}^L c^{ijkl}\right) & \text{gdy} \quad x_3 \in L \quad ; \\ \left({}^S\rho, {}^S c^{ijkl}\right) & \text{gdy} \quad x_3 \in S \quad ; \\ \left({}^U\rho, {}^U c^{ijkl}\right) & \text{gdy} \quad x_3 \in U \quad ; \end{cases} \qquad (1.2)$$

where ${}^L\rho$, ${}^L c^{ijkl}$, ${}^S\rho$, ${}^S c^{ijkl}$, ${}^U\rho$, ${}^U c^{ijkl}$ are material constants related to the parts $\Pi\times L$, $\Pi\times S$, $\Pi\times U$ of the region Ω, respectively.

We define the discrete functions [1], (K=1,2,...,N), such that:

$$\bar{\zeta}\left(h^- + \frac{Kh}{N}\right) \equiv \zeta_K; \quad \bar{\eta}\left(h^- + \frac{Kh}{N}\right) \equiv \eta_K; \quad \bar{\bar{\eta}}\left(h^- + \frac{Kh}{N}\right) \equiv \tilde{\eta}_K; \qquad (1.3)$$

Next, we "approximate" functions $\bar{\zeta}(\cdot)$, $\bar{\eta}(\cdot)$, $\bar{\bar{\eta}}(\cdot)$ .by certain smooth functions $\zeta(x_3)$, $\eta_1(x_3)$, $\eta_2(x_3)$, respectively, defined on the interval $[h^-, h^+]$:

$$\zeta:[h^-, h^+] \to [0, h]; \quad \eta_1:[h^-, h^+] \to (0,1); \quad \eta_2:[h^-, h^+] \to (0,1); \qquad (1.4)$$

where $\zeta(x_3)$ is a strongly monotone function, such that $\zeta(x_3=h^-)=0$, $\zeta(x_3=h^+)=h$ and $\eta_1(x_3)$, $\eta_2(x_3)$ must satisfy a condition: $\eta_1(x_3)+\eta_2(x_3)<1$.

2.The primary problem

The governing equations of the plate under consideration will be represented by:
(i) The strain-displacement relations

$$\varepsilon_{\alpha\beta}(\mathbf{x},\tau) = u_{(\alpha,\beta)}(\mathbf{x},\tau) + u_{3,\alpha}(\mathbf{x},\tau)u_{3,\beta}(\mathbf{x},\tau)/2;$$

$$\varepsilon_{\alpha 3}(\mathbf{x},\tau) = u_{(\alpha,3)}(\mathbf{x},\tau); \quad \varepsilon_{33}(\mathbf{x},\tau) = u_{3,3}(\mathbf{x},\tau); \quad \mathbf{x} \in \Omega; \quad \tau \in [\tau_0, \tau_f]; \quad (2.1)$$

(ii) The stress-strain relations

$$t^{\alpha\beta}(\mathbf{x},\tau) = \bar{c}^{\alpha\beta\gamma\delta}(x_3)e_{\gamma\delta}(\mathbf{u})(\mathbf{x},\tau); \quad t^{\alpha 3}(\mathbf{x},\tau) = 2c^{\alpha 3 3\delta}(x_3)e_{3\delta}(\mathbf{x},\tau); \qquad (2.2)$$

where: $\bar{c}^{\alpha\beta\gamma\delta}(x_3) = c^{\alpha\beta\gamma\delta}(x_3) - c^{\alpha\beta 33}(x_3)c^{\gamma\delta 33}(x_3)/c^{3333}(x_3)$;

(iii) The principle of virtual work

$$\int_{h^-}^{h^+}\int_{\Pi}\left[t^{\alpha\beta}\delta e_{\alpha\beta}+2t^{\alpha 3}\delta e_{\alpha 3}+t^{33}\delta e_{33}\right]d\Pi dx_3=\int_{\Pi}\left[p_+^3\delta u_3\Big|_{x_3=h^+}+p_-^3\delta u_3\Big|_{x_3=h^-}\right]d\Pi+$$

$$+\int_{h^-}^{h^+}\int_{\Pi}\rho b^3\delta u_3 d\Pi dx_3-\int_{h^-}^{h^+}\int_{\Pi}\rho\ddot{u}^i\delta u_i d\Pi dx_3;\quad d\Pi\equiv dx_1dx_2;\quad \delta u_i=0\quad\text{on}\quad\Gamma; \tag{2.3}$$

