Mathematical model of a multi-parameter oscillator based on a core-less three-phase linear motor with skewed magnets (MTR070-15)

Jakub Gajek, Radosław Kępiński, Jan Awrejcewicz

Abstract: This paper uses the example of a three-phase core-less linear motor to create a mathematical model of single-dimension multi-parameter oscillator. The studied linear motor consists of: a stator, an U-shaped stationary guide-way with permanent magnets placed askew to the motor's movement's direction; and a forcer, a movable set of three rectangular coils subjected to alternating external electrical voltage. The system's parameters are both mechanical (number of magnets and coils, size of magnets, distances between magnets, size of coils) and electromagnetic (auxiliary magnetic field, permeability, coil's resistance). Lorentz force allows for the transition from electromagnetic parameters to mechanical force and Faraday's law of induction creates a feedback between the forcer's speed and coils voltage. An Ampere's model of permanent magnet is used to determine the simplified function of auxiliary magnetic field distribution throughout the stator. In the model the external voltage applied to each coil serves as the excitation while displacement of the forcer is the output parameter. The solution to the introduced mathematical model of the system is compared with the experimental results showing a good coincidence.

1. Introduction

Over the years linear motors have been taking on bigger and bigger shares in the market for precise positioning systems. They provide a dynamically superior although costly alternative to standard drives such as feed screw conveyors. For a motor to correctly project a desired motion profile a specific close-looped controller should be introduced between the motor and mains. The quality of such system depends on, amongst other things, the precision of motor's model used for building the controller. Most industrial controllers today use simplified models while this paper focuses on construction of a model for a simple core-less motor from scratch.

A lab stand, with HIWIN's coreless linear motor and Copley-Controls' servo-drive, is used as both a physical base for constructing the model and a platform for its validation. The forcer (inductor) is capable of moving at speeds of up to 5 meters per second with a load of 45N. An analogue optical linear encoder can read the forcer's position with a resolution of 0,1 μm and the entire system's positioning error not trailing far behind. Three U-shaped stators with permanent magnets were used to create the motors magnetic guide-way giving the stand a theoretical maximum stroke of 830 mm and an actual stroke of 780 mm. Two high quality linear guide-way provide for a swift and quiet motion. The stand

program allows it to work with both manual input and automatically based on a signal from external devices, [1].
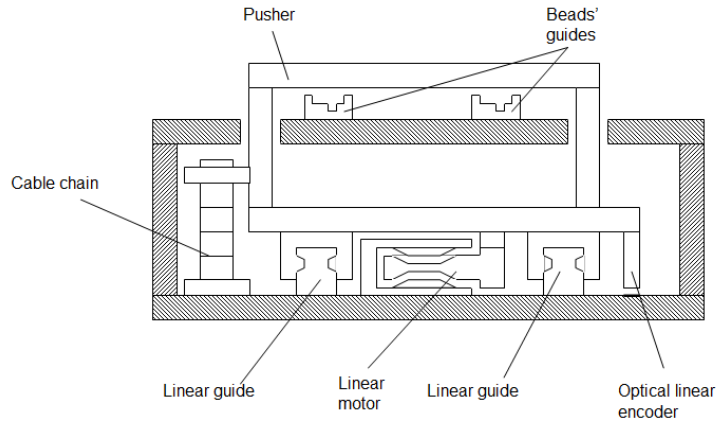

Figure 1. Lab stand's diagram

As the most interesting scientifically only the motor will be modeled in this paper. The stand will serve merely as a validation platform.

2.   Model construction

To construct the motor model a simple base model, consisting of a single winding and single magnet, was built first. The force acting on such a coil was calculated with respect to its position and of magnetic field strength. The field's distribution for a single magnet and an infinitely long guide-way was then evaluated using Ampere's model. The entire model for three windings inside the guide-way is then presented and encapsulated to a single ODE.

