ON A STABILITY OF THIN MICROPERIODIC PLATES

JAROSŁAW JĘDRYSIAK
Department of Structural Mechanics, University of Technology, Łódź
al. Politechniki 6, 90-924 Łódź, Poland

The structural macromechanics is applied to investigate the stability of a composite plate. Thin elastic plates with microperiodic structure in planes parallel to the midplane are examined. A new refined approach is used to the problem of the dynamic stability of periodic plates. Some special results are shown.

1. Introduction

The aim of our considerations is a dynamic stability of thin microperiodic plates. The plates are assumed to have material and/or inertial properties, which are periodic functions in planes parallel into the midplane. In these plates we can distinguish a small (comparing to the minimum characteristic size of the plate in the midplane) repeated element. The example of this plate is presented on Fig. 1.1.

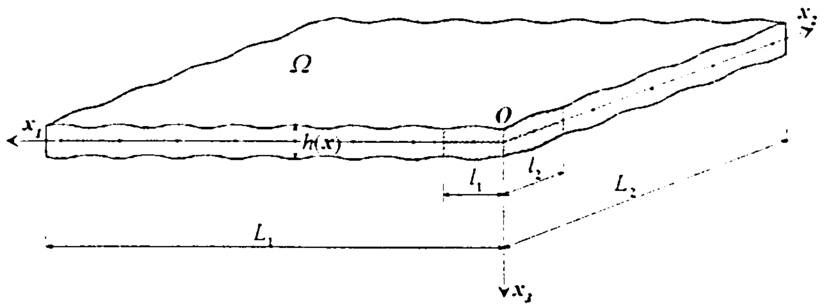

Fig. 1.1. An example of a periodic plate
Rys. 1.1. Przykład płyty periodycznej

The dynamic problems for periodic plates can be described by the equations of three-dimensional micromechanics, which are too complicated to constitute the basis for investigations of most engineering problems. Since they involve highly oscillating coefficients. This is why the simplified models, called homogenized or local models, are used to investigate these problems. Local models describe periodic plates using constant effective stiffnesses and averaged mass densities. However, these models are not able to describe some important features of the dynamic plate behaviour because they neglect the length-scale effect, which plays a crucial role in the description non-stationary processes.

In this paper we will take into account this effect of the microstructure size on the dynamic stability. Our considerations are based on the structural (refined) macrodynamics of periodic materials and structures (2, 3), in particular of microperiodic plates (1). Using assumptions of the refined theory, which were presented in the aforementioned papers, and the assumptions of the Kirchhoff plate theory the governing equations describing the dynamic plate behaviour were derived in (1).

These equations we will adapt here for problems of the dynamic stability, which makes it possible to investigate the length-scale effect on the dynamic macrobehaviour of microperiodic plates loaded in their midplanes.

2. Preliminaries

Let $0x_1x_2x_3$ be the orthogonal cartesian coordinate system in the physical space. Setting $x\equiv(x_1,x_2)$ and $z\equiv x_3$, we assume that the region of underformed plate is defined by $\Omega=\{(x,z):-h(x)/2 < z < h(x)/2, x\in\Pi\}$, where Π is the region of midplane and $h(x)$ is the plate thickness at a point $x\in\Pi$. We shall denote by $\Delta:=(0,l_1)\times(0,l_2)$ the periodicity unit cell on $0x_1x_2$ plane, where l_1, l_2 are length dimensions sufficiently small compared to L_Π, which is the minimum characteristic length dimension of Π ($L_\Pi=\min(L_1,L_2)$). The size of the cell is described by *the microstructure length parameter* l (defined by $l:=\sqrt{l_1^2+l_2^2}$, where $l<<L_\Pi$). Subscripts α, β, ...(i, j, ...) run over 1, 2 (over 1, 2, 3) and indices A, B,... run over 1,..., N. Summation convention holds for all aforementioned indices. We assume that $h(x)$ is a Δ-periodic function of x and all material and inertial properties of the plate (ρ - a mass density, $a_{\alpha\beta\gamma\delta}$ - terms of an elastic modulae tensor) are also Δ-periodic functions of x and even functions of z. For an arbitrary integrable Δ-periodic function $f(\cdot)$ we define $<f>:=(l_1l_2)^{-1}\int_\Delta f(x)da$, where $<f>$ is an averaged (constant) value of f. By p^+, p^- tractions (in the x_3-axis direction) on upper and lower plate boundaries, respectively, will be denoted and b stands for the constant body force. We also define t as the time coordinate.

