Double pendulum colliding with a rough obstacle
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Abstract: The externally excited and damped vibrations of the double pendulum in the vertical plane are considered. The pendulum can collide with a rough obstacle many times during its motion. The pendulum is modeled as a piecewise smooth system. The differential equations govern the motion of the system in the relatively long time between the collisions. When a contact with the obstacle occurs, the pendulum exhibits a discontinuous behaviour. The velocities of both parts of the pendulum and the reaction forces are changing stepwise. An important element of the solving algorithm is aimed on the continuous tracking of the position of the pendulum in order to detect the collision with the unilateral constraints and to determine the state vector of the pendulum at the impact time instant. A single collision is described by the Euler's laws of motion in the integral form. The equations are supplemented by the Poisson's hypothesis and Coulomb's law of friction. The friction law is formulated for the instantaneous values of the reaction forces. The values of their impulses depend on the existence of a slip between the contacting bodies. Furthermore, during the collision the dynamic behaviour may change. Therefore the Coulomb law cannot be generalized for the linear impulses of the forces in a simple way. We have applied the Routh method in order to solve the problem. The method has a simple geometrical interpretation in the impulse space.

1. Introduction

It is well known that in many mechanical engineering systems, a contact between individual elements is not continuous. This may be an intended effect or a result of gradual destruction of the mechanical parts. Regardless of the cause, the collisions between various elements entail significant increase of the dynamic loads what leads to the greater destruction of the colliding elements.

In many papers dealing with collisions in multi-body systems consisting of rigid bodies, the influence of friction forces on the kinematic state of the bodies after impact is omitted. An approach to three-dimensional analysis of the collision of rigid bodies with friction was originally formulated by Routh [1]. In recent years many researchers rediscover the Routh model [2], analysing it in detail [4] and also applying it in modelling of multi-body systems with impacts [5]. On the base of Routh's model some more advanced models have been recently developed [3].

In this paper the collision of a relatively simple system, which is a double pendulum, with a fixed obstacle is investigated. The friction forces accompanying the collision are taken into account and the Routh model is employed.

2. Model of the system

The behaviour of a double pendulum colliding with an obstacle is considered (see Figure 1). The system is constrained to move in a fixed vertical plane. The pendulum, suspended at the point *O*, consists of two rigid bodies linked by a joint *A*. One of the bodies is a rod of mass m_1 and length l_1. The second part of the pendulum is composed of a rod of length l_2 and a disc of radius *r*. The rod and the disc are fixed connected, and their common mass is equal to m_2. C_1 and C_2 are the mass centres of the both parts. I_1 and I_2 denote the moments of inertia of the parts with respect to the axes which pass through the points C_1 and C_2, respectively, and are perpendicular to the plane of motion. There is viscous damping at the both joints with a priori known coefficients c_1 and c_2. Moreover, two torques $M_1(t) = M_{01}\cos(\Omega_1 t)$ and $M_2(t) = M_{02}\cos(\Omega_2 t)$ are taken into consideration.

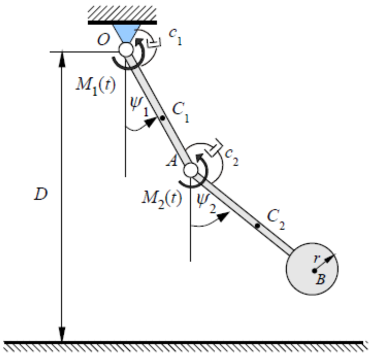

Figure 1. Double pendulum colliding with a rough obstacle.

The angles ψ_1 and ψ_2 are chosen as the general coordinates. The obstacle is motionless, and its surface is plane and rough. We assume that the distance *D* between the obstacle and the point *O* fulfils the following conditions

$$l_1 < D < l_1 + l_2 + 2r. \tag{1}$$

So, the motion of the pendulum is constrained by unilateral constraint

$$D - l_1\cos(\psi_1) - (l_2 + r)\cos(\psi_2) - r \geq 0, \tag{2}$$

and only the disk placed at the end of the pendulum may come into contact with the rough obstacle.