2.1.   Base model

A single motor winding can be modeled as a perfectly rectangular conductor loop, with a certain voltage function U_g applied to it. The means and exact spot of this application is omitted as unimportant for the workings of the model. Let this loop be placed in the vicinity of a C-shaped magnet in a way depicted in Figure 2.
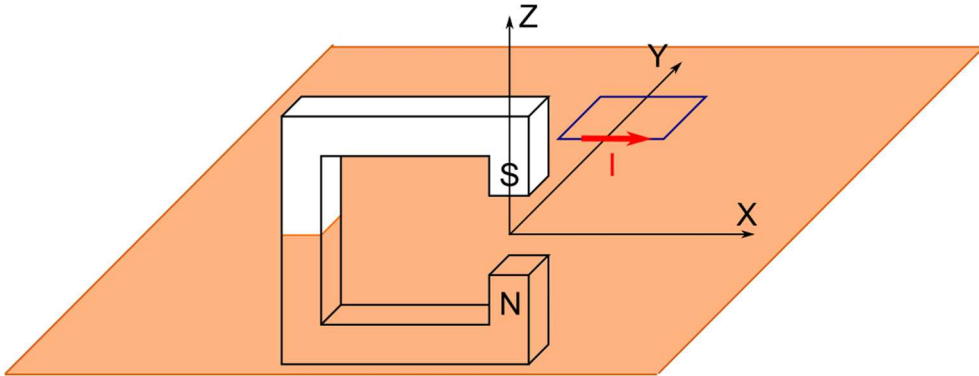
Figure 2. Model of a single winding

Assuming that the loop cannot deform or rotate and can only move along the direction of y-axis it shall always remain a rectangle with a center placed on y-axis and with shorter sides parallel to the direction of motion. Every infinitely small slice of the coil is subjected to Lorentz force acting perpendicularly to its side (see Figure 3).
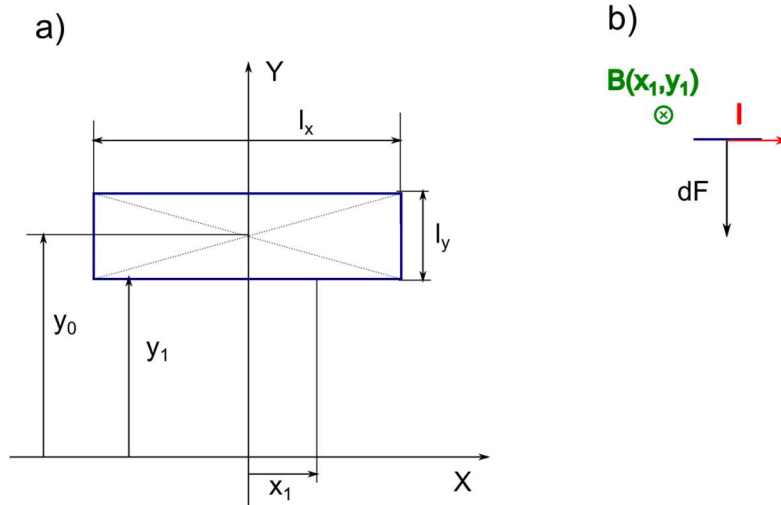

Figure 3. Single loop overview: (a) - coordinate system; (b) - Lorentz force of the elemental slice

The value of that force equals

$$d\vec{F} = \vec{B} \times \vec{\iota},$$ (1)

$$d\vec{F} = I\vec{B} \times d\vec{l},$$ (2)

where i and B are the vector of electric current and magnetic induction respectively. In every point of a conductor loop the i vector's direction is parallel to the coil while its value can be treated as constant equal to I.

Since the model cannot move in any direction other than along y-axis, the only significant component of the Lorentz force is one parallel to that axis. It is defined as

$$dF' = dF \cos \phi = B'Idl \cos \phi, \tag{3}$$

where ϕ is the angle between dl and the direction of motion. After integration, the equation takes the form

$$F = \oint_L B'I \cos \phi \, dl. \tag{4}$$

In equation (4) F is the y-component of total force acting on the coil, B' is the z-component of magnetic field acting on dl, and L is the conductor loop's path. The above method can be used for calculating the force working on any other closed loop conductor coils as well. For a specific case of rectangular shape the closed-loop integral can be rewritten as a sum of two definite integrals in the following form

$$F = I \left(\int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} B\left(x, y_0 - \frac{l_y}{2}\right) dx - \int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} B\left(x, y_0 + \frac{l_y}{2}\right) dx \right), \tag{5}$$

where $B(x, y)$ is the function of the magnetic field's z-component distribution, [2].