Now we formulate the following:

Problem P: for known Ω, p_+^3, p_-^3, b^3, initial and boundary conditions and ${}^L\rho$, ${}^Lc^{ijkl}$, ${}^S\rho$, ${}^Sc^{ijkl}$, ${}^U\rho$, ${}^Uc^{ijkl}$ as well as for known L,S,U, find the displacements $\mathbf{u}(\mathbf{x},\tau)$ and stresses $\mathbf{t}(\mathbf{x},\tau)$, $\mathbf{x}\in\Omega$, $\tau\in[\tau_0,\tau_f]$, such that Eqs.(2.1)-(2.3) under conditions (1.2) hold.

3. Passage to the nonstandard problem

Applying the approximation hypothesis [1] and the homogenization hypothesis [3] we will pass from P to the nonstandard problem $P^{(\tilde{\omega})}$.

The micro-macro localization hypothesis: the approximate solution to the nonstandard problem $P^{(\tilde{\omega})}$ can be expected in the class of functions given by:

$$u_\alpha^{(\tilde{\omega})}(x,\tau)={}^*W_\alpha(x_\alpha,\tau)+x_3\,{}^*D_\alpha(x_\alpha,\tau)+h_a(x_3)\,{}^*Q_\alpha^a(x_\alpha,\tau);$$

$$u_3^{(\tilde{\omega})}(x,\tau)={}^*W_3(x_\alpha,\tau);\quad x\in{}^*\Omega;\quad \tau\in{}^*[\tau_0,\tau_f];\quad \alpha=1,2,\quad a=1,2; \tag{3.2}$$

where

(i) ${}^*W_i$, ${}^*D_\alpha$, ${}^*Q_\alpha^a$ are (sufficiently regular) unknown standard functions [2], fields W_i, D_α are called macrodisplacements [3], vector Q_α^a is called microlocal (or correction) parameters [3],

(ii) $h_a(x_3)$ are postulated a priori, linear independent, nonstandard micro-shape functions [3]; they attain only infinitesimal values and hence the therms involving $h_a(x_3)$ can be neglected, but their derivatives attain values that are not infinitely small, so they play an essential role if we calculate the strains and stresses.

Taking into account the known theorems of the nonstandard integral calculus [2] and after neglecting the terms involving micro-shape functions (but not their derivatives!) we can pass from the nonstandard structure to the primary structure.

The approximate solution to the nonstandard problem $P^{(\tilde{\omega})}$ can be found as the solution to a certain problem $\tilde{P}$ for the macrodisplacements W_i, D_α and the correction parameters Q_α^a; $x_\alpha\in\Pi$, $\tau\in[\tau_0,\tau_f]$, the problem will be called the effective (microlocal or standard) problem and it does not involve any nonstandard entity.

4. The effective problem

Since the functions W_i, D_α, Q^a_α are arbitrary and independent, then after denotations:

$$N^{\alpha\beta}(x_\alpha,\tau)=\int_{h^-}^{h^+}T^{\alpha\beta}dx_3\,;\quad M^{\alpha\beta}(x_\alpha,\tau)=\int_{h^-}^{h^+}x_3T^{\alpha\beta}dx_3\,;\quad \hat{Q}^\alpha(x_\alpha,\tau)=\int_{h^-}^{h^+}T^{\alpha 3}dx_3\,;$$

$$p^3(x_\alpha,\tau)\equiv p^3_+ + p^3_- + \int_{h^-}^{h^+}\tilde{\rho}(x_3)b^3dx_3;\quad \tilde{f}\equiv\int_{h^-}^{h^+}\tilde{\rho}(x_3)dx_3\,;\quad \hat{f}\equiv\int_{h^-}^{h^+}\tilde{\rho}(x_3)x_3^2dx_3\,; \tag{4.1}$$

where the mean stress tensor $T^{ij}(\mathbf{x},\tau)$ and the mean mass density $\tilde{\rho}(x_3)$ have a form:

$$T^{\alpha j}(\mathbf{x},\tau)\equiv{}^Lt^{\alpha j}(\mathbf{x},\tau)\eta_1(x_3)+{}^St^{\alpha j}(\mathbf{x},\tau)\eta_2(x_3)+{}^Ut^{\alpha j}(\mathbf{x},\tau)\big(1-\eta_1(x_3)-\eta_2(x_3)\big); \tag{4.2}$$

$$\tilde{\rho}(x_3)\equiv{}^L\rho\eta_1(x_3)+{}^S\rho\eta_2(x_3)+{}^U\rho\big(1-\eta_1(x_3)-\eta_2(x_3)\big); \tag{4.3}$$

and using the divergence theorem as well as du Bois lemma, we obtain from the principle of virtual work tha following equations of homogenized model:

(i) the plate equations of motion

$$N^{\alpha\beta}{}_{,\beta}(x_\alpha,\tau)=\tilde{f}\,\ddot{W}^\alpha(x_\alpha,\tau);\quad M^{\alpha\beta}{}_{,\beta}(x_\alpha,\tau)-\hat{Q}^\alpha(x_\alpha,\tau=\hat{f}\,\ddot{D}^\alpha(x_\alpha,\tau)\,;$$

$$\hat{Q}^\alpha{}_{,\alpha}(x_\alpha,\tau)+\big(N^{\alpha\beta}(x_\alpha,\tau)W_{3,\alpha}(x_\alpha,\tau)\big)_{,\beta}+p^3(x_\alpha,\tau)=\tilde{f}\,\ddot{W}^3(x_\alpha,\tau)\,; \tag{4.4}$$

(ii) the following system of linear algebraic equations for correctors:

$$P^{\alpha 33\delta}_{ab}Q^b_\delta(x_\alpha,\tau)=-h[\![c^{\alpha 33\delta}_a]\!]\big(W_{3,\delta}(x_\alpha,\tau)+D_\delta(x_\alpha,\tau)\big); \tag{4.5}$$

where:

$$P^{\alpha 33\delta}_{ab}\equiv\begin{cases}\displaystyle\int_{h^-}^{h^+}\left(\frac{{}^Lc^{\alpha 33\delta}}{\eta_1(x_3)}+\frac{{}^Sc^{\alpha 33\delta}}{\eta_2(x_3)}\right)dx_3 & \text{gdy}\quad a=b=1;\\[2ex] \displaystyle\int_{h^-}^{h^+}\left(\frac{{}^Sc^{\alpha 33\delta}}{\eta_2(x_3)}+\frac{{}^Uc^{\alpha 33\delta}}{1-\eta_1(x_3)-\eta_2(x_3)}\right)dx_3 & \text{gdy}\quad a=b=2;\\[2ex] \displaystyle-\int_{h^-}^{h^+}\frac{{}^Sc^{\alpha 33\delta}}{\eta_2(x_3)}dx_3 & \text{gdy}\quad \begin{cases}a=1\text{ i }b=2\quad\text{lub}\\ a=2\text{ i }b=1\end{cases};\end{cases} \tag{4.6}$$

$$[\![c^{\alpha 33\delta}_1]\!]\equiv{}^Lc^{\alpha 33\delta}-{}^Sc^{\alpha 33\delta};\quad [\![c^{\alpha 33\delta}_2]\!]\equiv{}^Sc^{\alpha 33\delta}-{}^Uc^{\alpha 33\delta}; \tag{4.7}$$

A solution to the equations (4.5) can be written in the form

$$Q^b_\delta(x_\alpha,\tau)=-hK^{bd}_{\delta 33\gamma}[\![c^{\gamma 33\beta}_d]\!]\big(W_{3,\beta}(x_\alpha,\tau)+D_\beta(x_\alpha,\tau)\big); \tag{4.8}$$

where $K^{bd}_{\delta 33\gamma}$ are defined by: $\quad P^{\alpha 33\gamma}_{ab}K^{bd}_{\delta 33\gamma}=\delta^d_a\delta^\alpha_\delta$; (4.9)