2.1. *Fundamental relations of the refined plate theory*

We assume that every plane z=const is a material symmetry plane ($a_{3\alpha\beta\gamma}=0$, $a_{333\gamma}=0$) and define $c_{\alpha\beta\gamma\delta}:=a_{\alpha\beta\gamma\delta}-a_{\alpha\beta33}a_{\gamma\delta33}(a_{3333})^{-1}$.

Under the well known denotations this theory will be described by the following relations.

- *The kinematic constrains:*

$$u_\alpha(x,z,t)=-zw,_\alpha(x,t), \qquad u_3(x,z,t)=w(x,t), \tag{1}$$

where $w(x,t)$ are displacements of points of the midplane assumed in the form

$$w(x,t)=W(x,t)+g^A(x)V^A(x,t), \tag{2}$$

where functions $W(x,t)$, $V^A(x,t)$ are macrodeflections and inhomogeneity correctors, respectively, which are macrofunctions (cf. (1)). Moreover, $g^A(x)$ are postulated *a priori* microshape functions defining the class of disturbances of the plate deflections.

- *The strain-displacement equations:*

$$e_{\alpha\beta} = u_{(\alpha,\beta)} + \tfrac{1}{2}W_{,\alpha}W_{,\beta}, \tag{3}$$

where the non-linear term depends only on macrodeflections.

- *The stress-strain relations for the plane stress:*

$$s_{\alpha\beta}=c_{\alpha\beta\gamma\delta}e_{\gamma\delta}, \quad s_{33}=0. \tag{4}$$

- *The equation of motion (weak form):*

$$\begin{aligned}&\int_{\Pi}\int_{-h/2}^{h/2}\rho\ddot{u}_i\delta u_i dz da + \int_{\Pi}\int_{-h/2}^{h/2}(s_{\alpha\beta}\delta e_{\alpha\beta} + 2s_{\alpha 3}\delta e_{\alpha 3})dz da = \\ &= \int_{\Pi}[p^{+}\delta u_3(x,\tfrac{h}{2}) + p^{-}\delta u_3(x,-\tfrac{h}{2})]da + b\int_{\Pi}\int_{-h/2}^{h/2}\rho\delta u_3 dz da,\end{aligned} \tag{5}$$

which has to be satisfied for every admissible virtual fields δu and $\delta e_{\alpha\beta}$ restricted by (1), (2), (3).

For microperiodic plates from Eqs (1)÷(5) the governing equations of the refined (structural) theory were derived in (1). At the some time *the macromodelling hypothesis* was applied, which states that for every macrofunction F in calculation of averages over Δ terms $O(\varepsilon_F)$ can be neglected, where ε_F is a computational accuracy parameter related to an arbitrary macrofunction F.

In the subsequent section the governing equations of the structural theory for microperiodic plates with forces in the midplane will be obtained.

3. The governing equations

Under denotations

$$\mu \equiv \int_{-h/2}^{h/2}\rho dz, \quad j \equiv \int_{-h/2}^{h/2}\rho z^2 \rho dz, \quad d_{\alpha\beta\gamma\delta} \equiv \int_{-h/2}^{h/2} z^2 c_{\alpha\beta\gamma\delta}dz, \tag{6}$$

$$D_{\alpha\beta\gamma\delta} \equiv < d_{\alpha\beta\gamma\delta} >, \quad D^{A}_{\alpha\beta} \equiv < d_{\alpha\beta\gamma\delta}g^{A}_{,\gamma\delta} >, \quad D^{AB} \equiv < d_{\alpha\beta\gamma\delta}g^{A}_{,\alpha\beta}g^{B}_{,\gamma\delta} >, \tag{7}$$

where μ, j, $d_{\alpha\beta\gamma\delta}$ are Δ-periodic functions and using *the macromodelling hypothesis* we shall obtain

- *the constitutive equations*

$$\begin{aligned}M_{\alpha\beta} &= D_{\alpha\beta\gamma\delta}W_{,\gamma\delta} + D^{B}_{\alpha\beta}V^{B}, \\ M^{A} &= D^{A}_{\gamma\delta}W_{,\gamma\delta} + D^{AB}V^{B},\end{aligned} \tag{8}$$