The electric current I is the result of the external voltage function $U_G(t)$ and the voltage U_I induced in the loop due to Faraday's law of induction. The second component can be calculated as

$$U_I = -\frac{d}{dt} \iint_S B(x, y) dS, \tag{6}$$

where S is the area inside the conductor loop. For a rectangular coil (6) can be recast to the following form

$$U_I = -\frac{d}{dt} \int_{-\frac{l_y}{2}}^{\frac{l_y}{2}} \int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} B(x, y) dx dy. \tag{7}$$

Equation (5) can then be rewritten in the form

$$F = \frac{1}{R_e}\left(U_G(t) - \frac{d}{dt}\int\limits_{-\frac{l_y}{2}}^{\frac{l_y}{2}}\int\limits_{-\frac{l_x}{2}}^{\frac{l_x}{2}} B(x,y)dxdy\right)\left(\int\limits_{-\frac{l_x}{2}}^{\frac{l_x}{2}} B\left(x, y_0 - \frac{l_y}{2}\right)dx - \int\limits_{-\frac{l_x}{2}}^{\frac{l_x}{2}} B\left(x, y_0 + \frac{l_y}{2}\right)dx\right), \tag{8}$$

where R_e is the electric resistance of the coil.

2.2. C-Shape magnets' field distribution

With accordance to [3] the value of magnetic field of a permanent magnet can be approximated by using Ampere's model, that is by assuming that a magnet's magnetic field is the same as that of a perfect, tightly wound solenoid. Then Biot-Savart's law can be applied to calculate the exact value of magnetic field at any point P in the vicinity of the magnet via the following formula

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int\limits_L I\, \frac{d\vec{l} \times \vec{r'}}{\left|\vec{r'}\right|^3}, \tag{9}$$

where

$$\vec{r'} = \vec{r} - \vec{l}, \tag{10}$$

and $\vec{r}$ is the distance between point P and the center of the magnet, $\vec{l}$ is the distance between the magnet's center and the infinitely small solenoid length dl and μ_0 is the magnetic permeability of the magnet's environment.

In case of a C-shaped magnet the field can be calculated as a resultant fields of two bar magnets placed perpendicular to each other with opposing poles facing each other. For a single bar magnet the field can be then calculated by solving the following integral equation

$$B(x_p, y_p, z_p) = \frac{\mu_0 i}{4\pi}\left(B_1(x_p, y_p, z_p) + B_2(x_p, y_p, z_p) + B_3(x_p, y_p, z_p) + B_4(x_p, y_p, z_p)\right), \tag{11}$$

where

$$B_1(x_p, y_p, z_p) = \int\limits_{-\frac{\sigma_z}{2}}^{\frac{\sigma_z}{2}}\int\limits_{-\frac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} \frac{(z - z_p)\hat{y} + \left(y_p - \frac{\sigma_y}{2}\right)\hat{z}}{\sqrt{(x_p - x)^2 + \left(y_p - \frac{\sigma_y}{2}\right)^2 + (z_p - z)^2}^{\,3}}\, dxdz, \tag{12}$$

$$B_2(x_p, y_p, z_p) = -\int\limits_{-\frac{\sigma_z}{2}}^{\frac{\sigma_z}{2}}\int\limits_{-\frac{\sigma_x}{2}}^{\frac{\sigma_x}{2}} \frac{(z - z_p)\hat{y} + \left(y_p + \frac{\sigma_y}{2}\right)\hat{z}}{\sqrt{(x_p - x)^2 + \left(y_p + \frac{\sigma_y}{2}\right)^2 + (z_p - z)^2}^{\,3}}\, dxdz, \tag{13}$$