(iii) the following plate constitutive equations

$$N^{\alpha\beta}(x_\alpha,\tau) = \overline{B}^{\alpha\beta\gamma\delta}\left[W_{(\gamma,\delta)}(x_\alpha,\tau) + W_{3,\gamma}(x_\alpha,\tau)W_{3,\delta}(x_\alpha,\tau)/2\right] + \overline{F}^{\alpha\beta\gamma\delta}D_{(\gamma,\delta)}(x_\alpha,\tau);$$

$$M^{\alpha\beta}(x_\alpha,\tau) = \overline{F}^{\alpha\beta\gamma\delta}\left[W_{(\gamma,\delta)}(x_\alpha,\tau) + W_{3,\gamma}(x_\alpha,\tau)W_{3,\delta}(x_\alpha,\tau)/2\right] + \overline{G}^{\alpha\beta\gamma\delta}D_{(\gamma,\delta)}(x_\alpha,\tau);$$

$$\hat{Q}^{\alpha}(x_\alpha,\tau) = \left(B^{\alpha33\beta} - H^{\alpha33\beta}\right)\left[W_{3,\beta}(x_\alpha,\tau) + D_\beta(x_\alpha,\tau)\right]; \qquad (4.10)$$

where:

$$H^{\alpha33\beta} \equiv h^2 [\![c_a^{\alpha33\delta}]\!] K_{\delta33\gamma}^{ad} [\![c_d^{\gamma33\beta}]\!]; \quad \overline{B}^{\alpha\beta\gamma\delta} \equiv \int_{h^-}^{h^+} \overline{C}^{\alpha\beta\gamma\delta}(x_3)dx_3;$$

$$B^{\alpha33\delta} \equiv \int_{h^-}^{h^+} C^{\alpha33\delta}(x_3)dx_3; \quad \overline{F}^{\alpha\beta\gamma\delta} \equiv \int_{h^-}^{h^+} \overline{C}^{\alpha\beta\gamma\delta}(x_3)x_3dx_3; \quad \overline{G}^{\alpha\beta\gamma\delta} \equiv \int_{h^-}^{h^+} \overline{C}^{\alpha\beta\gamma\delta}(x_3)x_3^2dx_3;$$

$$\overline{C}^{\alpha\beta\gamma\delta} \equiv {}^{L}\overline{c}^{\alpha\beta\gamma\delta}\eta_1(x_3) + {}^{S}\overline{c}^{\alpha\beta\gamma\delta}\eta_2(x_3) + {}^{U}\overline{c}^{\alpha\beta\gamma\delta}\left(1-\eta_1(x_3)-\eta_2(x_3)\right); \qquad (4.11)$$

It can be proven that tensors $\overline{B}^{\alpha\beta\gamma\delta}$, $\overline{F}^{\alpha\beta\gamma\delta}$, $\overline{G}^{\alpha\beta\gamma\delta}$, $(B^{\alpha33\beta} - H^{\alpha33\beta})$ are positive definite.

Combining (4.10) and (4.4) we arrive at the system of five nonlinear differential equations for five basic unknows: W_i, D_α. However, in stability and vibration problems the governing system of equations will be written in the form:

$$\begin{cases} \left(B^{\alpha33\beta} - H^{\alpha33\beta}\right)\left[W_{3,\alpha\beta} + D_{(\alpha,\beta)}\right] + N^{\alpha\beta}W_{3,\alpha\beta} + p^3 - \tilde{f}\ddot{W}_3 = 0; & (4.12) \\ \underline{\overline{F}^{\alpha\beta\gamma\delta}\left[W_{\gamma,\delta\beta} + W_{3,\gamma}W_{3,\delta\beta}\right]} + \overline{G}^{\alpha\beta\gamma\delta}D_{\gamma,\delta\beta} - \left(B^{\alpha33\beta} - H^{\alpha33\beta}\right)\left[W_{3,\beta} + D_\beta\right] - \hat{\tilde{f}}\ddot{D}^{\alpha} = 0; \end{cases}$$

$$\begin{cases} N^{\alpha\beta}_{,\beta} - \tilde{f}\ddot{W}^{\alpha} = 0; \end{cases} \qquad N^{\alpha\beta} = \overline{B}^{\alpha\beta\gamma\delta}\left[W_{\gamma,\delta} + \frac{1}{2}W_{3,\gamma}W_{3,\delta}\right] + \underline{\overline{F}^{\alpha\beta\gamma\delta}D_{(\gamma,\delta)}}; \qquad (4.13)$$

The underlined terms in (4.12), (4.13) depend on $\overline{F}^{\alpha\beta\gamma\delta}$ and represent the coupling between $N_{\alpha\beta}$ and $M_{\alpha\beta}$ in the plate constitutive equations (4.10).