3. Mathematical model

Observe that the unilateral constraint (2) does not act in relatively long intervals. From the point of view of the slow time scale which is suitable to observe the pendulum motion, they become active only in some time instants. When the contact with the obstacle occurs, the pendulum exhibits discontinuous behaviour. The velocities of both parts of the pendulum as well as the reaction forces at the joints, and in the contact point with the obstacle exhibit stepwise changes. The pendulum, the motion of which is constrained by inequality (2) will be treated as piecewise smooth system. In such an approach, all the intervals in which kinematic and dynamic quantities change continuously are separated from each other by some time instants t_1, t_2,…, in which some from them change in stepwise way.

3.1. Motion between collisions

In each relatively long term between the collisions the pendulum motion is determined by differential equations and initial conditions. The equations of motion of the pendulum derived from the Lagrange equations of the second kind and written in matrix notation are as follows

$$\mathbf{A}\cdot\mathbf{\Psi}=\mathbf{Q}, \tag{3}$$

where:

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12}(\psi_1,\psi_2) \\ a_{21}(\psi_1,\psi_2) & a_{22}\end{bmatrix},\quad \mathbf{\Psi}=\begin{bmatrix}\ddot{\psi}_1\\ \ddot{\psi}_2\end{bmatrix},\quad \mathbf{Q}=\begin{bmatrix} q_1(t,\psi_1,\psi_2,\dot{\psi}_1,\dot{\psi}_2) \\ q_2(t,\psi_1,\psi_2,\dot{\psi}_1,\dot{\psi}_2)\end{bmatrix},$$

$$a_{11}=I_1+l_1^2\left(\frac{m_1}{4}+m_2\right),\quad a_{22}=I_2+l_c^2 m_2,\quad a_{12}=a_{21}=l_1 l_c m_2\cos(\psi_1-\psi_2),$$

$$q_1=M_{01}\cos(\Omega_1 t)-(c_1+c_2)\dot{\psi}_1+c_2\dot{\psi}_2-l_1 l_c m_2\sin(\psi_1-\psi_2)\dot{\psi}_2^2-\left(\frac{m_1}{2}+m_2\right)g l_1\sin\psi_1,$$

$$q_2=M_{02}\cos(\Omega_2 t)+c_2(\dot{\psi}_1-\dot{\psi}_2)+l_1 l_c m_2\sin(\psi_1-\psi_2)\dot{\psi}_1^2-m_2 g l_c\sin\psi_2,$$

where $l_c=AC_2$.

The configuration of the pendulum at time instant t_i and its kinematic state after i-th collision are assumed as the initial values for the motion in (i+1)-th interval. Taking into account also the conditions at $t = 0$, we can write

$$\mathbf{\Psi}(t_i)=\mathbf{\Psi}_{0i}\quad \dot{\mathbf{\Psi}}(t_i)=\mathbf{\Theta}_{0i}\quad i=0,1,2,... \tag{4}$$

The vectors $\mathbf{\Psi}_{00},\mathbf{\Theta}_{00}$ are known, while both the vectors $\mathbf{\Psi}_{0i},\mathbf{\Theta}_{0i}$ (for $i>0$) are determined, using the Routh method, for each of the collision event.

Each of the initial value problems (3) with appropriate initial conditions (4) is solved numerically using the fourth order Runge-Kutta method. Usefulness of the method as well as the choice of the appropriate time step h have been tested numerically.

3.2. Kinematic aspect of the collision

An important element of the algorithm used to solving the initial problem (3)-(4) is the continuous tracking of the position of the pendulum in order to detect the collision with the obstacle. The collision occurs when

$$f(\psi_1,\psi_2) = D - l_1\cos(\psi_1) - (l_2 + r)\cos(\psi_2) - r = 0. \tag{5}$$

Every time when function f changes its sign the Runge-Kutta algorithm is discontinued. The function f in equation (5) is interpolated linearly in the found interval of length h what allows to determine approximately the instant t_i of starting of i-th collision. Then the configuration of the pendulum as well as all necessary velocities at the instant t_i may be calculated.