189

$$B_3\left(x_p, y_p, z_p\right) = \int\limits_{-\frac{\sigma_z}{2}}^{\frac{\sigma_z}{2}} \int\limits_{-\frac{\sigma_y}{2}}^{\frac{\sigma_y}{2}} \frac{\left(z - z_p\right)\hat{x} + \left(x_p - \frac{\sigma_x}{2}\right)\hat{z}}{\sqrt{\left(x_p - \frac{\sigma_x}{2}\right)^2 + \left(y_p + y\right)^2 + \left(z_p - z\right)^2}^{\,3}}\, dydz, \tag{14}$$

$$B_4\left(x_p, y_p, z_p\right) = -\int\limits_{-\frac{\sigma_z}{2}}^{\frac{\sigma_z}{2}} \int\limits_{-\frac{\sigma_y}{2}}^{\frac{\sigma_y}{2}} \frac{\left(z - z_p\right)\hat{x} + \left(x_p + \frac{\sigma_x}{2}\right)\hat{z}}{\sqrt{\left(x_p + \frac{\sigma_x}{2}\right)^2 + \left(y_p + y\right)^2 + \left(z_p - z\right)^2}^{\,3}}\, dydz, \tag{15}$$

and (x_p, y_p, z_p) are the coordinates of point P, $\sigma_x, \sigma_y, \sigma_z$ are the bar magnet dimensions in their respective axes, $\hat{x}, \hat{y}, \hat{z}$ are the unit vectors of axes x, y and z and i is the current density over the diameter of wire of the solenoid.

Figures 4-5 show the C-shaped magnet's magnetic field's z-component distribution in a yz-plane calculated with equations (11) - (15) for a sample magnet.
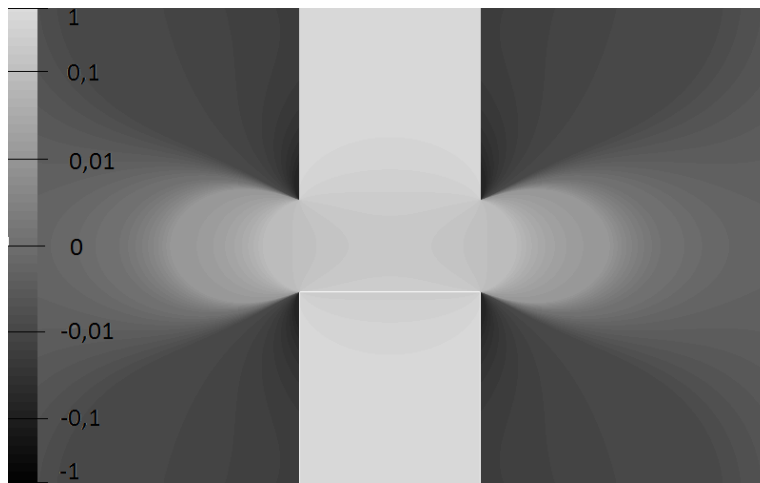

Figure 4. Z-axis coefficient of magnetic field distribution of
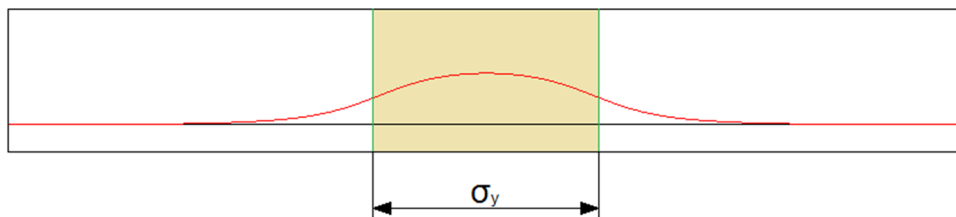a C-shaped permanent magnet (in Teslas)

Figure 5. Distribution of C-shaped permanent magnet's magnetic field (z-axis coefficient) on a line
ran parallel to y-axis in between the magnet's poles, highlighted part marks
the position of magnet's poles

For the ease of use and because of the good coincidence with actual values, the magnetic field's distribution on a xy-plane (coordinate system set as in Figure 3) will be approximated with 2D Gaussian function in the following form

$$B_c(x, y) = B_0 e^{-\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)}, \tag{16}$$

where B_0 is the given magnet's constant.