- *the equations of motion*

$$\begin{aligned}M_{\alpha\beta,\alpha\beta} - N_{\alpha\beta}W_{,\alpha\beta} + <\mu>\ddot{W} - <j>\ddot{W}_{,\alpha\alpha} + \underline{<\mu g^{B}>\ddot{V}^{B}} - \underline{<jg^{B}_{,\alpha}>\ddot{V}^{B}_{,\alpha}} = p + b<\mu>, \\ M^{A} + \underline{<\mu g^{A}>\ddot{W}} + \underline{<jg^{A}_{,\alpha}>\ddot{W}_{,\alpha}} + \underline{<\mu g^{A}g^{B}>\ddot{V}^{B}} + \underline{<jg^{A}_{,\alpha}g^{B}_{,\alpha}>\ddot{V}^{B}} = \underline{b<\mu g^{A}>},\end{aligned} \tag{9}$$

where $N_{\alpha\beta}$ are membrane forces acting in the plate midplane. The underlined terms in (9) describe the length-scale effect on the dynamic stability of periodic plates. The new basis unknowns are macrodeflections W and inhomogeneity correctors V^A, A=1,..., N.

Let us consider a thin plate made of an isotropic homogeneous material and having the Δ-periodic thickness h. In this case under the denotation

$$B \equiv \frac{Eh^3}{12(1-\nu^2)},$$

where E, ν are the constant Young modulus and the constant Poisson ratio, respectively, from Eq $(6)_3$ and Eq $(7)_1$ we obtain

$$D_{\alpha\beta\gamma\delta} = < B[\delta_{\alpha\gamma}\delta_{\beta\delta}(1-\nu) + \delta_{\alpha\beta}\delta_{\gamma\delta}\nu] >,$$

and after substituting the right-hand sides of Eqs. (8) to Eqs. (9) the resulting system of governing equations can be written in the form

$$\begin{aligned} &< B > W_{,\alpha\alpha\beta\beta} - N_{\alpha\beta}W_{,\alpha\beta} + < \mu > \ddot{W} - < j > \ddot{W}_{,\alpha\alpha} + \\ &+ D^B_{\alpha\beta}V^B_{,\alpha\beta} + \underline{< \mu g^B >} \ddot{V}^B - \underline{< j g^B_{,\alpha} >} \ddot{V}^B_{,\alpha} = p + b < \mu >, \\ &D^A_{\gamma\delta}W_{,\gamma\delta} + \underline{< \mu g^A >} \ddot{W} + \underline{< j g^A_{,\alpha} >} \ddot{W}_{,\alpha} + \\ &+ D^{AB}V^B + \underline{< \mu g^A g^B >} \ddot{V}^B + \underline{< j g^A_{,\alpha} g^B_{,\alpha} >} \ddot{V}^B = b \underline{< \mu g^A >}. \end{aligned} \qquad (10)$$

The above equations constitute the basis for subsequent investigations.

4. Applications

In the order to estimate the length-scale effect on the dynamic stability of a microperiodic plate we will investigate a stability of the rectangular isotropic homogeneous plate, simply supported on the opposite edges, and having the Δ-periodic thickness. We will assume that body force b can be neglected. To simplify the model only one microshape function $g(x_1,x_2)=g^1(x_1,x_2)=l^2[\cos(2\pi x_1/l_1)\cos(2\pi x_2/l_2)+c]$, which satisfies the condition $<\mu g>=0$ will be introduced. It can be shown that $<jg_{,1}>=0$. Hence, setting $V \equiv V^1$, $D_{\alpha\beta} \equiv D^A_{\alpha\beta}$, $D \equiv D^{AB}$ equations (10) take the form

$$\begin{aligned} &< B > W_{,\alpha\alpha\beta\beta} - N_{\alpha\beta}W_{,\alpha\beta} + < \mu > \ddot{W} - < j > \ddot{W}_{,\alpha\alpha} + D_{\alpha\beta}V_{,\alpha\beta} = p, \\ &D_{\alpha\beta}W_{,\alpha\beta} + DV + \underline{< \mu(g)^2 >} \ddot{V} + \underline{< j(g_{,\alpha}) >} \ddot{V} = 0. \end{aligned} \qquad (11)$$