3.3. Collision laws

Modelling the collision, we assume commonly used assumptions. All displacements and rotations during the collision are negligible. Impulses of contact forces are bounded, and impulses of all other forces may be neglected. The colliding bodies remain rigid and interact on each other only at one point. In time scale adapted to motion of the pendulum the duration of the collision is very short. However, during the collision velocities, accelerations and forces change, and these changes require to be described in a different time scale. We introduce the fast scale time τ suitable for collision modeling. For each collision, the time τ starts from 0.

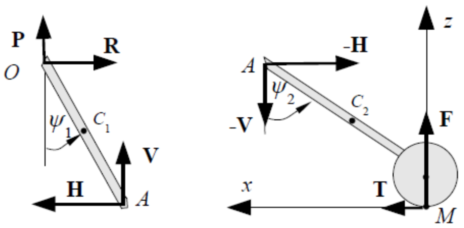

Figure 2. Forces taken into account during collision.

Integrating the Euler laws of motion with respect to time τ, and taking into account only the forces at the contact point M, the internal forces at the joint A, and the reaction forces at the joint O (see Figure 2), we can write

$$m_1(v_{1x}(\tau) - v_{1x}(0)) = \overline{H} - \overline{R}, \tag{6}$$

$$m_1(v_{1z}(\tau) - v_{1z}(0)) = \overline{V} + \overline{P}, \tag{7}$$

$$m_2(v_{2x}(\tau) - v_{2x}(0)) = \overline{T} - \overline{H}, \tag{8}$$

$$m_2(v_{2z}(\tau)-v_{2z}(0))=\overline{F}-\overline{V}, \quad (9)$$

$$I_1(\omega_{1y}(\tau)-\omega_{1y}(0))=(\overline{V}-\overline{P})\frac{l_1}{2}\sin(\psi_1)-(\overline{R}+\overline{H})\frac{l_1}{2}\cos(\psi_1), \quad (10)$$

$$I_2(\omega_{2y}(\tau)-\omega_{2y}(0))=(\overline{V}l_c+\overline{F}d)\sin(\psi_2)-(\overline{H}l_c+\overline{T}d)\cos(\psi_2)-\overline{T}r, \quad (11)$$

where: $d=l_2+r-l_c$, $\overline{F},\overline{T},\overline{H},\overline{V},\overline{R},\overline{P}$ are impulses of the components of the forces shown in Figure 2 (for instance $\overline{F}=\int_0^{\tau}F(\tilde{\tau})d\tilde{\tau}$), $v_{1x},v_{1z},v_{2x},v_{2z}$ are components of the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the mass centres C_1 and C_2, ω_{1y},ω_{2y} are components of the angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ of the bodies (perpendicular to the motion plane), whereas $v_{1x}(0),v_{1z}(0),v_{2x}(0),v_{2z}(0)$ and $\omega_{1y}(0),\omega_{2y}(0)$ are known.

Equations (6) - (11) are valid for any instant of time $\tau\in(0,\tau_f)$, whereas τ_f is the time instant in which the collision ends. Due to rigidity of the pendulum, there the following relationships hold

$$\mathbf{v}_1=\boldsymbol{\omega}_1\times\overline{OC_1}, \quad (12)$$

$$\mathbf{v}_2=\boldsymbol{\omega}_1\times\overline{OA}+\boldsymbol{\omega}_2\times\overline{AC_2}. \quad (13)$$

Therefore, among the six kinematic quantities in equations (6) - (11), only two are independent, for example $\omega_{1y}(\tau)$ and $\omega_{2y}(\tau)$.