2.3. Magnetic guide-way field distribution

An infinitely long magnetic guide-way can be modeled as a set of C-shape magnets placed in equal distance χ_T from one another along the y-axis. All of the magnets are placed askew from the x-axis by the angle ϕ and each two neighboring ones have their poles set oppositely (see Figure 6).
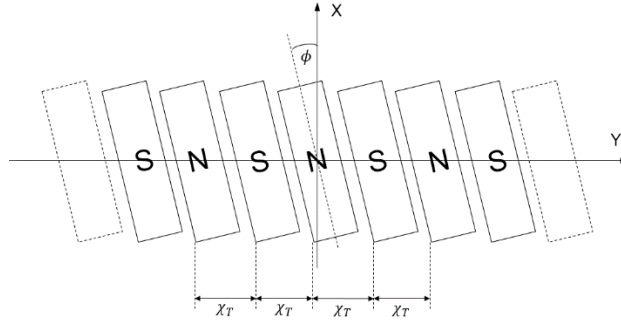

Figure 6. Magnetic guide-way modeled as infinite set of C-shaped magnets

For such a guide-way the magnetic field distribution can be written as

$$B_G(x, y) = B_0 e^{-\left[\frac{x^2}{\sigma_{xs}^2}\right]} \sum_{n=-\infty}^{\infty} (-1)^n e^{-\left[\frac{(y+n\chi_T)^2}{\sigma_{ys}^2} + \frac{x(y+n\chi_T)}{\sigma_{xys}^2}\right]}, \tag{17}$$

where

$$\frac{1}{\sigma_{ys}^2} = \frac{\sigma_y^2 sin(\phi)^2 + \sigma_x^2 cos(\phi)^2}{\sigma_y^2 \sigma_x^2}, \tag{18}$$

$$\frac{1}{\sigma_{xs}^2} = \frac{\sigma_y^2 cos(\phi)^2 + \sigma_x^2 sin(\phi)^2}{\sigma_y^2 \sigma_x^2}, \tag{19}$$

$$\frac{1}{\sigma_{xys}^2} = \frac{sin(2\phi)(\sigma_y^2 + \sigma_x^2)}{\sigma_y^2 \sigma_x^2}, \tag{20}$$

are the relative pole sizes for tilted magnets.

After introducing a correlation factor between the relative pole size in y (σ_{ys}) and magnets' displacement (χ_T) in the form

$$k = \frac{\chi_T}{\sigma_{ys}}, \tag{21}$$

the equation (17) can be rewritten in the form

$$B_G = B_{G0} e^{-\left(\frac{x^2}{\sigma^2}\right)} \Theta(x, y), \tag{22}$$

where

$$\frac{1}{\sigma^2} = \left(\frac{\sigma_y^2}{4\sigma_{xy}^4} - \frac{1}{\sigma_x^2}\right), \qquad B_{G0} = \frac{B_0\sqrt{\pi}}{2k}, \tag{23}$$

and $\Theta(x, y)$ is the sum of two Jacobi theta functions of the third kind ($\vartheta_3(z, \tau)$) in the form

$$\Theta(x, y) = \left[\vartheta_3\big(xz_x + yz_y, \tau_M\big) - \vartheta_3\left(xz_x + yz_y - \frac{1}{2}, \tau_M\right)\right], \tag{24}$$

where

$$z_x = \frac{\pi\sigma_{ys}}{4\sigma_{xys}^2}, \qquad z_y = \frac{\pi}{2k\sigma_{ys}}, \qquad \tau_M = e^{-\frac{\pi^2}{4k^2}}. \tag{25}$$

2.4. Three-phase motor model

For a set of three same size, stiffly connected, rectangular conductor loops inside the magnetic guide-way the total force acting on this set can be calculated as a sum of forces acting on each coil. That is

$$F_M = \sum_{j=1}^{3} F_j, \tag{26}$$

where F_j is calculated with (8) assuming the filed distribution is equal to (22). This single force can be written as

$$F_j = \frac{B_{G0}}{R_e}\big(U_{Gj}(t) - U_{ij}\big)\left(\int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} e^{-\frac{x^2}{\sigma^2}}\left(\Theta\left(x, y_j - \frac{l_y}{2}\right) - \Theta\left(x, y_j + \frac{l_y}{2}\right)\right) dx\right), \tag{27}$$

where y_j is the position of individual coil and is equal to

$$y_j = y_0 + a(j - 2), \tag{28}$$

and y_0 is the position of the motor's forcer and a is the displacement of coils in the forcer. The induced voltage for j-coil $\big(U_{ij}\big)$ is equal to

$$U_{ij} = -B_G z_y \dot{y} \int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} \int_{y_j-\frac{l_y}{2}}^{y_j+\frac{l_y}{2}} e^{-\left(\frac{x^2}{\sigma^2}\right)} \Theta'(x,y) dx dy. \tag{29}$$