We can show that $D_{12}=D_{21}=0$. Solutions to Eqs. (11) can be assumed in the form

$$\begin{aligned} W &= \sum_{m=1,n=1}^{\infty} a_{mn} \sin(\alpha_m x_1)\sin(\beta_n x_2)\cos(\omega t), \\ V &= \sum_{m=1,n=1}^{\infty} b_{mn} \sin(\alpha_m x_1)\sin(\beta_n x_2)\cos(\omega t), \end{aligned} \qquad (12)$$

provided that external loads are given by

$$p = \sum_{m=1,n=1}^{\infty} p_{mn} \sin(\alpha_m x_1)\sin(\beta_n x_2)\cos(\omega t), \tag{13}$$

where $\alpha_m \equiv m\pi/L_1$, $\beta_n \equiv n\pi/L_2$, $m,n=1, 2, 3,\dots$ and ω is a vibration frequency. Substituting the right-hand sides of formulae for W, V, p into Eqs. (11) and assuming that $N_{12}=0$ we obtain the system of linear algebraic equations for coefficients a_{mn}, b_{mn}

$$\begin{aligned}&\{<B>[(\alpha_m)^2+(\beta_n)^2]^2+N_{11}(\alpha_m)^2+N_{22}(\beta_n)^2+\\&-\{<\mu>+<j>[(\alpha_m)^2+(\beta_n)^2]\}\}\omega^2 a_{mn}-[D_{11}(\alpha_m)^2+D_{22}(\beta_n)^2]b_{mn}=p_{mn},\\&-[D_{11}(\alpha_m)^2+D_{22}(\beta_n)^2]a_{mn}+\{D-[\underline{<\mu(g)^2>}+\underline{<j(g_{,\alpha})^2>}]\omega^2\}b_{mn}=0.\end{aligned} \tag{14}$$

Denoting by

$$N_1 \equiv -N_{11}, \quad N_2 \equiv -N_{22}, \tag{15}$$

the critical values of forces in the plate midplane the criterion of the loss of dynamic stability can be written in the form

$$\begin{vmatrix} \begin{matrix}<B>[(\alpha_m)^2+(\beta_n)^2]^2-N_1(\alpha_m)^2-N_2(\beta_n)^2+\\-\{<\mu>+<j>[(\alpha_m)^2+(\beta_n)^2]\}\omega^2\end{matrix} & -[D_{11}(\alpha_m)^2+D_{22}(\beta_n)^2] \\ -[D_{11}(\alpha_m)^2+D_{22}(\beta_n)^2] & D-[\underline{<\mu(g)^2>}+\underline{<j(g_{,\alpha})^2>}]\omega^2 \end{vmatrix}=0. \tag{16}$$

From the above formula, assuming that the critical forces are acting only in the x_1-axis direction ($N_2=0$) we will obtain the following formula for N_1

$$\begin{aligned}N_1=&<B>\frac{[(\alpha_m)^2+(\beta_n)^2]^2}{(\alpha_m)^2}-\frac{<\mu>+<j>[(\alpha_m)^2+(\beta_n)^2]}{(\alpha_m)^2}\omega^2+\\&-\frac{[D_{11}(\alpha_m)^2+D_{22}(\beta_n)^2]^2}{(\alpha_m)^2[D-(\underline{<\mu(g)^2>}+\underline{<j(g_{,\alpha})^2>})\omega^2]},\end{aligned} \tag{17}$$

which takes into account the length-scale effect in the dynamic stability. This effect is described by the underlined terms.

The above analysis was carried out in the framework of the structural theory. In the order to pass to the local theory we will neglect the underlined terms in Eqs. (11). After some manipulations the formula for the critical force $\tilde{N}_1$ in the x_1-axis direction takes the form

$$\begin{aligned}\tilde{N}_1=&<B>\frac{[(\alpha_m)^2+(\beta_n)^2]^2}{(\alpha_m)^2}-\frac{<\mu>+<j>[(\alpha_m)^2+(\beta_n)^2]}{(\alpha_m)^2}\omega^2+\\&-\frac{[D_{11}(\alpha_m)^2+D_{22}(\beta_n)^2]^2}{(\alpha_m)^2 D}.\end{aligned} \tag{18}$$

We can observe qualitative differences between formulae (17) and (18) for the value of a critical force. More detailed analysis of the obtained results will be investigated in a separate paper.

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