Let us decompose the velocity at the contact point M into two components in directions of the axes of the collision reference frame shown in Figure 2 in the following way

$$\mathbf{v}_M=\mathbf{s}+\mathbf{c}=s\mathbf{i}+c\mathbf{k}, \quad (14)$$

where $\mathbf{s}=s\mathbf{i}$ is called the *sliding velocity*, and $\mathbf{c}=c\mathbf{k}$ is the *closing velocity*. The friction force **T** is modeled using the Coulomb law, and hence both the magnitude as well as the direction of the force **T** depend on the sliding velocity **s** at the contact point M, namely

$$s\neq 0 \;\Rightarrow\; T=-\mu F\frac{s}{|s|}, \qquad s=0 \;\Rightarrow\; T\leq\mu F, \quad (15)$$

where μ is the friction coefficient, and $T=\mathbf{T}\cdot\mathbf{i}$ is the projection of the vector **T** on the common tangent at the contact point.

In accordance with the Poisson hypothesis we distinguish two phases of the collision. They are separated from each other by an instant τ_m in which the magnitude of the normal force **F** achieves the maximum, and the closing velocity **c** is equal to zero. The impulses of the force **F** in the first and in the second phase are coupled as follows

$$\overline{F}_{II} = k\,\overline{F}_m, \tag{16}$$

where *k* is the coefficient of restitution, and $\overline{F}_m = \int_0^{\tau_m} F(\tilde{\tau})d\tilde{\tau}$, $\overline{F}_{II} = \int_{\tau_m}^{\tau_f} F(\tilde{\tau})d\tilde{\tau}$.

Equations (6) - (13), written for time instant τ_f , together with equation (16) and relationships (15) create the set of collision laws. They allow to determine the kinematic state of the pendulum after collision. All of the laws, except the friction laws, have the global nature (in other words they are written in the integral formulation) and concern the whole collision. The law given by (15) is formulated for instantaneous values and it cannot be generalized for the impulses $\overline{F}$ and $\overline{T}$ in a simple way, i.e. by integrating. This is why due to the dual form of the friction law the value of the impulse $\overline{T}$ at the moment τ_f depends on existence of the slip at the contact point *M*. In what follows we illustrate how to solve this difficulty applying the Routh method.

3.4. Routh's method and collision of double pendulum

It should be emphasized that Routh's method is an exact one. Application of this method requires solving the system of equations (6) - (13) with respect to $\omega_{1y}, \omega_{2y}, \overline{H}, \overline{V}, \overline{R}, \overline{P}$. Then, substituting the solution for ω_{1y} and ω_{2y} into formula

$$\mathbf{v}_M = \boldsymbol{\omega}_1 \times \overline{OA} + \boldsymbol{\omega}_2 \times \overline{AM}, \tag{17}$$

we obtain two equations

$$s(\tau) - s(0) = \alpha\overline{T} + \varepsilon\overline{F}, \tag{18}$$

$$c(\tau) - c(0) = \varepsilon\overline{T} + \gamma\overline{F}, \tag{19}$$

where $\alpha, \varepsilon, \gamma$ are constant dependent on the geometric parameters, the inertia parameters and the configuration of the pendulum at $\tau = 0$. This means that they must be determined for each collision.

The Routh method has a clear geometrical interpretation in the impulse space. In the case of planar motion, this space is two-dimensional. Accordingly to the nature of a normal force **F**, its impulse is always non-negative. Therefore, we will say further about the impulse semi-plane.

Assuming that in Eq. (18) $s(\tau)=0$, one can derive the condition for the collision without slip. A line called the line of sticking corresponds to this condition on the semi-plane. Similarly, supposing $c(\tau)=0$ in Eq. (19), we obtain the equation of the line of maximum compression. The relationship between impulses $\overline{F}$ and $\overline{T}$, when $s(\tau)\neq 0$, derived from (15) is as follows

$$\overline{T}=-\mu\overline{F}\frac{s(0)}{|s(0)|}. \tag{20}$$

The line of friction is its image on the impulse semi-plane.