The $\Theta'(x,y)$ is the sum of two Jacobi theta prime functions of the third kind

$$\Theta'(x,t) = \left[\vartheta_3'\left(xz_x + yz_y, \tau_M\right) - \vartheta_3'\left(xz_x + yz_y - \frac{1}{2}, \tau_M\right) \right]. \tag{30}$$

The complete model's ODE can be written as

$$\ddot{y}_0(t) = \frac{1}{m} \sum_{j=1}^{3} F_j, \tag{31}$$

where m is the mass of the forcer. (31) can also be written in developed form

$$\ddot{y}_0(t) = \frac{1}{mR_E} \sum_{j=1}^{3} \left[U_{Gj}(t) - B_G z_y \dot{y}_0 \int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} \int_{y_j-\frac{l_y}{2}}^{y_j+\frac{l_y}{2}} e^{-\left(\frac{x^2}{\sigma^2}\right)} \Theta'(x, y_0 + a(j-2)) dx dy \right]$$

$$\left[\int_{-\frac{l_x}{2}}^{\frac{l_x}{2}} e^{-\frac{x^2}{\sigma^2}} \left(\Theta\left(x, y_0 + a(j-2) - \frac{l_y}{2}\right) - \Theta\left(x, y_0 + a(j-2) + \frac{l_y}{2}\right) \right) \right]. \tag{32}$$

As the integration of Jacobi theta functions $\Theta(x,y)$ and Jacobi theta prime functions $\Theta'(x,y)$ over x and y are analytically insolvable, it is likewise only possible to solve (32) using numerical methods. The nature of Jacobi theta function makes the equation highly non-linear.

3. Validation

Based on the equation (32) a computer simulation was created and conducted in Wolfram Mathematica. The external voltage functions $U_{Gj}(t)$ were used as excitation while the position of the forcer $y_0(t)$ was the output parameter. The size and displacement of magnets and coils, magnet's skew angle and magnetic field constant and coils' resistance, were treated as constant parameters.

Mathematica gives a wide variety of possible excitation functions to be supplied to the model. Likewise the stand's servo-drive can be set in "maintenance mode" giving, amongst other options, a direct control over the motor from a PC desktop. This function function allows subjecting the coils to a given voltage function. The variety of functions available from the servo-drive manufacturers is scarce but sufficient. All of them have the form of

$$U_{Gj}(t) = U_M(t) \sin\left(\omega t + (j-1)\frac{2\pi}{3}\right),$$ (33)

with $U_M(t)$ (maximum voltage) and ω (angular frequency) changeable in time along a step, a sawtooth or a sinusoidal function. For the purpose of validation a step function of maximum voltage was used. The computer model was supplied with the following set of parameters, taken from the motor's documentation as well as from direct measurement, so that they resemble the actual motor as closely as possible.

Table 1 - Parameters used for the model

Parameter	Symbol	Value
Coils' width	lx	5,2 [mm]
Coils' length	l_y	49 [mm]
Coils' displacement	a	7,36 [mm]
Magnets' field strength	B_0	1,5 [T]
Magnets' width (x-axis)	σx	4,7 [mm]
Magnets' length (y-axis)	σ_y	53 [mm]
Electrical resistance	R_e	6,7 [Ω]
Magnets' displacement correlation	k	2,35 [-]
Forcer's mass	m	0,31 [kg]
Maximum voltage	U_{M0}	1,5 [V]
Voltage frequency	ω	20 [rad/s]
Voltage step function frequency	f	1 [Hz]

Figure 6 displays the function of maximum voltage and the resulting voltage on one of the coils taken from the simulation.
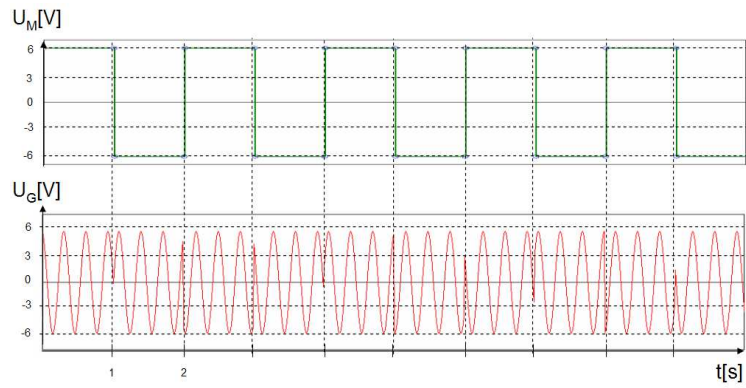
Figure 7. Function of maximum voltage (top most graph) and voltage
on the first coil (bottom most graph)