5. Applications

In order to illustrate the general results obtained in the paper we shall apply equations (4.12), (4.13) to the analysis of the stability and free vibrations of a rectangular plate which is simply suported on edges $x_1=0$, $x_1=a_1$. We shall treat this problem as onedimensional, setting $x_\alpha \equiv x_1$. For simplicity we shall neglect the inertia terms $\tilde{f}\ddot{W}^{\alpha}$ and the body forces. We also assume that $p^3=0$ and $N_{11}=N_{11}(\tau)$. Let:

$$W_3(x_1,\tau) = \sum_{m=1}^{\infty} A_m \sin\frac{m\pi}{a_1}x_1 e^{-i\omega_m\tau}; \quad D_1(x_1,\tau) = \sum_{m=1}^{\infty} B_m \cos\frac{m\pi}{a_1}x_1 e^{-i\omega_m\tau}; \qquad (5.1)$$

Using the aforementioned assumptions and subsituting (5.1) into (4.12) we obtain for $A_m \neq 0$, $B_m \neq 0$:

$$\begin{vmatrix} \tilde{f}\omega_m^2 - \left(B^{1331} - H^{1331}\right)\lambda_m^2 + \overline{N}^{11}\lambda_m^2 & -\left(B^{1331} - H^{1331}\right)\lambda_m \\ -\left(B^{1331} - H^{1331}\right)\lambda_m & \hat{f}\omega_m^2 - \hat{G}^{1111}\lambda_m^2 - \left(B^{1331} - H^{1331}\right) \end{vmatrix} = 0\,; \qquad (5.2)$$

where: $\overline{N}^{11}(\tau) = -N^{11}(\tau)$; $\hat{G}^{1111} \equiv \overline{G}^{1111} - \left(\overline{F}^{1111}\right)^2 / \overline{B}^{1111}$; $\lambda_m = m\pi/a_1$.

Let us introduce parameters: $\xi \equiv H^{1331}\left(B^{1331}\right)^{-1}$; $s^2 = B^{1331}\left(a_1\right)^2\left(\hat{G}^{1111}\pi^2\right)^{-1}$;

where ξ characterizes the relative heterogenity of laminated plate structure (for $\zeta=0$ we are dealing with a homogeneous plate) and s is the plate slenderness parameter.

a) if $\omega_m^2 = 0$, then for a critical force we obtain the condition:

$$\overline{N}_{kr}^{11} = B^{1331}(1-\xi)\left[1+(1-\xi)s^2\right]^{-1}; \qquad (5.3)$$

which describes the effect of the heterogeneity of a laminated plate structure on the plate stability.

(ii) if $\overline{N}^{11} = 0$ then, after neglecting terms $\hat{f}\omega_m^2$ we obtain the formula:

$$\omega_1^2 = \pi^2\left(\tilde{f}a_1\right)^{-1} B^{1331}(1-\xi)\left[1+(1-\xi)s^2\right]^{-1}; \qquad (5.4)$$

which has a form similar to (5.3) and characterizes the effect of a laminated plate structure on the plate free vibration frequency.

6. Conclusions

From numerical example, it follows that the effects of heterogenity of the plate under consideration on a critical force and a free vibration frequency are negligibly small. However, if ξ is close to 1 then the heterogenity of laminated plate strucutre leads to the sudden decrease of the critical force and the free vibration frequency.

References

1. Konieczny S., Tomczyk B., *Modelowanie płyt o nieperiodycznie warstwowej strukturze*, XXXVI Sympozjon Modelowanie w Mechanice, t.3 Gliwice (1997).
2. Robinson A., *Non-standard Analysis*, North-Holland Publ. Comp., Amsterdam, (1996).
3. Woźniak C., *Nonstandard Analysis and Micromorphic Effects in Multilayered Elastic Composites*, Arch. Mech. **Vol. 43** (1991), 385-392.