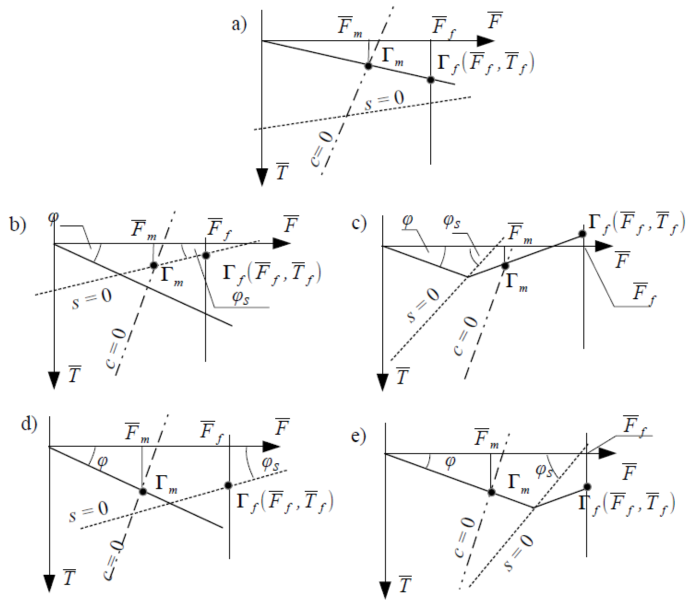

Figure 3. Interpretation of Routh's method on the impulse semi-plane.

It should be noted that all lines are strictly determined by a configuration state of the pendulum as well as the velocity $\mathbf{v}_M$ at the moment in which the collision starts. Mutually location of these lines is of decisive importance for the values of impulses at the end of each collision. There are some possibilities of this arrangement. Five of them are presented in Figure 3. The axes of the coordinate system are scaled in units of impulse. The starting point of the line of friction (depicted by bold solid line) lies always at the origin of the system and represents the beginning of the collision (we assume that at $\tau = 0$, the slip always appears). On the graph a), the line of friction and the line of sticking (depicted by dotted line) do not cross each other in the first phase of collision (i.e. on the left side of the line of maximum compression c = 0). Therefore, the point Γ_m in which the line of friction and the line of maximum compression intersect each other determines the end of the first phase of the collision. In accordance with Eq. (16), the abscissa of the point Γ_m multiplied by (1+*k*) specifies the line of end of collision (it is always parallel to the vertical axis). The line of friction crosses it at the point Γ_f the coordinates of which are sought values of impulses $\bar{F}$ and $\bar{T}$ at the end of the collision. On the graphs b) and c) the line of friction crosses the line of sticking in the first phase of the collision. That means then the sliding velocity **s** becomes equal zero. The relation between the angles φ and φ_s is of decisive importance for the possibility of continuation the collision accompanied by the slip. If $\varphi > \varphi_s$, what is shown in Figure 3b), the slip vanishes and the collision without the sliding lasts to the end of the contact with the obstacle. Since that moment, the line of sticking represents the collision and its crossing-points with the lines of maximum compression as well as the line of end of collision determine the point Γ_f. Otherwise, what is shown on the graph c), the friction forces are too small to eliminate the slip. So, the disc having changed the direction still slides on the obstacle. The change of the sign of the vectors **s** and **T** causes that on the line of friction appears the discontinuity in slope. Graphs d) and e) concern the situation when the line of friction crosses the line of sticking in the second phase of the collision. For each of the graphs shown in Figure 3 exists its counterpart, that presents an arrangement with the line of friction which is symmetric in respect to the horizontal axis.

4. Results

Let us consider the motion of double pendulum whose parameters are as follows: $m_1 = 3\text{kg}$, $m_2 = 2.4\text{kg}$, $l_1 = 0.5\text{m}$, $l_c = 0.34\text{m}$, $r = 0.04\text{m}$. The moments of inertia due to central principal axis: $I_1 \approx 0.64\text{kgm}^2$, $I_2 \approx 0.94\text{kgm}^2$. During the motion, the pendulum collides with a fixed obstacle that is placed at the distance *D* = 0.8m from the point *O*. The chosen values of the coefficients of friction and restitution are: $\mu = 0.2$, $k = 0.9$. It is assumed that do not act the external torques and there is no damping at the both joints. The values for the initial conditions are as follows: $\psi_1(0) = 130°$, $\psi_2(0) = 90°$, $\dot{\psi}_1(0) = 0.6\text{rad/s}$, $\dot{\psi}_2(0) = 0.5\text{rad/s}$. The numerical simulations have been done using the original software written in Fortran 95.