Exactly the same function of voltage was applied to actual motor coils. The result of both the experiment and simulation are presented below.
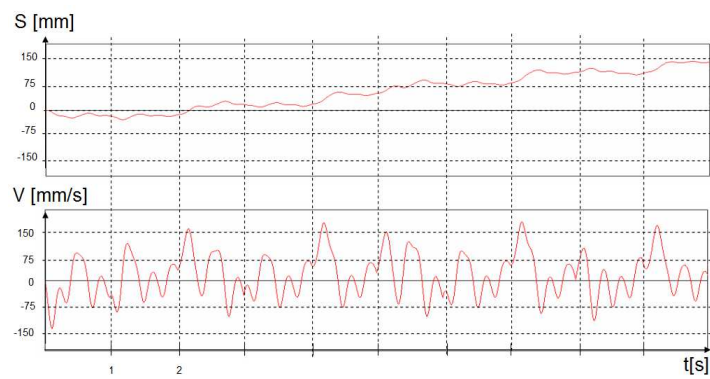

Figure 8. Response of computer simulation:
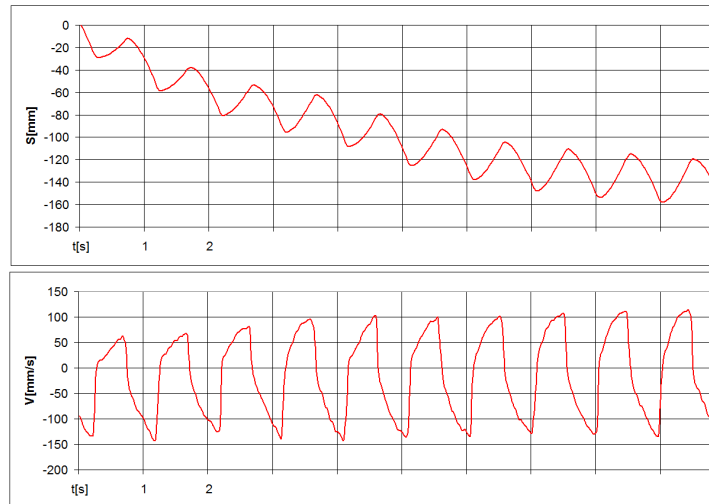the position and velocity of inductor

Figure 9. Response of the actual motor

In light of the result above further validation was abandoned.

4. Conclusions

The prepared and studied model does not yet reassemble actual motor with satisfactory precision. Most likely cause of this is the usage of documentation data for parameter identification. Instead the parameters should be identified with numerical methods. The model clearly expresses interesting chaotic behaviors, however the complexity of Jacobi theta functions necessitate a largely time consuming simulations. A further simplification of the model might be required to better study it.

References

[1] Gajek J.: *Construction of a laboratory stand based on the PVC Glazing beads saw feeder design*, M.Sc. thesis, Lodz University of Technology, 2012, Lewandowski D.

[2] Chari M. V. K., *Finite elements in electrical and magnetic field problems*, Somerset, John Wiley & Sons, 1980

[3] DERBY N.; OLBERT S.: Cylindrical magnets and ideal solenoids, *American Journal of Physics*, Volume 78, Issue 3, 2010, pp. 229-235

Jakub Gajek, M.Sc. (Ph.D. student): Lodz University of Technology, 90-924, POLAND (gajek.jakub@gmail.com), the author presented this work at the conference.

Radosław Kępiński, M.Sc. (Ph.D. student): Lodz University of Technology, 90-924, POLAND (radoslaw.kepinski@dokt.p.lodz.pl).

Jan Awrejcewicz, Professor: Lodz University of Technology, 90-924, POLAND (jan.awrejcewicz@p.lodz.pl).