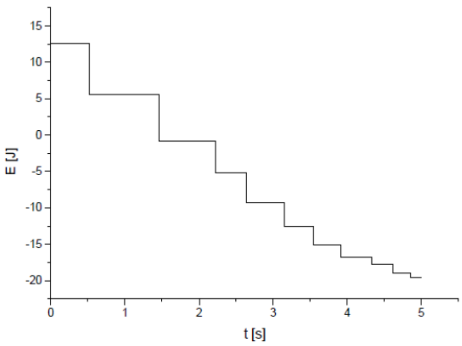

Figure 4. Stepwise changes of mechanical energy of the pendulum.

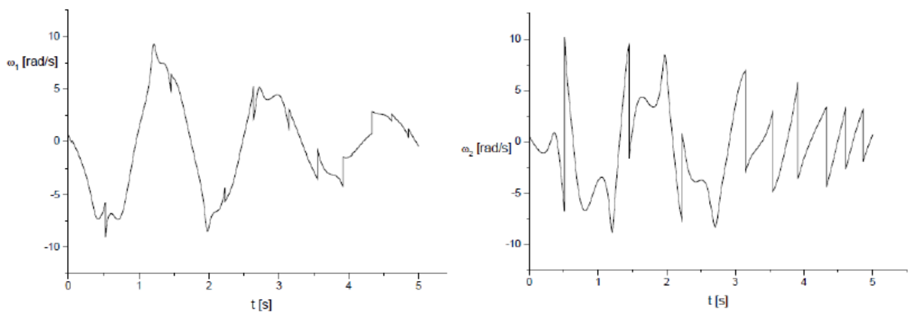

Figure 5. Stepwise changes of angular velocities of the pendulum parts.

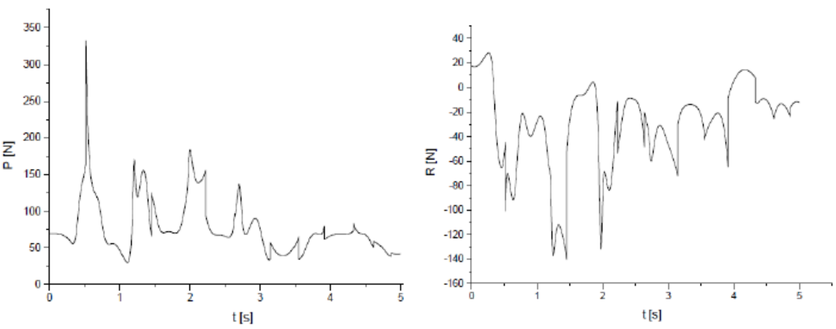

Figure 6. Stepwise changes of reaction forces.

In Figure 4, the loss of mechanical energy versus time of motion is shown. Under the given above assumptions, the system may lose the energy only as a result of collisions. Indeed, between the collisions the total energy is constant. It can be seen that the pendulum collides ten times in time of simulation. The energy decreases stepwise after each contact with the obstacle (for comparison, the energy of the pendulum at the stable equilibrium position is about of -27.13 J).

In Figure 5, the time histories of the angular velocities of the both part of the pendulum are shown. The observed stepwise changes of their values (in number of ten) are the results of the collisions with the fixed and rough plane. The horizontal and vertical components of the reaction force at the joint *O* as functions of time are presented in Figure 6.

5. Conclusions

Models of collision phenomena taking into account friction forces at contact are more suitable for real problems. Possibility of the non-sliding type of contact significantly complicates the solving of the problem, even for relatively simple models with only one point of contact. In the paper, the Routh method has been applied to solve the problem concerning the collision of multi-body system. Combination of an exact method with numerical calculations in the framework of the piecewise smooth model allows us to investigate multiple collisions of mechanical systems under consideration.

Acknowledgments

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